Fractional Combinatorial Games

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Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

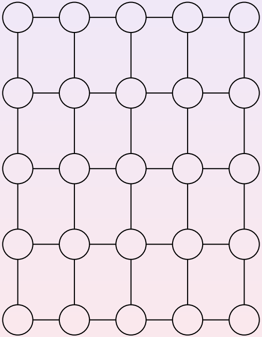
Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:
A cop at same node as robber.

Goal:

Cop-number=minimum number of cops



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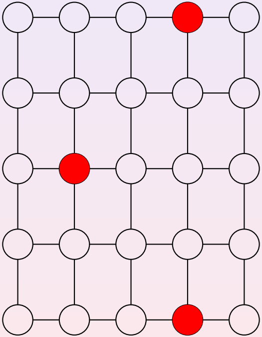

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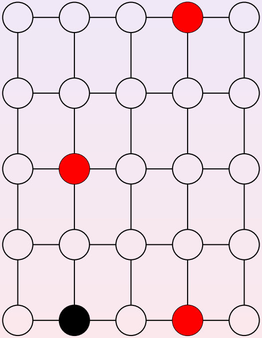

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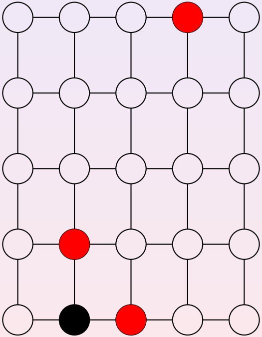

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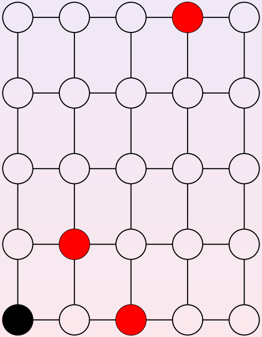

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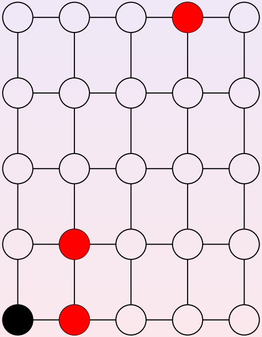

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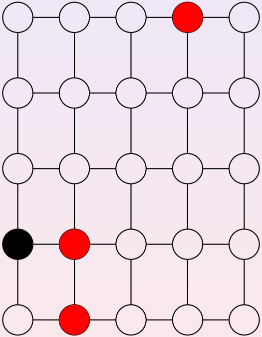

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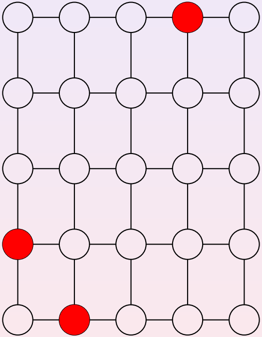
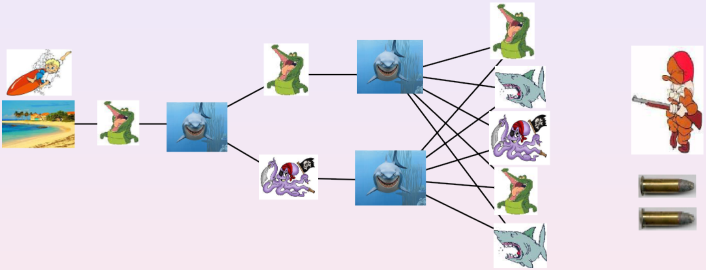


Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



An Observer must ensure that a Surfer never reaches a dangerous node

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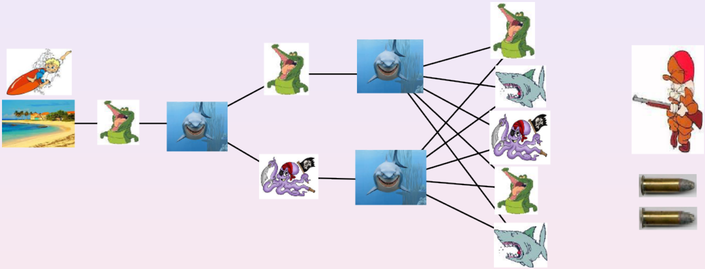

