Practical computation of pathwidth and vertex separation

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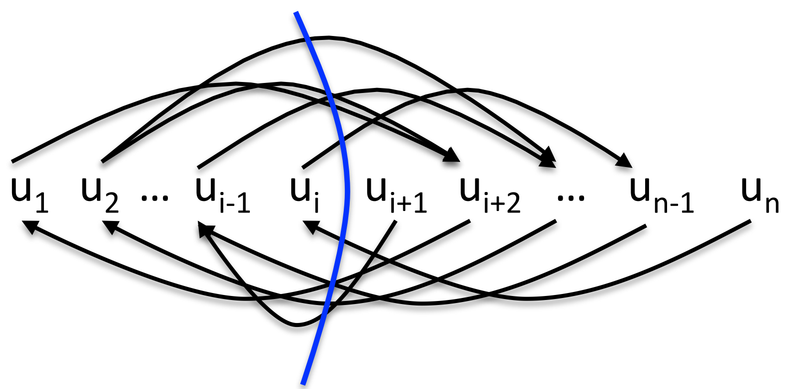
Layout, vertex separation, pathwidth

Layout

- $L = u_1 u_2 \ldots u_n$ is an **ordering** of the elements of a set V
- $\mathcal{L}(V)$, set of all orderings of V

Vertex-separation of a digraph D = (V,A)

- $\omega(L,i) = |\ N_D^+ (u_1 u_2 \ldots u_i)\ |$
- $\omega(L) = \max_{i \leq |L|} \omega(L,i)$
- **$vs(D) = \min_{L \text{ in } \mathcal{L}(V)} \omega(L)$**



Pathwidth of a graph G = (V,E)

- Symmetric digraph D, pw(G) = vs(D)

Existing implementations

Dynamic Programming:

- [Bodlaender, Fomin, Koster, Kratsch, Thilikos - ToCS 2012]
- Worse case time and space complexity in $O(2^n)$
- $O(n^{vs(D)})$ algorithm implemented by N. Cohen in Sage (http://www.sagemath.org)
- (di)graphs with up to 32 nodes in 1 sec.
  - 32 nodes: use up to 4GB of RAM

ILP and SAT formulations:

- [Biedl, Bläsius, Niedermann, Nöllenburg, Prutkin, Rutter - Graph Drawing 2013]
- 20-30 nodes (minutes to hours)
- Another MILP formulation in Sage

Branch-and-Bound:

- [Solano & Pioro – IEEE/OSA JOCN 2010]
- 30 nodes (sec. to hours)

Outline

1. **Pre-processing / reduction rules**
   1. Pathwidth
   2. Vertex separation

2. **Generic branch-and-bound**

3. **Improvements**
   1. Greedy steps
   2. Storing prefixes

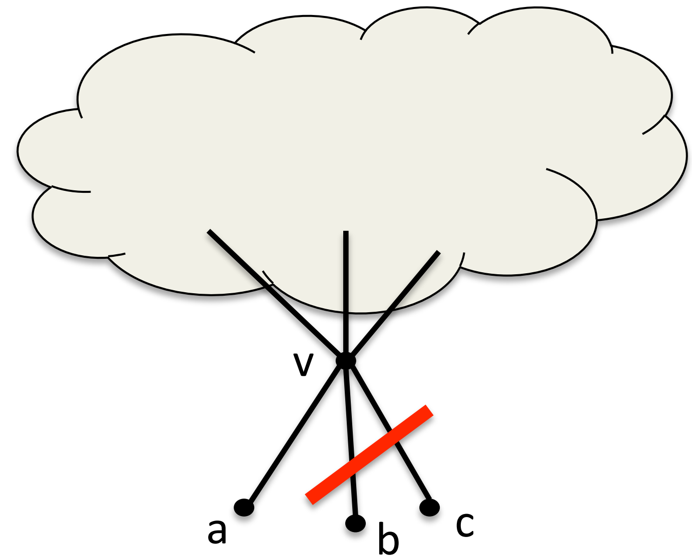
4. **Numerical results**

(Simple) Reduction rules for pathwidth

Simple: **easy to implement** **and** **small computation time**

Safe: pw(G) = pw(G')

Rule 1: If two degree-one vertices share their neighbor, then remove one of them.

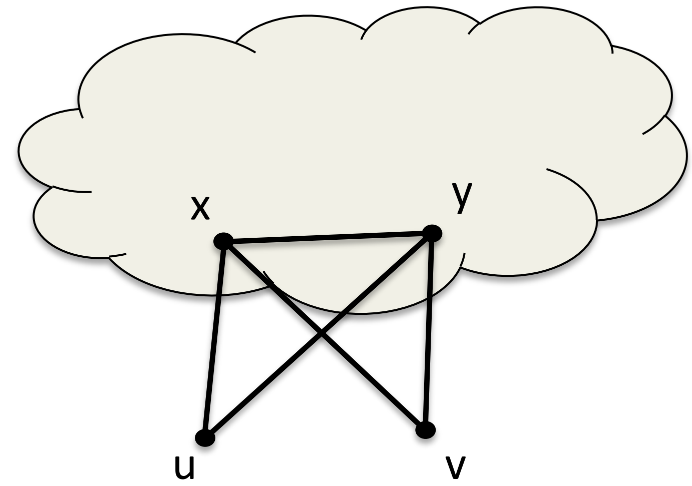


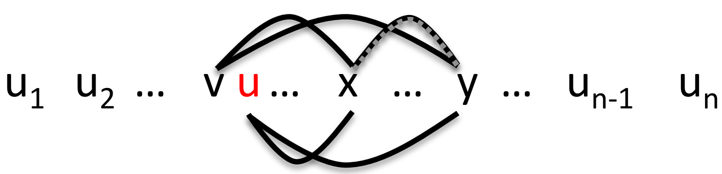
u_1 u_2 … u_{i-1} u_i a … b c v … u_{n-1} u_n

