Higher Level Orderings on Modules

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Abstract

The aim of this paper is to investigate higher level orderings on modules over commutative rings. Based on the theory of higher level orderings on fields and commutative rings, some results involving existence of higher level orderings are generalized to the category of modules over commutative rings. Moreover, a strict intersection theorem for higher level orderings on modules is established.

Keywords higher level ordering, commutative ring, prime ideal, real module, real prime submodule.

1 Introduction

In Artin's solution to the Hilbert's 17th problem, the notion of orderings on fields played an important role. Since E. Artin and O. Schreier published two papers [1, 2], the notion of orderings has been generalized in many directions as a basic concept in real algebra. M. Coste and M.-F. Coste-Roy introduced orderings on commutative rings with identity (see [8]). Later, the notion of orderings on modules over commutative rings was proposed by G. Zeng [11]. Another remarkable direction of the generalization of orderings is the introduction of higher level orderings on fields by E. Becker [4, 5]. Subsequently, the notion of higher level orderings was further introduced into the category of commutative rings with identity by E. Becker and S. M. Barton (see [6]

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and [3]). The study of orderings and other relevant notions, e.g. valuations and quadratic forms, is a principal task of real algebra.

In view of the development of the study of orderings, such a question naturally arises: can we introduce higher level orderings on modules over commutative rings? This paper is aimed to answer this question.

In this paper, the symbol $\mathbb{N}$ stands for the set of all positive integers, all rings are supposed to be commutative rings with identity and all modules are unitary.

For two subsets A and B of a set S, denote by $A \setminus B$ the complement of B in A, i.e. $A \setminus B = \{x \in S \mid x \in A \text{ but } x \notin B\}$. Let R be a ring. For two subsets A and B of R, set $A \cdot B = \{ab \mid a \in A, b \in B\}$. Let M be an R-module. For a prime ideal $\wp$ of R, denote by $M_\wp$ the localization of M at $\wp$, i.e. $M_\wp = \{\frac{\eta}{s} \mid \eta \in M \text{ and } s \in R \setminus \wp\}$ and write $\wp M_\wp$ for the submodule $\{\frac{\eta}{s} \mid \eta \in \wp M \text{ and } s \in R \setminus \wp\}$ of $M_\wp$. For two nonempty subsets A and B of M, set $(A : B) = \{r \in R \mid rb \in A \text{ for all } b \in B\}$. In particular, if $A = \{0\}$ (or respectively, $B = \{b\}$) we write $\mathrm{anni}(B)$ (or respectively, $(A : b)$) instead of $(A : B)$. Note that $(A : B)$ is an ideal of R if A is a submodule of M. For a nonempty subset S of R and a nonempty subset A of M, set $S \cdot A = \{sa \mid s \in S, a \in A\}$ and $(A : S) = \{x \in M \mid sx \in A \text{ for all } s \in S\}$. Note that $(A : S)$ is a submodule of M if A is a submodule of M. For an ideal I of R and a submodule A of M, set $I \cdot A = \{\sum_{i=1}^{s} r_i x_i \mid r_i \in I \text{ and } x_i \in A \text{ for } i = 1, \ldots, s\}$, a submodule of M. A submodule $\mathcal{P}$ of M is called *prime* if $ax \in \mathcal{P}$ with $a \in R$ and $x \in M$ implies either $a \in (\mathcal{P} : M)$ or $x \in \mathcal{P}$. We observe that if $\mathcal{P}$ is a prime submodule of M, then $(\mathcal{P} : M)$ is a prime ideal of R and $(\mathcal{P} : M) = (\mathcal{P} : x)$ for any $x \in M$ satisfying $x \notin \mathcal{P}$. For properties of modules and prime submodules, we refer the reader to [9] and [10].

2 A Review of Higher Level Orderings on Rings

In this section, we recall the definition and properties of higher level orderings on rings.