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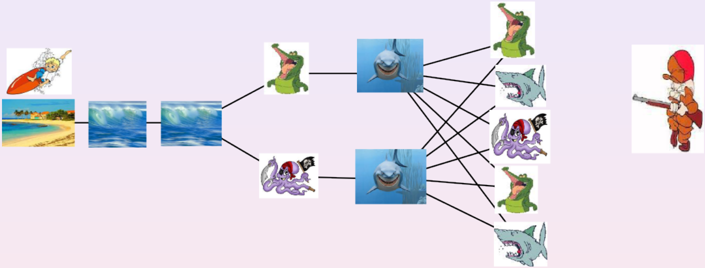

Turn by turn: Observer marks $k = 2$ nodes

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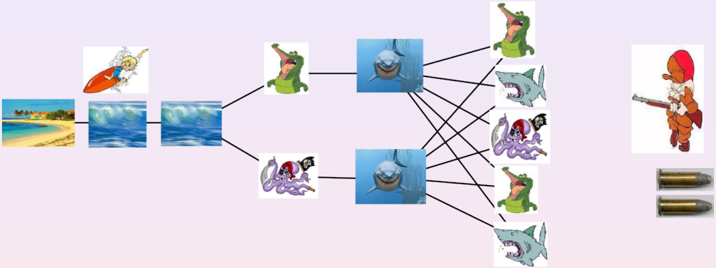

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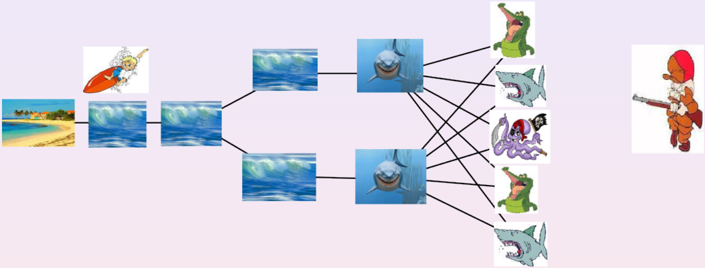

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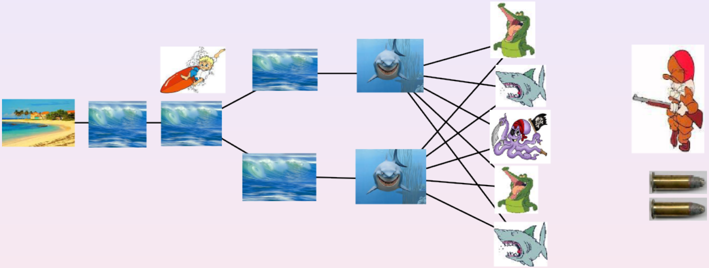

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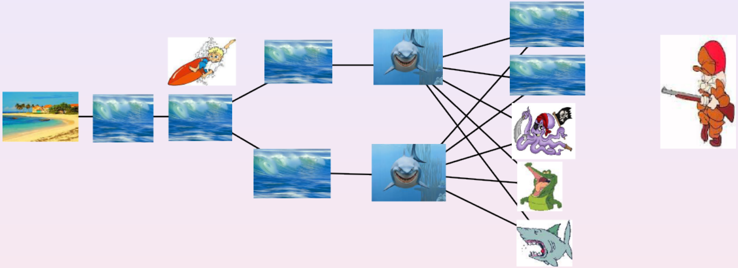

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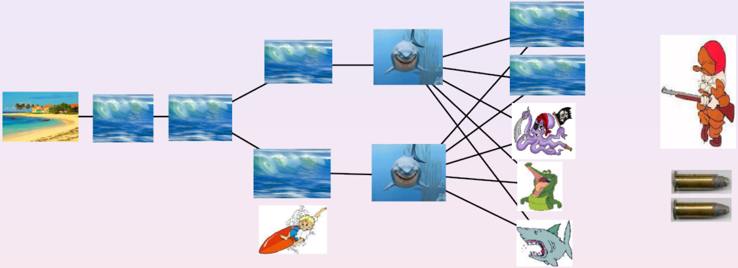

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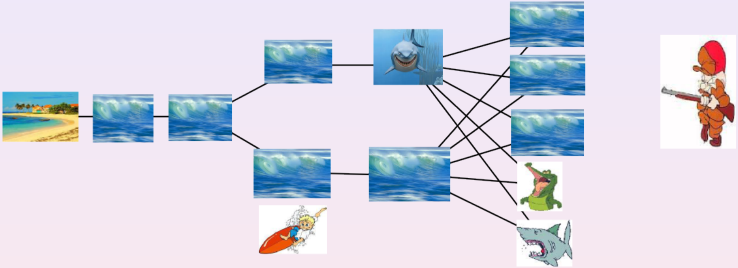