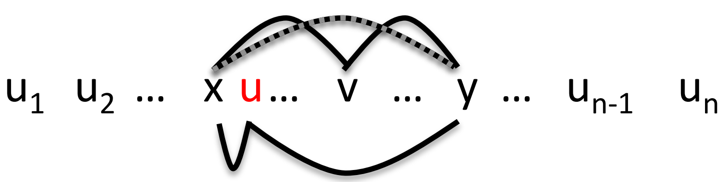
u_1 u_2 … u_{i-1} u_i b c v … a … u_{n-1} u_n

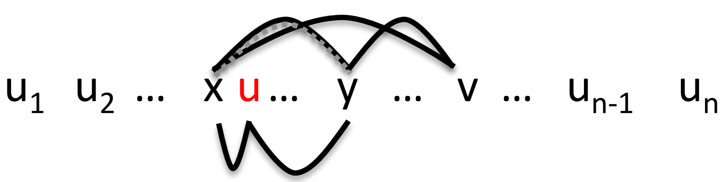
Reduction rules for pathwidth (2)

Rule 2: If two degree-two vertices u and v have the same neighbors x and y, then contract edge ux.









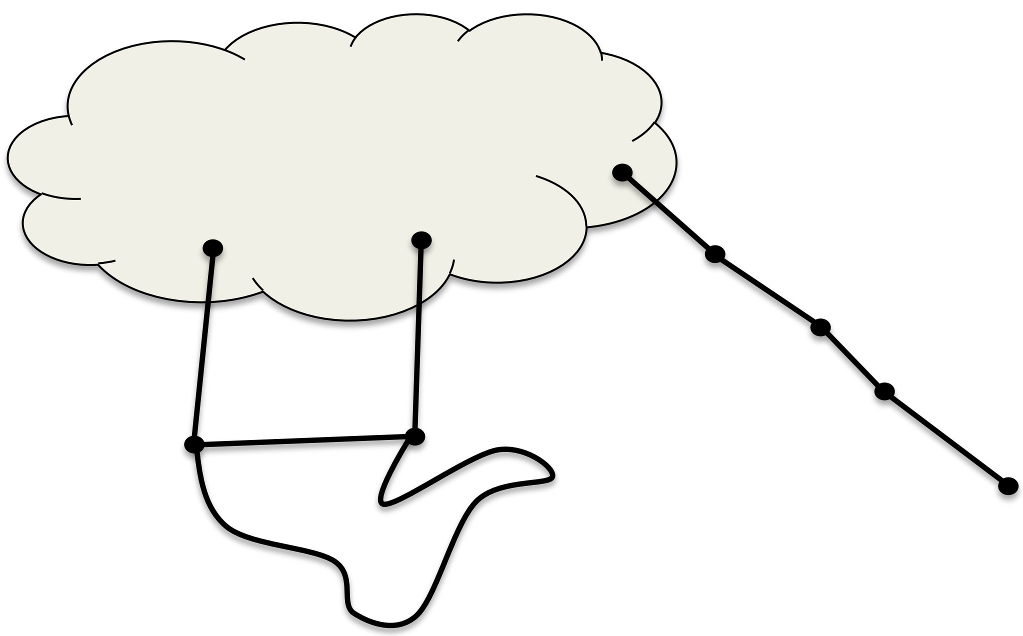
Insert u after leftmost of v, x, y

Value of $\omega(L, i)$ unchanged

Reduction rules for pathwidth (3)

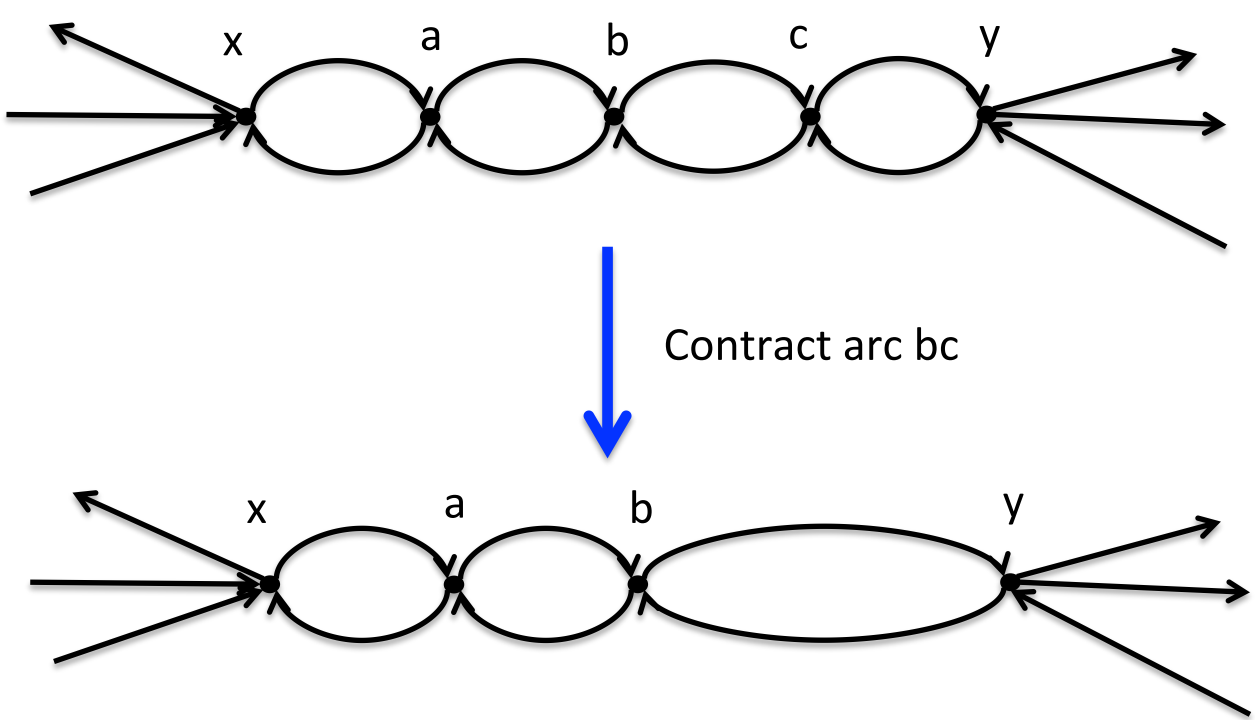
Rule 3: Contract long chains

- Contract long chains of degree-two vertices to chains of length 3 (edges).
- Contract pending chains of degree-two vertices to chains of length 2.