Let R be a ring and let $k(\wp)$ be the quotient field of $R/\wp$ where $\wp$ is a prime ideal of R. In [6], Becker defined

Definition 0 *A subset P of R is called an ordering of level n on R if P satisfies the following conditions:*

(1) $P + P \subseteq P,\ P \cdot P \subseteq P,\ R^{2n} \subseteq P$;

(2) $\wp := P \cap -P$ *is a prime ideal of R;*

(3) $xy^{2n} \in P$ *with* $x,\, y \in R$ *implies either* $x \in P$ *or* $y \in \wp$;

(4) $\overline{P} := \{\sum_{i=1}^{s} a_i^{2n}\bar{p} \mid a_i \in k(\wp),\ i = 1,\ldots,s,\ s \in \mathbb{N},\ \bar{p} = p + \wp$ *for* $p \in P\}$ *is an ordering of level* n *on* $k(\wp)$.

From Definition 0, we can set up a one-to-one correspondence between orderings of level n on R and pairs $(\wp, \chi)$ where $\wp$ is a prime ideal of R and χ is a signature sending $k(\wp)$ into the cyclic group of all $2n$-th roots of unity as defined in [3]. Accordingly, such a pair $(\wp, \chi)$ may be equivalently regarded as an ordering of level n on R.

In Definition 1.5 of [3], Barton defined higher level orderings on rings in terms of families of subsets of rings as follows

Definition 1 *A family of subsets* $\{\, \alpha_1, \ldots, \alpha_{2n}\,\}$ *of a ring* R *is called an ordering of level* n *on* R *if the following conditions are satisfied:*

(1) $R = \bigcup\limits_{i=1}^{2n} \alpha_i$;

(2) $\alpha_i \bigcap \alpha_j = \wp$ *for all* $i \neq j$, *where* $\wp$ *is a prime ideal of* R *and we denote* $\alpha_i^* = \alpha_i \setminus \wp$ *for* $i = 1, \ldots, 2n$;

(3) $\alpha_i^* + \alpha_i^* \subseteq \alpha_i^*$ *for* $i = 1, \ldots, 2n$;

(4) $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_{i+j}^*$ *for all* $i,\, j \in \{1, \ldots, 2n\}$, *where the following convention is adopted:* $\alpha_i = \alpha_j$ *whenever* $i \equiv j \pmod{2n}$.

In Lemma 1.7 of [3], Barton gave a slightly different definition of higher level orderings:

Definition 2 *A family of subsets* $\alpha = \{\alpha_1, \ldots, \alpha_{2n}\}$ *of a ring* R *is called an ordering of level* n *on* R *if the following conditions are satisfied:*

(1) $R = \bigcup\limits_{i=1}^{2n} \alpha_i$;

(2) $\alpha_i \bigcap \alpha_j = \wp$ *for all* $i \neq j$, *where* $\wp$ *is a prime ideal of* R ;

(3) $\alpha_i + \alpha_i \subseteq \alpha_i$ *for* $i = 1, \ldots, 2n$;

(4) $\alpha_i \cdot \alpha_j \subseteq \alpha_{i+j}$ *for all* $i,\, j \in \{1, \ldots, 2n\}$, *where the same convention as in Definition* 1 *is adopted:* $\alpha_i = \alpha_j$ *whenever* $i \equiv j \pmod{2n}$;

(5) $-1 \notin \alpha_{2n}$ (*or equivalently,* $-1 \in \alpha_n$).

The prime ideal $\wp$ *is called the support of* α *and denoted by* $\mathrm{supp}(\alpha)$.

Barton claimed that Definition 1 and Definition 2 are equivalent. However, on this "equivalence", she did not give a rigorous proof. In fact, even if the claim is true, the proof is not necessarily trivial. But once the claim does not hold, both the proof and the conclusion of Theorem 1.3 in [3] are incorrect. So, we do not adopt Definition 1 as the definition of higher level orderings on rings. However, from Definition 2 we can still establish the following

Proposition 1 *Let R be a ring and let $\{\alpha_1, \ldots, \alpha_{2n}\}$ be a family of subsets of R satisfying the conditions $(1-5)$ in Definition 2. Set $\alpha_i^* = \alpha_i \setminus \wp$ for $i = 1, \ldots, 2n$. Then for an arbitrary $2n$-th primitive root ξ, the map*

$$\chi : \dot{k}(\wp) \longrightarrow \{\xi^i \mid i = 1, \ldots, 2n\} \quad \textit{where } \dot{k}(\wp) := k(\wp) \setminus \{0\}$$

determined by $\frac{\overline{a}}{\overline{b}} \mapsto \xi^{i-j}$ with $\overline{a} = a + \wp$ and $\overline{b} = b + \wp$ for $a \in \alpha_i^*$ and $b \in \alpha_j^*$, is a signature.