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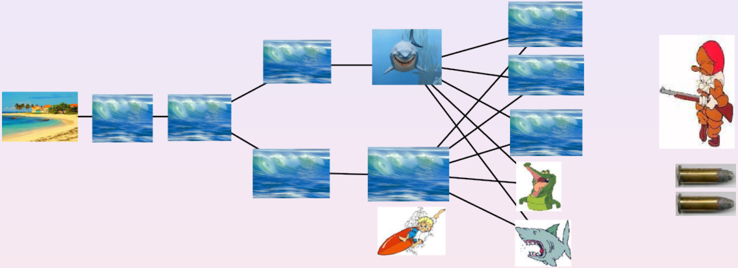

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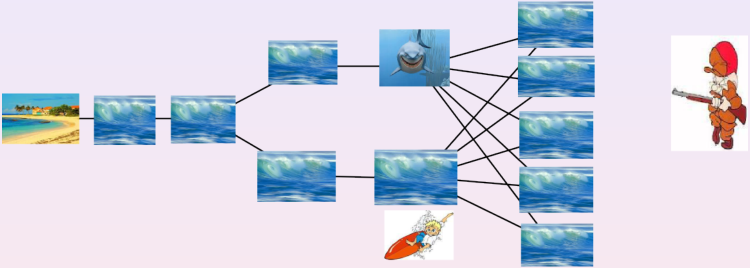

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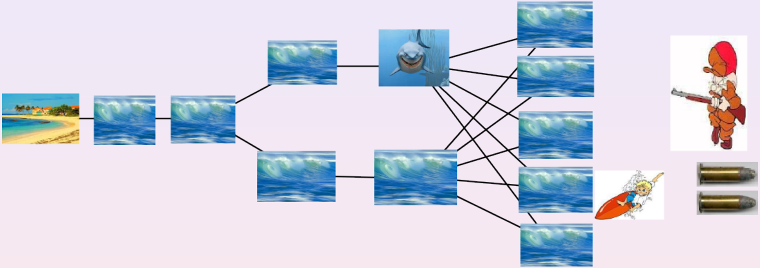

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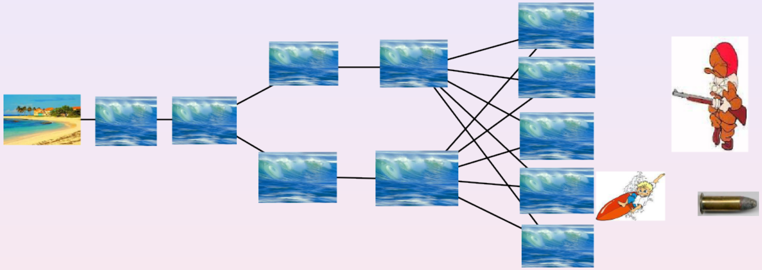

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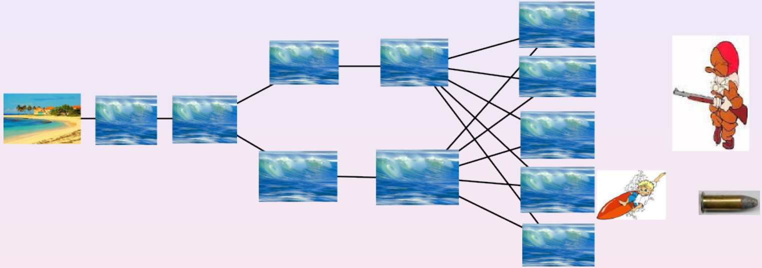
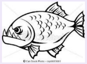


In this example, all nodes are marked

Victory of the Observer using 2 marks per turn

Model: another Two players game

- a *Surfer*  starts from safe homebase v_0 

  in G, a dangerous graph     
- a *Guard*  with some amount k of bullets 

Turn by turn:

1. the guard secures $\leq k$ nodes;
2. then, the Surfer may move to an adjacent node.