Reduction rules for vertex separation

Rule 4: Contract long **symmetric** chains

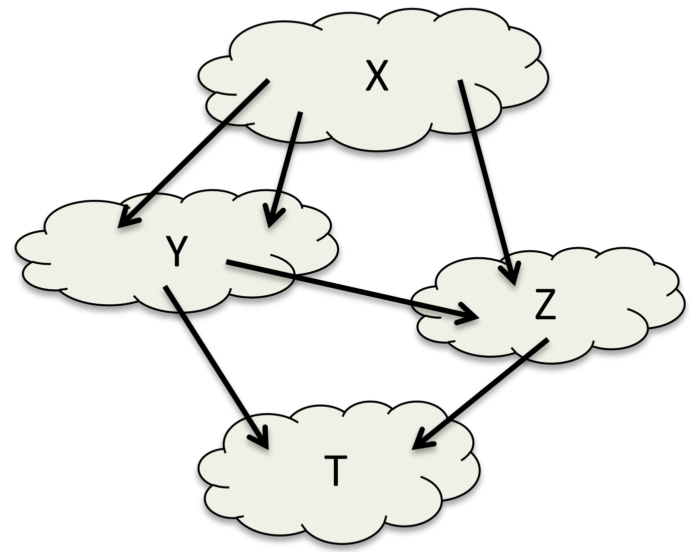


Reduction rules for vertex separation (2)

Rule 5: Decomposition into **strongly connected components**

$L(D) = L(T) \odot L(Z) \odot L(Y) \odot L(X)$

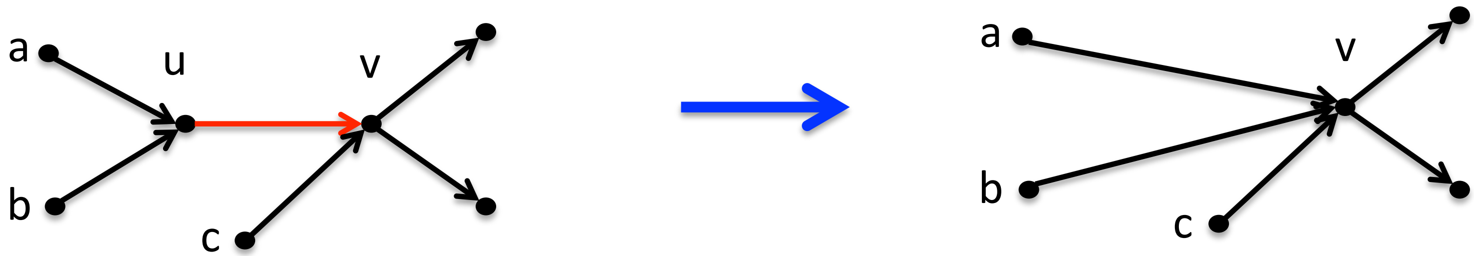
$$\underbrace{t_1\ t_2\ \dots\ t_{|T|}}_{L(T)}\ \underbrace{z_1\ z_2\ \dots\ z_{|Z|}}_{L(Z)}\ \underbrace{y_1\ y_2\ \dots\ y_{|Y|}}_{L(Y)}\ \underbrace{x_1\ x_2\ \dots\ x_{|X|}}_{L(X)}$$

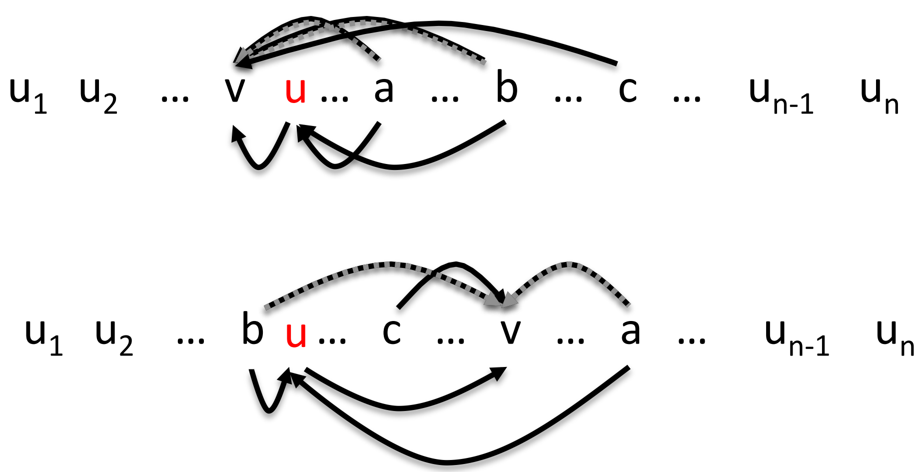


$vs(D) = \max \{ vs(X), vs(Y), vs(Z), vs(T) \}$

Reduction rules for vertex separation (3)

Rule 6: Contract arc *uv* if *u* has out-degree one or *v* has in-degree one unless it creates a loop or parallel arcs