Proof. We first show that χ is well-defined. Suppose that $\overline{a} = \overline{b} \neq 0$ with $a, b \in R \setminus \wp$, then $b - a \in \wp$. Since $R = \bigcup_{i=1}^{2n} \alpha_i$, a belongs to α_i for some $i \in \{1, \ldots, 2n\}$. Hence, $b = a + (b - a) \in \alpha_i + \wp \subseteq \alpha_i$ by the condition (3) of Definition 2. Thus, $\chi(\overline{a}) = \xi^i = \chi(\overline{b})$.

Now we assert that

$(3')$ $\alpha_i^* + \alpha_i^* \subseteq \alpha_i^*$ for $i = 1, \ldots, 2n$;

$(4')$ $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_{i+j}^*$, for all $i, j \in \{1, \ldots, 2n\}$.

Indeed, if $x, y \in \alpha_i^*$ we have $x + y \in \alpha_i$ by the condition (3) of Definition 2. Suppose that $x+y \in \wp$. Note that $-1 \in \alpha_n$, then $x \in -y + \wp \subseteq \alpha_n \cdot \alpha_i + \wp \subseteq \alpha_{n+i} + \wp = \alpha_{n+i}$, which leads to $x \in \alpha_i \cap \alpha_{n+i} = \wp$, a contradiction. Hence, the assertion $(3')$ is verified. Obviously, $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_{i+j}$. Thus, the assertion $(4')$ follows immediately from the fact $\alpha_i^* \cdot \alpha_j^* \cap \wp = \emptyset$.

Following an argument similar to the one used for proving Theorem 1.3 of [3], we get that χ is a signature. $\square$

From Proposition 1, it is then reasonable to adopt Definition 2 as the definition of higher level orderings on rings. In the sequel, by higher level orderings on rings we mean those given by Definition 2.

We now study properties of higher level orderings on rings. First, we have

Lemma 1 *If $\alpha = \{\alpha_1, ..., \alpha_{2n}\}$ is an ordering of level n on a ring R, then* $\mathrm{supp}(\alpha)$ *is a real ideal of R.*

Proof. It is an immediate consequence of Proposition 1 and Korollar 2.3 in [5]. □

Another property of higher level orderings on rings is described in

Lemma 2 *Let $\pi : R \to R'$ be a ring homomorphism. If $\{\alpha'_1, \dots, \alpha'_{2n}\}$ is an ordering of level n on R', then $\{\alpha_1, \dots, \alpha_{2n}\}$ is an ordering of level n on R where $\alpha_i = \pi^{-1}(\alpha'_i)$, the preimage of α'_i, for $i = 1, \dots, 2n$.*

Proof. Obviously, we have $R = \bigcup_{i=1}^{2n} \alpha_i$ and $\alpha_i + \alpha_i \subseteq \alpha_i$ for $i = 1, \dots, 2n$. Observe that for all $i, j \in \{1, \dots, 2n\}$ with $i \neq j$, we have

$$x \in \alpha_i \cap \alpha_j \Longleftrightarrow \pi(x) \in \alpha'_i \cap \alpha'_j = \wp' \Longleftrightarrow x \in \pi^{-1}(\wp').$$

Then, $\alpha_i \cap \alpha_j = \pi^{-1}(\wp')$ is a prime ideal of R.

If $x \in \alpha_i$ and $y \in \alpha_j$, then $\pi(x) \in \alpha'_i$ and $\pi(y) \in \alpha'_j$. Therefore, we get that $\pi(x\,y) \in \alpha'_i \cdot \alpha'_j \subseteq \alpha'_{i+j}$ and so $xy \in \alpha_{i+j}$. Hence, $\alpha_i \cdot \alpha_j \subseteq \alpha_{i+j}$.