Defeat: Surfer in unsafe node  **Victory:** G safe 

Minimize amount of bullets to win for any Surfer's trajectory

Surveillance number of G (**connected**) from v_0: $sn(G, v_0)$

Two Players Combinatorial Games

- Two players play a game on a graph.
- Game is played turn-by-turn.
- Players play by moving and/or adding tokens on vertices of the graph.
- Optimization problem:

  minimizing number of tokens to achieve some goal

All these games are hard

- Cops and Robber: k cops are enough?
  - PSPACE-complete in general graphs [Mamino, 2012].
- Surveillance Game: k marks per turn are enough?
  - $k = 2$ NP-complete for Chordal/Bipartite Graphs [Fomin et al, 2012].
  - $k = 4$ PSPACE-complete for DAGS [Fomin et al, 2012].
- Angels and Devils: Does an Angel of power k wins?
  - 1-Angel loses in (infinite) grids [Conway, 1982].
  - 2-Angel wins in (infinite) grids [Máthé, 2007].
- Eternal Dominating set.
- Eternal Vertex Cover.
  - NP-hard [Fomin et al, 2010].

New tools/approaches are required

Several questions remain open

- Meyniel conjecture: $cn(G) = O(\sqrt{n})$ in any n-node graphs?
- Polynomial-time Approximation algorithms?

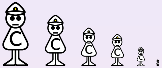
less difficult but still intriguing

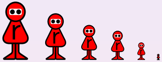
- number of cops to capture fast robber in grids?
- cost of connectivity in surveillance game?
- etc.

Here, we present preliminary results of our new approach

Fractional Combinatorial Game

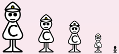
- Fractional games:

  both players can use "fractions" of tokens. 

- Semi-Fractional games:

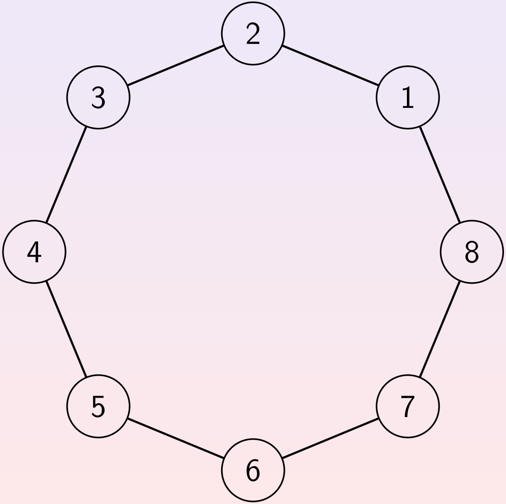
  only one player (Player C) can use fractions of tokens. 

- Integral games: classical games, token are unsplittable

Example: Fractional Cops and Robber

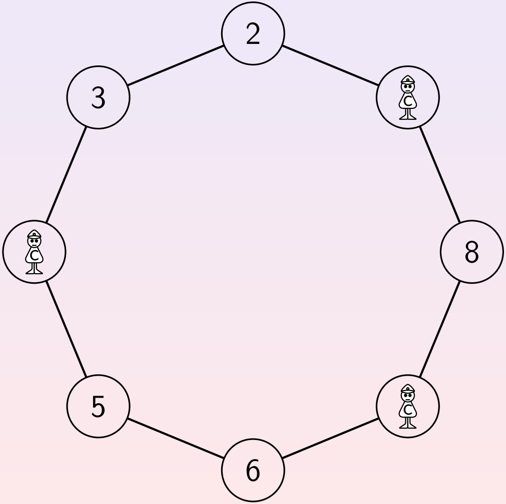
integral game:
cop-number $= 2$



Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

semi fractional:
cop-number $\leq 3/2$



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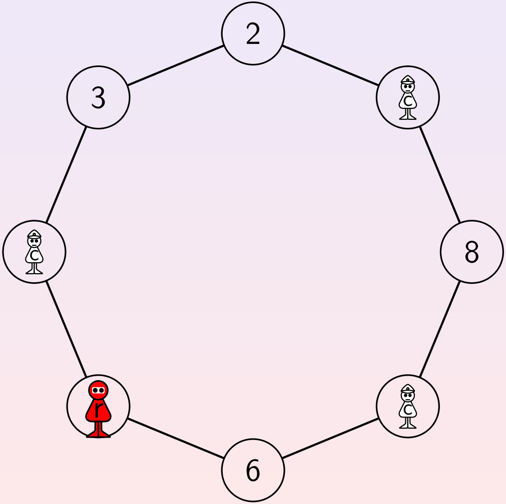