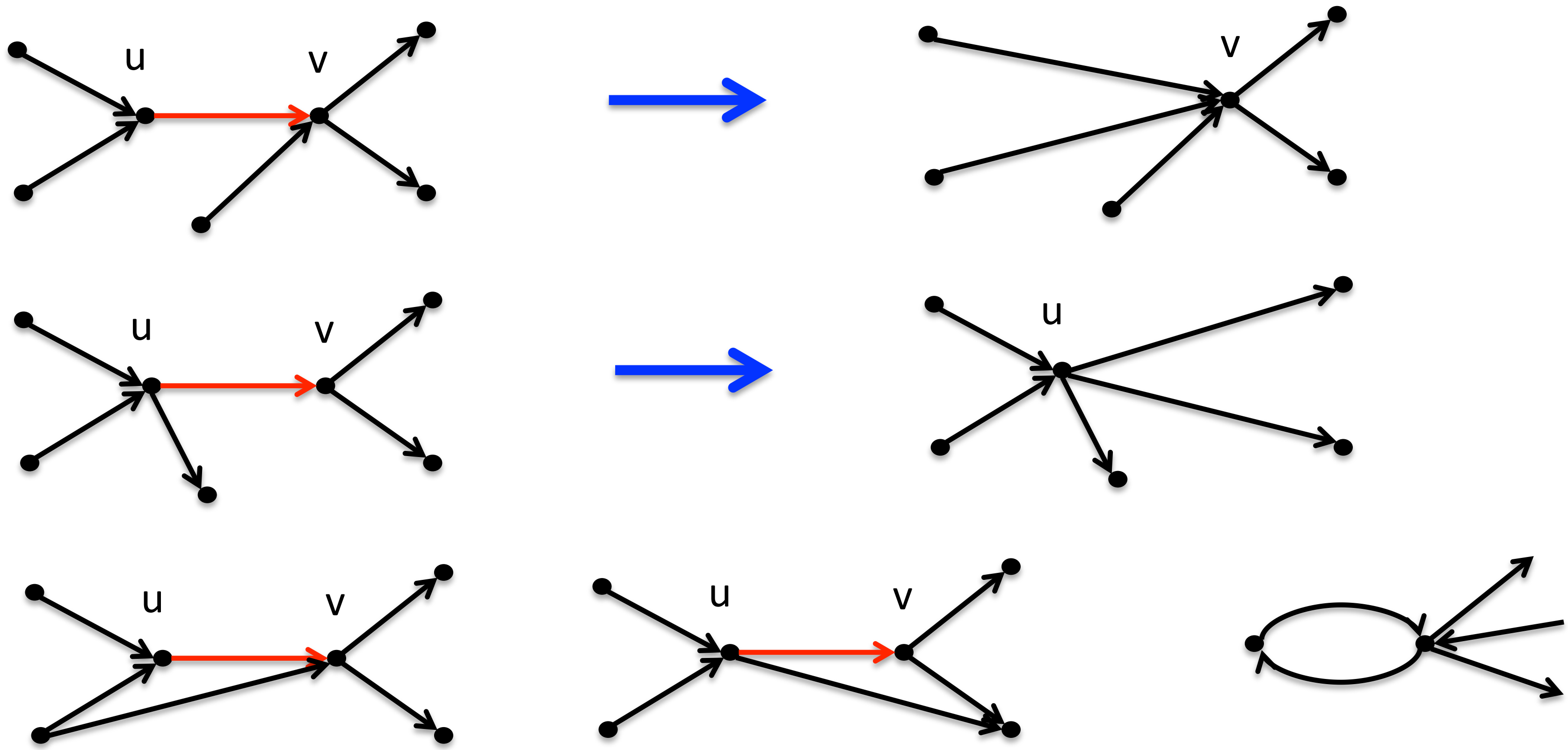




Insert after leftmost of v and the in-neighbors of v

Reduction rules for vertex separation (3)

Rule 6: Contract arc *uv* if *u* has out-degree one or *v* has in-degree one unless it creates a loop or parallel arcs



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Generic Branch and Bound

[Solano & Pioro – IEEE/OSA JOCN 2010]

Principle:

- Given current prefix $P = u_1\ u_2 \dots u_k$
- Try all possible extensions $P \odot v$ for v in $V\text{-}V(P)$ using *sort-and-prune*

Sort:

- Try most promising extensions first
- Sort vertices in $V\text{-}V(P)$ by increasing values of $\omega(P \odot v)$

$\omega(P) = \max_{1 \leq i \leq |P|} \omega(L, i)$
for any layout L with prefix P

Prune:

- Given best layout found so far, L^*
- Try only extensions such that $\omega(P \odot v) < \omega(L^*)$

$\omega(P \odot v)$ is a lower bound on the value of any layout with prefix $P \odot v$

Generic Branch and Bound

[Solano & Pioro – IEEE/OSA JOCN 2010]

B&B(D, P, L*):
if V(P) == V and ω(P) < ω(L*) :
    L* := P
else:
    for all v in V-V(P) by increasing value of ω(P ⊙ v) **do**
        if ω(P ⊙ v) < ω(L*) :
            P' := B&B(D, P ⊙ v, L*)
            if ω(P') < ω(L*):
                L* := P'
return L*

sort

prune

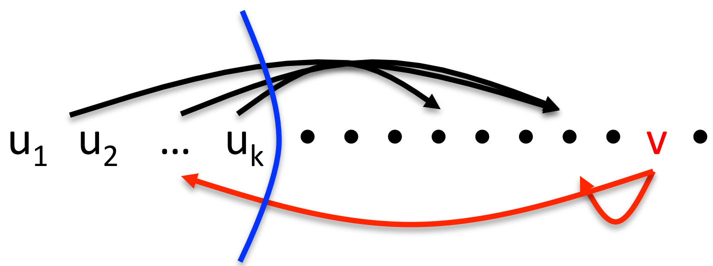
L* := B&B(D, Ø, L)   for some ordering L of V