Finally, we have $-1 \notin \alpha_{2n}$ because otherwise $-1 = \pi(-1) \in \alpha'_{2n}$. Then by Definition 2, $\{\alpha_1, \dots, \alpha_{2n}\}$ is an ordering of level n on R. □

3 Higher Level Orderings on Modules

In this section, we introduce higher level orderings on modules over rings and then establish some results about higher level orderings on modules.

Let R be a ring and let M be an R-module. We define

Definition 3 *A family of subsets $Q = \{Q_1, \dots, Q_{2n}\}$ of M is called an ordering of level n on M if the following conditions are satisfied:*

(1) $M = \bigcup_{i=1}^{2n} Q_i$;

(2) $Q_i + Q_i \subseteq Q_i$;

(3) $Q_i \cap Q_j = \mathcal{P}$ *for all $i \neq j$, where $\mathcal{P}$ is a prime submodule of M;*

(4) *R admits an ordering $\alpha = \{\alpha_1, \dots, \alpha_{2n}\}$ of level n such that*

$$\alpha_i \cdot Q_j \subseteq Q_{i+j}, \quad \text{for all } i, j \in \{1, \dots, 2n\},$$

and $\mathrm{supp}(\alpha) = (\mathcal{P} : M)$ where we adopt the convention that $Q_i = Q_j$ whenever $i \equiv j \pmod{2n}$.

The prime submodule $\mathcal{P}$ is called the support of Q and denoted by $\mathrm{supp}(Q)$. Also, α is called an ordering on R associated with Q.

Remark 1

(1) *If* $Q = \{Q_1, \ldots, Q_{2n}\}$ *is an ordering of level* n *on an* R*-module* M*, then for any* $k \in \{1, \ldots, 2n\}$*,* $Q_{(k)} = \{Q_{k+1}, \ldots, Q_{2n}, Q_1, \ldots, Q_k\}$ *is also an ordering of level* n *on* M *with the same associated ordering as* Q *on* R*.*

(2) *If* $\alpha := \{\, \alpha_1, \ldots, \alpha_{2n} \,\}$ *is an ordering of level* n *on a ring* R*,* α *is naturally an ordering of level* n *on* R *as an* R*-module. Conversely, however, an ordering* $\{Q_1, \ldots, Q_{2n}\}$ *of level* n *on* R *as an* R*-module is not necessarily an ordering of level* n *on* R *as a ring.*

Nevertheless, we have

Proposition 2 *Let* $Q = \{Q_1, \ldots, Q_{2n}\}$ *be a family of subsets of a ring* R*. Then, the following two statements are equivalent:*

(i) Q *is an ordering of level* n *on* R *as a ring;*

(ii) Q *is an ordering of level* n *on* R *as an* R*-module satisfying* $1 \in Q_{2n}$*.*

Proof. From Remark 1 (2), we know that (i) implies (ii). Conversely, let Q be an ordering of level n on R as an R-module satisfying $1 \in Q_{2n}$ and let $\alpha = \{\, \alpha_1, \ldots, \alpha_{2n} \,\}$ be an associated ordering on R with Q. Then, we have $\alpha_i = \alpha_i \cdot 1 \subseteq \alpha_i \cdot Q_{2n} \subseteq Q_i$ for $i = 1, \ldots, 2n$. If there exists $x \in Q_i$ but $x \notin \alpha_i$, then $x \in \alpha_j$ for some $j \neq i$ and thus $x = x \cdot 1 \in \alpha_j \cdot Q_{2n} \subseteq Q_j$. Hence, $x \in Q_i \bigcap Q_j = \mathcal{P}$. Since $\mathcal{P}$ is an R-submodule of R, we have $Rx \subseteq \mathcal{P}$ and therefore $x \in (\mathcal{P} : R) = \mathrm{supp}(\alpha) \subseteq \alpha_i$, a contradiction. So, $Q_i = \alpha_i$ and $\{Q_1, \ldots, Q_{2n}\}$ is an ordering of level n on R as a ring. $\square$