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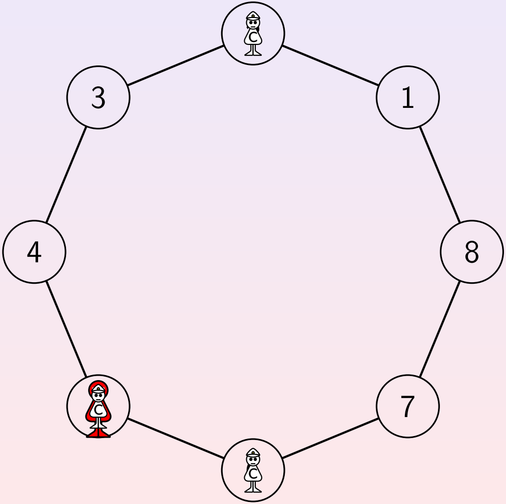

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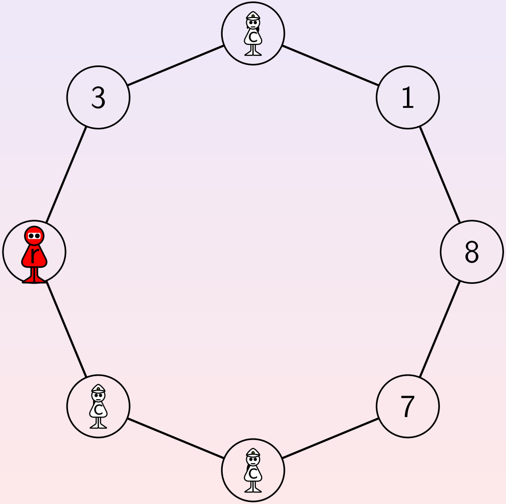

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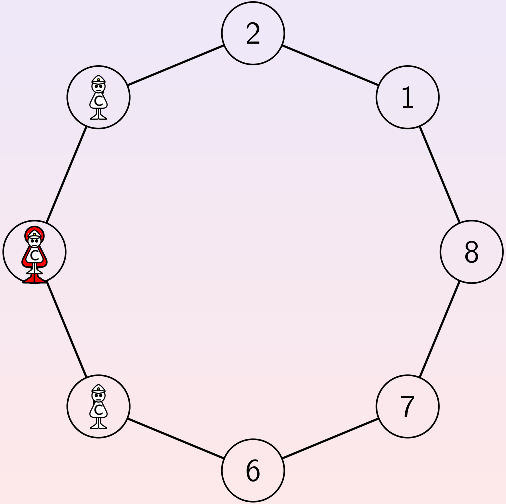

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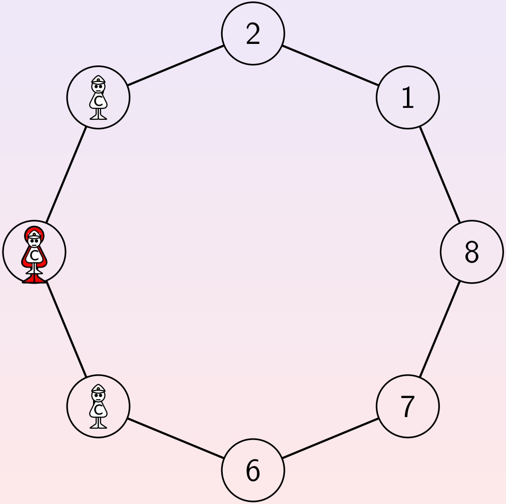

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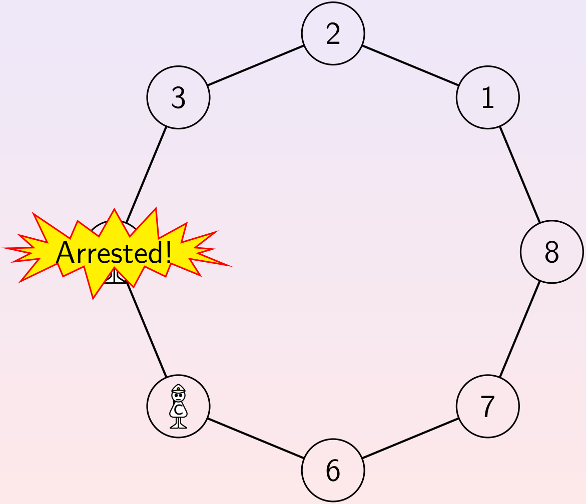