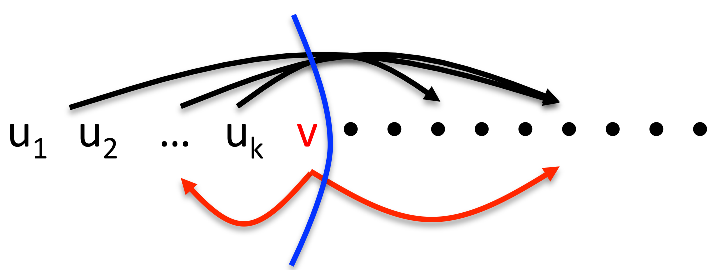
Greedy steps

Lemma 1:

- Given D = (V, A) and prefix $P = u_1\, u_2 \dots u_k$
- Set $\mathcal{L}(V, P)$ of layouts with prefix P
- If there exists v in V-V(P) such that $N^+(v) \subseteq V(P) \cup N^+(V(P))$

  then $\min_{L \text{ in } \mathcal{L}(V,\, P)} \omega(L) = \min_{L \text{ in } \mathcal{L}(V,\, P \odot v)} \omega(L)$



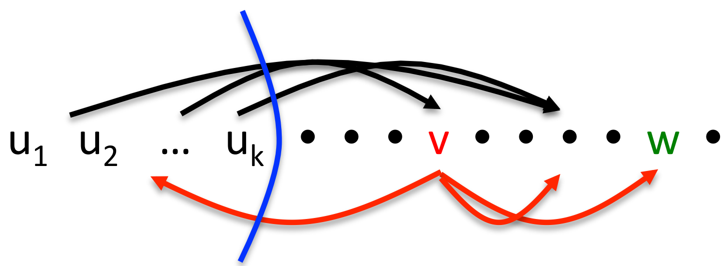


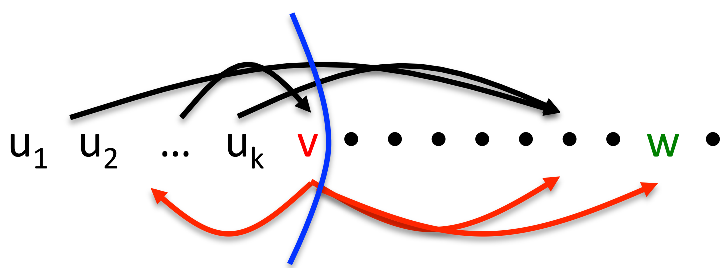
Greedy steps

Lemma 2:

- Given $D = (V, A)$ and prefix $P = u_1\ u_2 \dots u_k$
- Set $\mathcal{L}(V, P)$ of layouts with prefix P
- If there exists v in $N^+(V(P))$ such that $N^+(v) - (V(P) \cup N^+(V(P))) = \{w\}$

  then $\min_{L \text{ in } \mathcal{L}(V,\ P)} \omega(L) = \min_{L \text{ in } \mathcal{L}(V,\ P \odot v)} \omega(L)$





Branch and Bound + Greedy steps

```
B&B(D, P, L* ):
  P := Greedy_steps(D, P)
  if  V(P) == V  and  ω(P) < ω(L*) :
        L* := P
  else:
        for all v in V-V(P) by increasing value of ω(P ⊙ v) do
              if ω(P ⊙ v) < ω(L*) :
                    P' := B&B(D, P ⊙ v, L*)
                    if  ω(P') < ω(L*):
                          L* := P'
  return L*
```

$L^* := B\&B(D, \emptyset, L)$ for some ordering L of V

Observation

L = a b c d *R* best layout with prefix a b c d

1. ω(a b c d) < ω(L) ⟶ the layout of a,b,c,d has no impact on the solution
2. ω(a b c d) = ω(L) ⟶ possible improvement with better layout for a,b,c,d

Lemma 3:

- Given D = (V, A) and prefixes P, P' such that V(P)=V(P')
- Set $\mathcal{L}(V, P)$ of layouts with prefix P
- If $\omega(P) < \min_{L \text{ in } \mathcal{L}(V,\,P)} \omega(L)$ or $\omega(P) \leq \omega(P')$

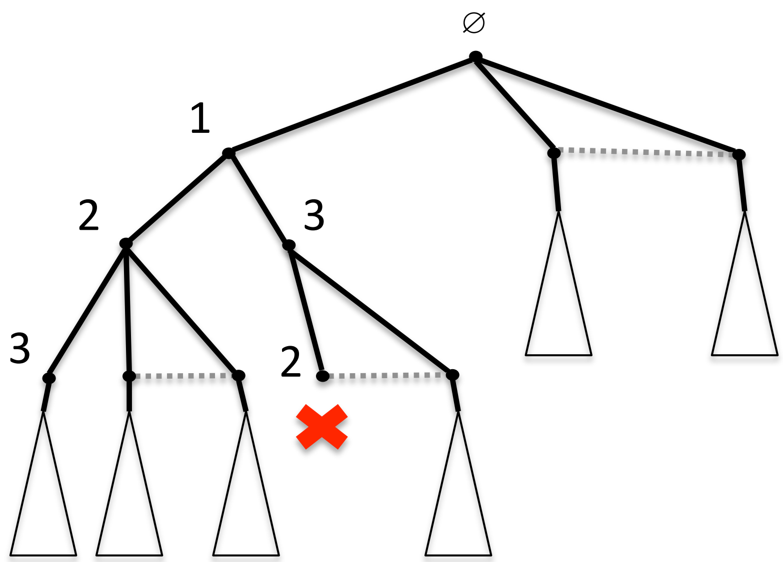
  then $\min_{L \text{ in } \mathcal{L}(V,\,P)} \omega(L) \leq \min_{L \text{ in } \mathcal{L}(V,\,P')} \omega(L)$