Now, let us look at the following

Example 1 *Let* F *be a real field and let* V *be a nonzero* F*-vector space with a basis* $\Gamma = \{\gamma_\lambda \mid \lambda \in \Lambda\}$ *where* Λ *is an index set. By Zermelo's well-ordering theorem,* Λ *admits a linear ordering* $\prec$*. Then, each nonzero* $x \in V$ *can be represented uniquely in the form:* $x = a_1\, \gamma_{\lambda_1} + \cdots + a_s\, \gamma_{\lambda_s}$ *where* $a_i \in F \setminus \{0\}$ *and* $\lambda_i \in \Lambda$ *ordered as* $\lambda_1 \prec \cdots \prec \lambda_s$*. We call* a_s *the leading coefficient of* x*.*

By Korollar 2.3 in [5], F *admits an ordering of level* n *for any* $n \in \mathbb{N}$*. Now let* $\alpha = \{\alpha_1, \ldots, \alpha_{2n}\}$ *be an ordering of level* n *on* F*, then the support of* α *is* $\{0\}$*. Consider the following subsets of* V*:*

$$Q_i = \{x \in V \mid x = 0 \text{ or the leading coefficient of } x \text{ lies in } \alpha_i^*\},$$

for $i = 1, \ldots, 2n$*.*

One can show that $\{Q_1, \ldots, Q_{2n}\}$ *is an ordering of level* n *on* V *with the support* $\{0\}$ *and* α *is its associated ordering on* F*.*

Now we begin to investigate properties of higher level orderings on modules. By Definition 3, we can verify without difficulty the following

Proposition 3 *If $Q = \{ Q_1, \ldots, Q_{2n} \}$ is an ordering of level n on an R-module M with an associated ordering $\alpha = \{ \alpha_1, \ldots, \alpha_{2n} \}$ on R, then*

(i) $\mathcal{P} = \mathrm{supp}(Q)$ *is a real submodule of M;*

(ii) $(Q_i : Q_k) = \alpha_{i-k}$ *if $Q_k \neq \mathcal{P}$;*

(iii) $(Q_i : \alpha_k) = Q_{i-k}$ *if $\alpha_k \neq \mathrm{supp}(\alpha)$. In particular, $(Q_i : \alpha_{2n}) = Q_i$.*

Proof. (i) Consider the quotient module $M/\mathcal{P}$ over R and suppose that $(\sum_{i=1}^s a_i^2)\overline{x} = 0$ where $a_i \in R$ for $i = 1, \ldots, s$ and $\bar{x} = x + \mathcal{P}$ for $x \in M$. Then, we have $(\sum_{i=1}^s a_i^2)x \in \mathcal{P}$. If $x \in \mathcal{P}$, it follows that $a_i\, x \in \mathcal{P}$ and therefore $a_i\overline{x} = 0$ for $i = 1, \ldots, s$. If $x \notin \mathcal{P}$, then $\sum_{i=1}^s a_i^2 \in (\mathcal{P} : M)$ because $\mathcal{P}$ is a prime submodule of M. By the condition (4) in Definition 3, we have $\sum_{i=1}^s a_i^2 \in (\mathcal{P} : M) = \mathrm{supp}(\alpha)$. By Lemma 1, $\mathrm{supp}(\alpha)$ is a real ideal of R. Then, $a_i \in \mathrm{supp}(\alpha) = (\mathcal{P} : M)$ and naturally $a_i\, x \in \mathcal{P}$ for $i = 1, \ldots, s$. So, we have $a_i\overline{x} = 0$ for $i = 1, \ldots, s$, which shows that $M/\mathcal{P}$ is a real R-module. By Definition 3 in [11], $\mathcal{P}$ is a real submodule of M.

(ii) Clearly, $\alpha_{i-k} \subseteq (Q_i : Q_k)$. If there exists $a \in (Q_i : Q_k)$ but $a \notin \alpha_{i-k}$, then $a \in \alpha_j^*$ for some $j \not\equiv i-k \pmod{2n}$. By supposition, for any $x \in Q_k \setminus \mathcal{P}$ we have $ax \in Q_i$. On the other hand, $ax \in \alpha_j \cdot Q_k \subseteq Q_{j+k}$. Hence, $a\, x \in Q_i \cap Q_{j+k} = \mathcal{P}$ because $i \not\equiv j+k \pmod{2n}$. Since $\mathcal{P}$ is a prime submodule of M not containing x, then $a \in (\mathcal{P} : M) = \mathrm{supp}(\alpha) \subseteq \alpha_{i-k}$, a contradiction. So, we have $(Q_i : Q_k) = \alpha_{i-k}$.