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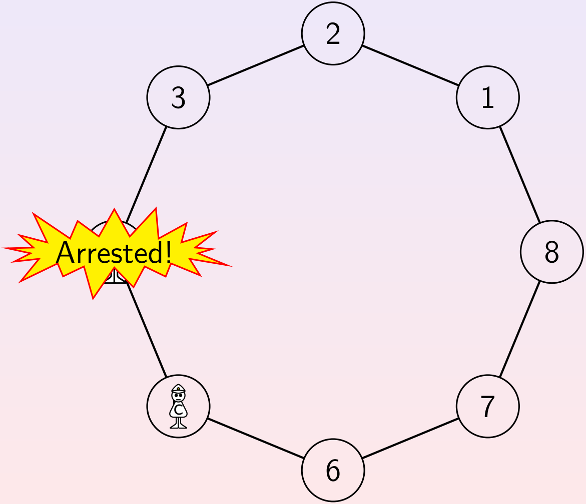
Remark:
by definition:
semi-fractional $\leq$ integral

gap?
relationship with fractional?



Preliminary results

Fractional games

general framework: fractional relaxation of turn-by-turn games
important property: *"convexity" of winning states*

Semi-fractional = fractional (properties of robber's moves)

solutions of fractional games provides lower bounds for integral games

Algorithm $\mathcal{A}$ to decide which player wins

tools: linear programming techniques.

Bad news: *one step of $\mathcal{A}$ is exponential (exponent: length of the game)*
Hope: *use specifities of games to reduce time-complexity*

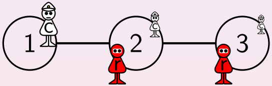
Integrality gap

Bad news: *fractional cop-number $\leq 1+\epsilon$ for any graph and any $\epsilon > 0$*
Hope: *surveillance game: fractional game gives a probabilistic* $\log n$*-approximation*

States of the Game

In n-node graph

- $c \in \mathbb{R}^n_+$ represents the tokens of Player C.
- $r \in \mathbb{R}^n_+$ represents the tokens of Player R.
- $(c, r) \in \mathbb{R}^{2n}_+$ represents the state of the game.



$c = (0.7, 0.2, 0.1)$ and $r = (0, 0.5, 0.5)$;

set of states = polytope Examples:

cops and robber: $\sum_{i \leq n} r_i = 1$ and $\sum_{i \leq n} c_i = k$ (# of cops)
surveillance game: $\sum_{i \leq n} r_i = 1$

Winning states and moves

winning states = convex subset of states

cops and robber: $\{(c,r) \mid c_i \geq r_i, i = 1 \cdots, n\}$
surveillance game: $\{(c,r) \mid c_i \geq 1, i = 1 \cdots, n\}$

moves

slide tokens along edges = multiplication by stochastic matrix in

$$\left\{ [\alpha_{i,j}]_{1 \leq i,j \leq n} \;\middle|\; \begin{array}{r} \forall 1 \leq i,j \leq n, \alpha_{i,j} \geq 0, \text{ and} \\ \forall j \leq n, \sum_{1 \leq i \leq n} \alpha_{i,j} = 1, \text{ and} \\ \text{if } \{i,j\} \notin E(G) \text{ then } \alpha_{i,j} = 0 \end{array} \right\}$$

if $ij \in E$, an amount $\alpha_{i,j}$ of the token in v_j goes to v_i

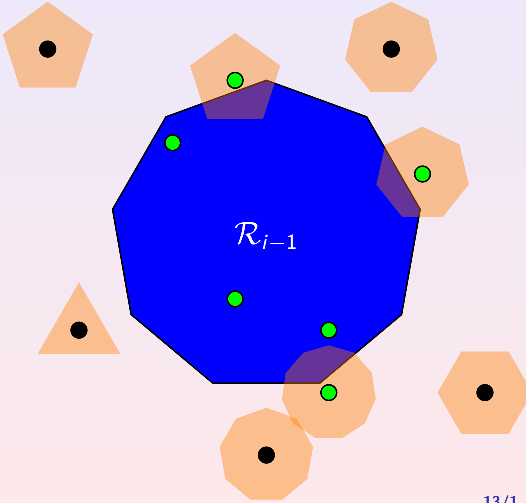
mark nodes = add to c a vector in

$$\{(m_1, \cdots, m_n) \mid \sum_{i \leq n} m_i \leq k\}$$

Main Idea of Algorithm 1/2

$\mathcal{R}_{i-1}$: states from which Player C always wins in at most $i-1$ rounds, when Player R is the first to play.