Prune branches with prefix P' such that V(P)=V(P') when best possible layout L* with prefix P is such that ω(P) < ω(L*)

Prune branches starting with prefix P' such that ω(P) ≤ ω(P')

Idea: Use a table to store prefixes

Store prefixes



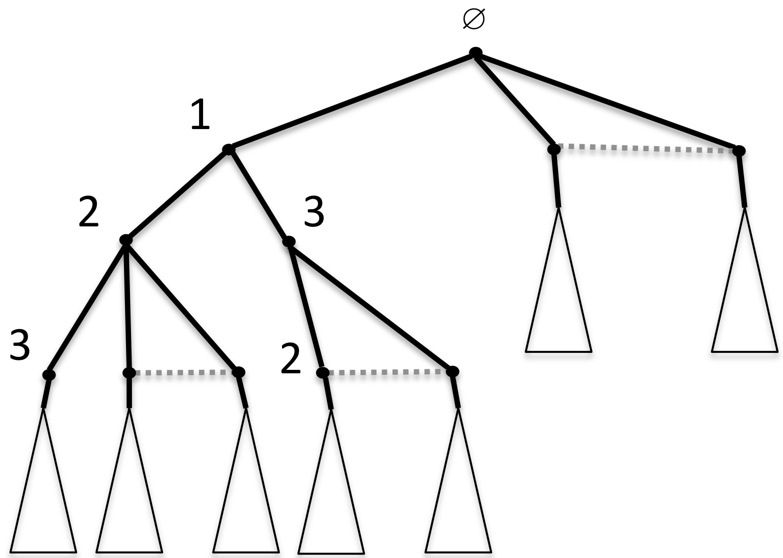
V(P)	ω(P)	b
1	ω(1)	0
1,2	ω(1,2)	0
1,2,3	ω(1,2,3)	0 1
…	…	…

P = (1,2,3)

If $\omega(P) < \min_{L \text{ in } \mathcal{L}(V,\, P)} \omega(L)$

then no possible improvement with other layouts of V(P)

Store prefixes



$V(P)$	$\omega(P)$	b
1	$\omega(1)$	0
1,2	$\omega(1,2)$	0
1,2,3	$\omega(1,3,2)$	0
…	…	…

$P = (1,2,3)$

If $\omega(P) = \min_{L \text{ in } \mathcal{L}(V,\ P)} \omega(L)$

then do nothing

We must try with other layouts of V(P)

$P' = (1,3,2)$

If $\omega(P') < \omega(P)$ then update table

Final branch and Bound

Prefix table

```
B&B(D, 𝒫, P, L* ):
 If not (V(P), ω(P), 1) in 𝒫:
     P := Greedy_steps(D, P)
     if V(P) == V and ω(P) < ω(L*) :
         L* := P
     else:
         Orig := ω(L*)
         for all v in V-V(P) by increasing value of ω(P ⊙ v) do
             if ω(P ⊙ v) < ω(L*) :
                 P' := B&B(D, 𝒫, P ⊙ v, L*)
                 if ω(P') < ω(L*):
                     L* := P'
         Update-prefix-table (D, 𝒫, V(P), Orig, ω(L*) )
 return L*
```

L* := B&B(D, ∅, ∅, L)   for some ordering L of V

Implementation

- Sage open source mathematical software (Python/Cython)
  - Several graph algorithms
  - Existing implementation of dyn. prog. and MILP formulation for vertex separation

- Use bitsets for fast operations on neighborhoods
  - Union (or), intersection (and), size (popcount), etc.
  - Use int on 32 bits or 64 bits if n smaller than 64.

- Prefix table: tree structure
  - Parameterized maximum prefix length (e.g. 10 or n/3)
  - Limit on the number of stored prefixes (e.g. 10 millions)

Performances on directed GNP

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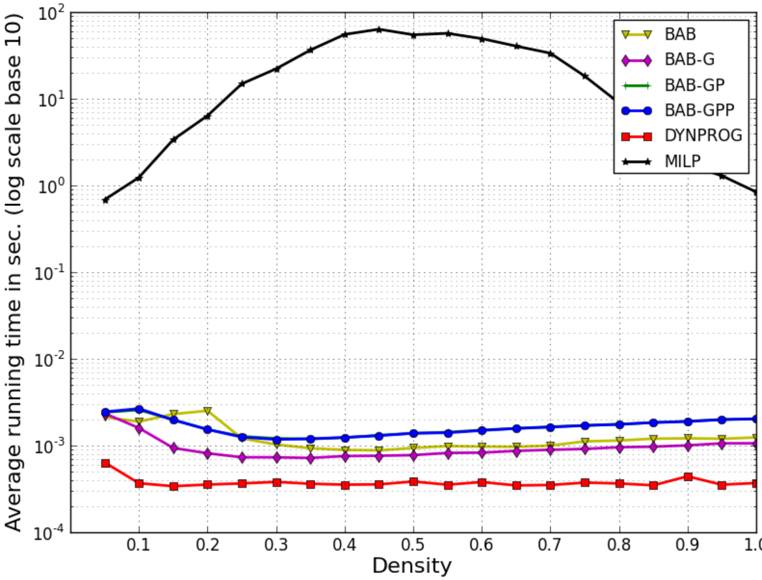