(iii) Obviously, $Q_{i-k} \subseteq (Q_i : \alpha_k)$. If there exists $x \in (Q_i : \alpha_k)$ such that $x \notin Q_{i-k}$, then $x \in Q_j^*$ for some $j \not\equiv i-k \pmod{2n}$. By supposition, for any $a \in \alpha_k \setminus \mathrm{supp}(\alpha)$ we have $a\, x \in Q_i$. On the other hand, we get that $a\, x \in \alpha_k \cdot Q_j \subseteq Q_{k+j}$. Hence, $a\, x \in Q_i \cap Q_{k+j} = \mathcal{P}$. Since $\mathcal{P}$ is a prime submodule of M not containing x, $a \in (\mathcal{P} : M) = \mathrm{supp}(\alpha)$, a contradiction. So, we have $(Q_i : Q_k) = \alpha_{i-k}$. □

Remark 2 *Proposition* 3 (ii) *implies that for an ordering of level n on an R-module M, its associated ordering on R is unique. Accordingly, the word "associated ordering" should be preceded by the definite article "the".*

Proposition 4 *If an R-module M admits an ordering of level n, then M is a semireal module.*

Proof. Let Q be an ordering of level n on M. Observe that each prime submodule of M is a proper subset of M, so we can pick out $e \in M \backslash \mathrm{supp}(Q)$. Set $T_R^{2n} = \{\sum_{i=1}^{s} a_i^{2n}$ with $a_i \in R$ and $s \in \mathbb{N}\}$. We claim that $(1+t)\,e \neq 0$ for any $t \in T_R^{2n}$. Indeed, otherwise, suppose that there exists $t_0 \in T_R^{2n}$ such that $(1+t_0)\,e = 0 \in \mathrm{supp}(Q)$. By Proposition 3, $\mathrm{supp}(Q)$ is a real submodule of M. Then, we have $e \in \mathrm{supp}(Q)$, a contradiction. So, M is a semireal module. $\square$

We note that the converse of Proposition 4 is not true. A counterexample for the case $n = 1$ can be found in [11]. In what follows, we propose some necessary and sufficient conditions for modules to admit higher level orderings.

Theorem 1 *An R-module M admits an ordering of level n if and only if there exists $\wp \in \mathrm{Spec}_r(R)$ such that $\wp M_\wp \neq M_\wp$ where $\mathrm{Spec}_r(R)$ denotes the set of all real prime ideals of R.*

Proof. For the necessity, suppose that $Q = \{\, Q_1, \ldots, Q_{2n}\,\}$ is an ordering of level n on M with the associated ordering $\alpha = \{\,\alpha_1, \ldots, \alpha_{2n}\,\}$ on R. Set $\mathcal{P} = \mathrm{supp}(Q)$ and $\wp = \mathrm{supp}(\alpha)$, then $\wp \in \mathrm{Spec}_r(R)$. Since $\mathcal{P}$ is a proper submodule of M, there exists $e \in M \setminus \mathcal{P}$. We claim that $\frac{e}{1} \notin \wp M_\wp$. Indeed, otherwise, suppose that there exists $s \in R \setminus \wp$ such that $se \in \wp M$. By Definition 3, $\wp M \subseteq \mathcal{P}$. Then, $se \in \mathcal{P}$. Since $\mathcal{P}$ is a prime submodule of M not containing e, we have $sM \subseteq \mathcal{P}$ and hence $s \in (\mathcal{P} : M) = \wp$, a contradiction to our supposition.

Conversely, for the sufficiency, suppose that there exists $\wp \in \mathrm{Spec}_r(R)$ such that $\wp M_\wp \neq M_\wp$. Set $V = M_\wp/\wp M_\wp$ and $F = R_\wp/\wp R_\wp$. Note that $\wp R_\wp$ is a maximal ideal of $R_\wp$, then F is a field and therefore V is a nonzero F-vector space. Since $\wp$ is a real ideal of R, $\wp R_\wp$ is a real ideal of $R_\wp$ and so F is a real field.