$\mathcal{C}_i = \{(c,r) \mid \exists move, (move(c), r) \in \mathcal{R}_{i-1}\}$



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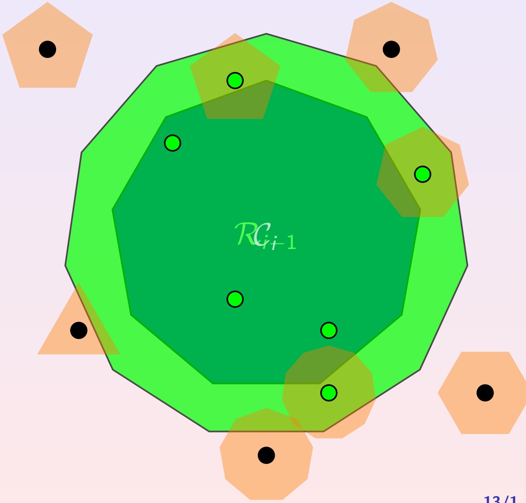
$\mathcal{C}_i$: states from which Player C always wins in at most i rounds when playing first

$\mathcal{C}_i$ = polytope, computable from $\mathcal{R}_{i-1}$, polynomial-size but in higher dimension

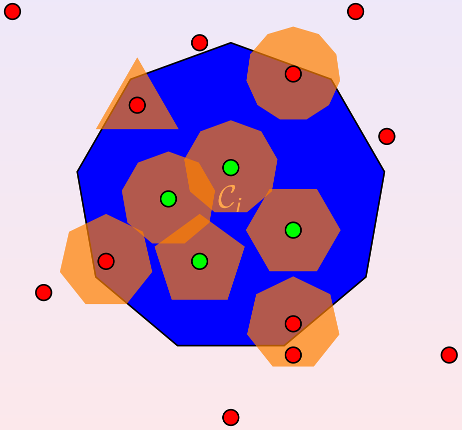
Problem: projection



Main Idea of Algorithm 2/2

$\mathcal{C}_i$: states from which Player C always wins in at most i rounds when playing first

$\mathcal{R}_i = \{(c, r) \mid \forall move, (c, move(r)) \in \mathcal{C}_i\}$



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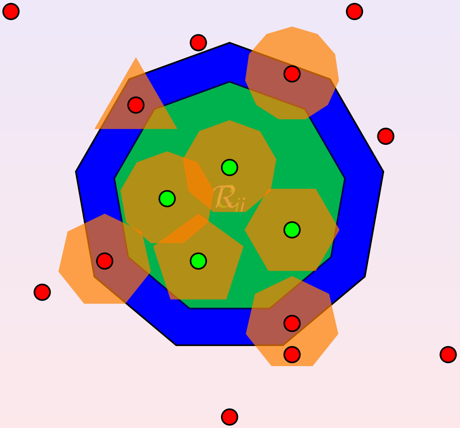
$\mathcal{R}_i$: states from which Player C always wins in at most i rounds, when Player R is the first to play.

$\mathcal{R}_i$ = polytope, computable from $\mathcal{C}_i$, polynomial-size.
Trick: “just" have to reenforce each constraint



Bad news and Good news

Fractional cop-number is one :(

Strategy: place $f_1 = 1/n$ cop per node

At step i,

1. $h_i = \sum_{j \leq i} f_j$ cop "follows" the robber.
   $1 - h_i$ cop remains
2. place $f_{i+1} = \frac{1-h_i}{n}$ cop per node.

$h_i \to_{i \to \infty} 1$

Meyniel conjecture seems safe...

approximation for surveillance number? :)

Surveillance game: inequality defining the polytopes are similar to set cover

Proof based on approximation of set cover

for the moment: only probabilistic strategy

Conclusion and Future Work

Promising framework (we hope)

Lot of work remains:

- polynomial algorithm? (it is if length of game is bounded)
- approximation?
- fast robber?
- careful analysis of polytopes in each game
- etc.

Thank you