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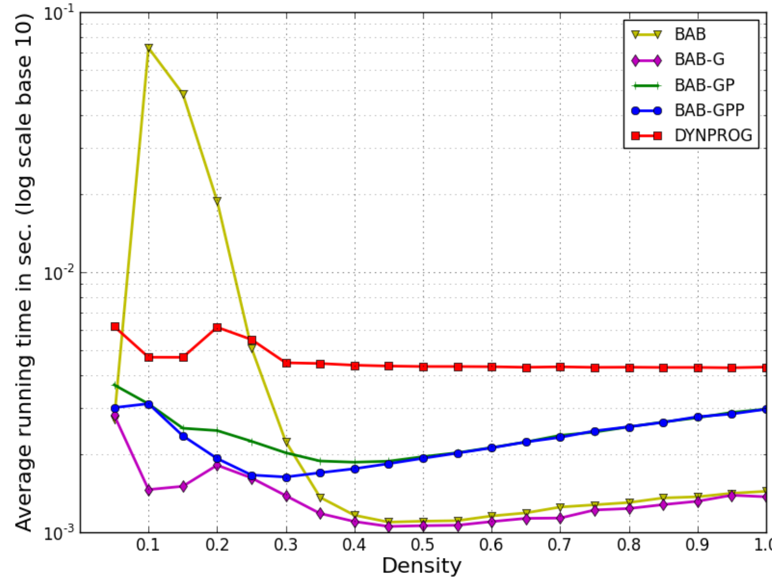

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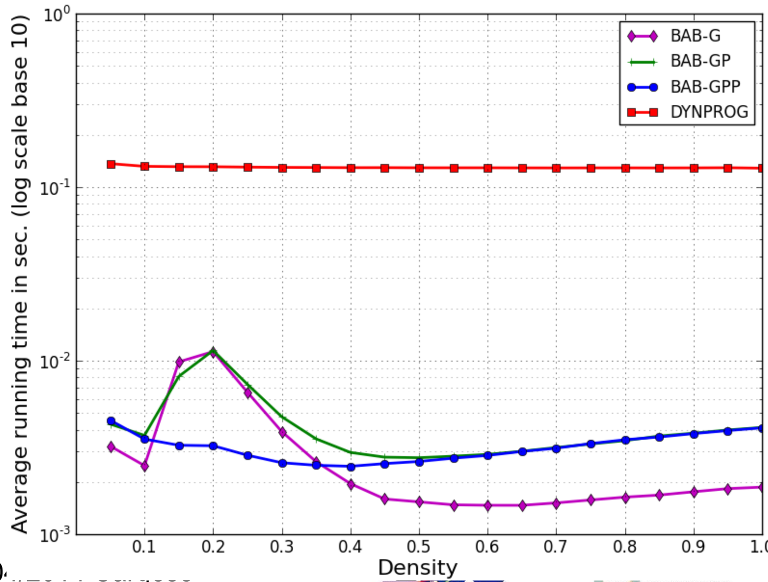

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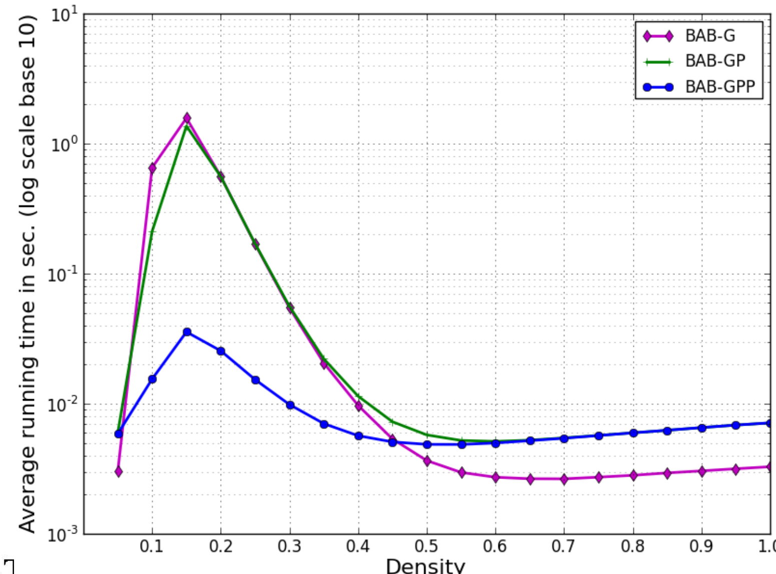