By Example 1, F admits an ordering $\alpha' = \{\alpha'_1, \ldots, \alpha'_{2n}\}$ of level n with $\mathrm{supp}(\alpha') = \{0\}$ and V admits an ordering $Q' = \{Q'_1, \ldots, Q'_{2n}\}$ of level n with $\mathrm{supp}(Q') = \{0\}$ such that α' is its associated ordering on F.

Denote by π the composition of two canonical homomorphisms $R \to R_\wp$ and $R_\wp \to F$ given by $r \mapsto \frac{r}{1}$ and $\frac{r}{s} \mapsto \frac{\frac{r}{s}}{\frac{1}{s}}$, respectively. Let $\alpha_i = \pi^{-1}(\alpha'_i)$ for $i = 1, \ldots, 2n$. By Lemma 2, we know that $\alpha = \{\alpha_1, \ldots, \alpha_{2n}\}$ is an ordering of level n on R and $\mathrm{supp}(\alpha) = \pi^{-1}(\mathrm{supp}(\alpha')) = \pi^{-1}(\{0\}) = \wp$.

Through the action of π, V can be viewed as an R-module. For any $r \in \wp$, we have $r\,V = \pi(r) \cdot V = 0 \cdot V = \{0\}$ and $r \in (0 : V)$. So, $\wp \subseteq (0 : V)$. Conversely, if $a \in (0 : V)$ then $\pi(a)V = aV = \{0\}$. Since V is a nonzero F-vector space, we must have $\pi(a) = 0$, i.e. $a \in \wp$. Then, $(0 : V) \subseteq \wp$ and so $\wp = (0 : V) = (\mathrm{supp}(Q') : V)$.

Let $a \in \alpha_i$ and $v \in Q'_j$, then $av = \pi(a)\, v \in \pi(\alpha_i) \cdot Q'_j \subseteq \alpha'_i \cdot Q'_j \subseteq Q'_{i+j}$ and therefore $\alpha_i \cdot Q'_j \subseteq Q'_{i+j}$. So, $Q' = \{Q'_1, \ldots, Q'_{2n}\}$ is an ordering of level n on the R-module V with the associated ordering α on R.

Denote by φ the composition of two canonical module homomorphisms $M \to M_\wp$ and $M_\wp \to V$ given by $m \mapsto \frac{m}{1}$ and $\frac{m}{s} \mapsto \frac{m}{s} + \wp M_\wp$, respectively. Set $Q_i = \{x \in M \mid \varphi(x) \in Q'_i\}$ for $i = 1, \ldots, 2n$. By Lemma 3 below, we know that $Q = \{Q_1, \ldots, Q_{2n}\}$ is an ordering of level n on M. Moreover, the support of Q is $\mathrm{supp}(Q) = \{x \in M \mid \frac{x}{1} \in \wp M_\wp\}$ and α is the associated ordering on R with Q. This completes the proof. $\Box$

Theorem 1 then induces

Corollary 1 *An R-module M admits an ordering of level n if and only if M admits an ordering of level 1.*

Proof. It is a direct consequence of Theorem 1 which holds for any n. $\Box$

The following lemma is a counterpart of Lemma 2 for higher level orderings on modules.

Lemma 3 *Let $\varphi : M \to M'$ be an R-module homomorphism. If $Q' = \{Q'_1, \ldots, Q'_{2n}\}$ is an ordering of level n on M' such that $\varphi(M) \nsubseteq \mathrm{supp}(Q')$, then $Q = \{\, \varphi^{-1}(Q'_1), \ldots, \varphi^{-1}(Q'_{2n})\,\}$ is an ordering of level n on M with the same associated ordering as Q' on R.*

Proof. Set $Q_i = \varphi^{-1}(Q'_i)$ for $i = 1, \ldots, 2n$. Clearly, $M = \bigcup_{i=1}^{2n} Q_i$ and $Q_i + Q_i \subseteq Q_i$ for $i = 1, \ldots, 2n$. For all $i,\, j \in \{1, \ldots, 2n\}$ with $i \neq j$, we have $Q_i \cap Q_j = \varphi^{-1}(Q'_i) \cap \varphi^{-1}(Q'_j) = \varphi^{-1}(Q'_i \cap Q'_j) = \varphi^{-1}(\mathrm{supp}(Q'))$. Set $\mathcal{P}' = \mathrm{supp}(Q')$ and $\mathcal{P} = \varphi^{-1}(\mathcal{P}')$. From $\varphi(M) \nsubseteq \mathcal{P}'$, it follows that $\mathcal{P}$ is a proper submodule of M.