Performances on directed GNP

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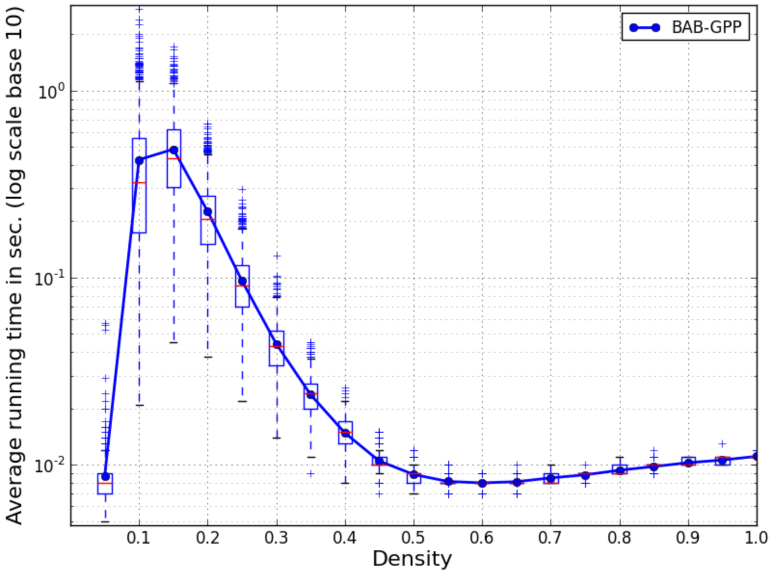
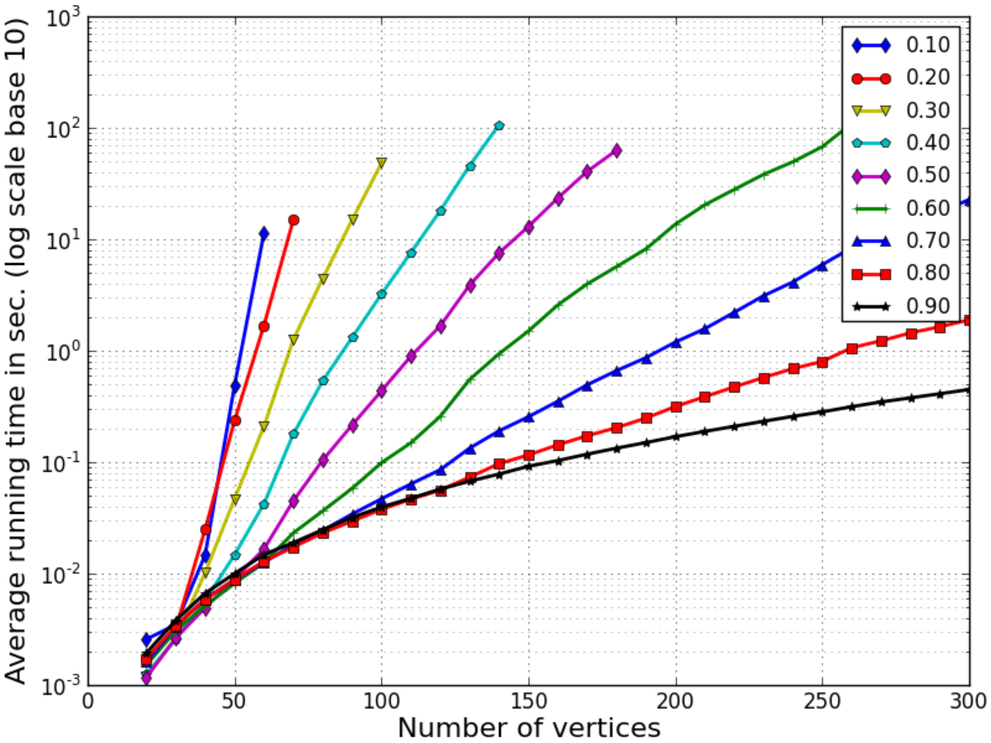




Performances on Mycielski graphs

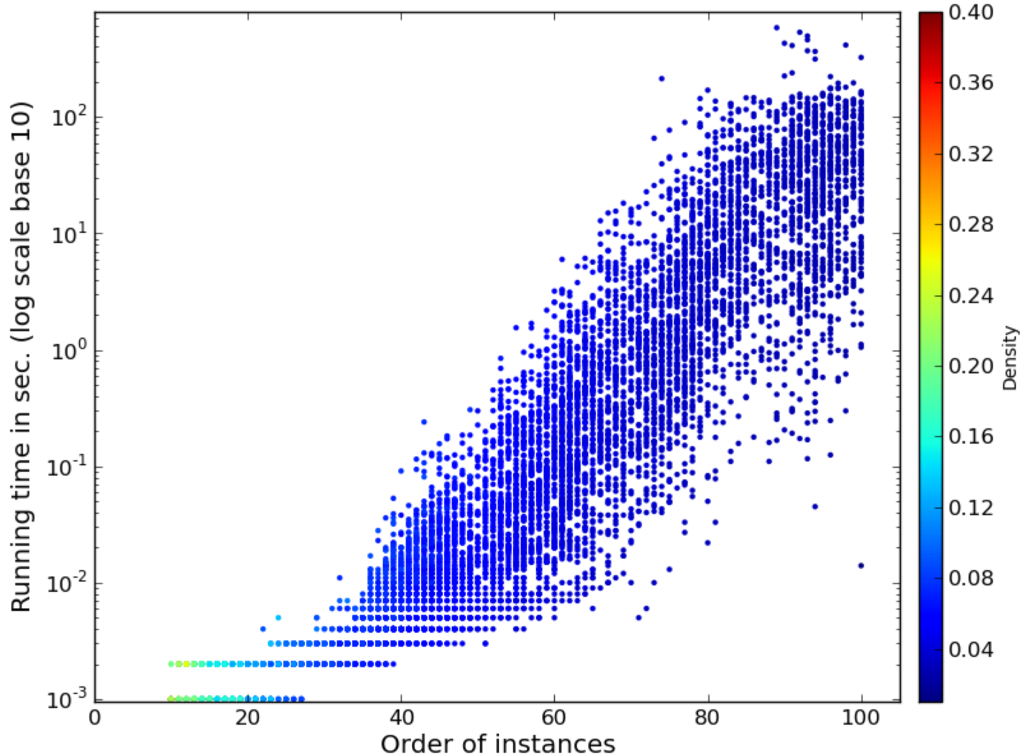
Triangle free graphs with increasing chromatic number (and large pathwidth)

$\mathbf{M_6}$ with 47 nodes and pw(M_6) = 20

Max. prefix length	Time (in sec.)	Visited nodes	Stored prefix
1	2226.45	1 256 780 074	44
5	144.93	82 417 700	16 386
8	5.81	3 319 482	47 756
10	1.07	664 677	61 466
15	0.77	496 482	65 252
20	0.78	496 482	65 253

Performances on Rome dataset

- 11 529 undirected n-node graphs with 10 ≤ n ≤ 100
- ILP/SAT [Biedl *et al.* - Graph Drawing 2013]
  - Time limit per graph: 10 min
  - 17 % graphs solved
  - Almost all such that n+m ≤ 45 and almost no graphs s.t. n+m ≥ 70
- B&B
  - Same time limit
  - 95.6 % graphs solved
  - All graphs with n ≤ 82



Ongoing & future work

Various heuristics:

- First solution of B&B
- Best solution of B&B in x sec.
- Local Search, simulated annealing

Future work:

- Inclusion of B&B in Sage
- Other reduction rules, cuts, new ideas, etc.
- Lower bounds
  - Almost all computation time used to prove optimality of the solution
  - Important for optimality gap

- Extend to other width: cutwidth, bandwidth, etc.

References

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