Let $ax \in \mathcal{P}$ with $a \in R$ and $x \in M \setminus \mathcal{P}$. Then, $a\varphi(x) = \varphi(ax) \in \mathcal{P}'$ but $\varphi(x) \notin \mathcal{P}'$. Since $\mathcal{P}'$ is a prime submodule of M', we have $aM' \subseteq \mathcal{P}'$ and $\varphi(aM) \subseteq \mathcal{P}'$. This implies that $a \in (\mathcal{P} : M)$ and $\mathcal{P}$ is a prime submodule of M.

Let $\alpha = \{\alpha_1, \ldots, \alpha_{2n}\}$ be the associated ordering on R with Q'. Then, it follows that $\mathrm{supp}(\alpha) = (\mathcal{P}' : M')$. If $a \in (\mathcal{P} : M)$, then $a\,M \subseteq \mathcal{P}$ and $a\,\varphi(M) = \varphi(a\,M) \subseteq \varphi(\mathcal{P}) = \mathcal{P}'$. Since $\mathcal{P}'$ is a prime submodule of M' and $\varphi(M) \nsubseteq \mathcal{P}'$, we have $a \in (\mathcal{P}' : M')$. Hence, $(\mathcal{P} : M) \subseteq (\mathcal{P}' : M')$. Conversely, if $a \in (\mathcal{P}' : M')$ then $a\,M' \subseteq \mathcal{P}'$, and $\varphi(a\,M) = a\,\varphi(M) \subseteq a\,M' \subseteq \mathcal{P}'$. So, $(\mathcal{P}' : M') \subseteq (\mathcal{P} : M)$ and then $(\mathcal{P} : M) = (\mathcal{P}' : M') = \mathrm{supp}(\alpha)$.

Let $a \in \alpha_i$ and $x \in Q_j$, then $\varphi(a\,x) = a\,\varphi(x) \in \alpha_i \cdot Q'_j \subseteq Q'_{i+j}$, which yields that $a\,x \in Q_{i+j}$ and hence $\alpha_i \cdot Q_j \subseteq Q_{i+j}$.

By Definition 3, $Q = \{ Q_1, \ldots, Q_{2n} \}$ is an ordering of level n on M and α is the associated ordering on R with Q. $\square$

Theorem 2 *An R-module M admits an ordering of level n if and only if M has a real prime submodule.*

Proof. The necessity follows immediately from Proposition 3 (i). It remains to show the sufficiency. Let $\mathcal{P}$ be a real prime submodule of M. Set $\wp = (\mathcal{P} : M)$, then $\wp$ is a real prime ideal of R by Proposition 1.1 (3) in [11]. We claim that $\wp M_\wp \neq M_\wp$. Indeed, if otherwise, then $\frac{x}{1} \in M_\wp = \wp M_\wp$ for any $x \in M \setminus \mathcal{P}$. Then, there exists $s \in R \setminus \wp$ such that $sx \in \wp M$. From $\wp M \subseteq \mathcal{P}$, it follows that $sx \in \mathcal{P}$. Since $\mathcal{P}$ is a prime submodule of M not containing x, we get that $s \in \wp$, a contradiction. Thus, $\wp M_\wp \neq M_\wp$. By Theorem 1, M admits an ordering of level n. $\square$

For finitely generated modules, the converse of Proposition 4 is also true. Before we give a rigorous proof of it, we first establish a more general result as follows

Proposition 5 *Let M be a finitely generated R-module and let T be a preordering of level n on R. If there exists $e \in M$ such that $0 \notin (1+T)e$, then M admits an ordering of level n such that its associated ordering $\{\alpha_1, \ldots, \alpha_{2n}\}$ satisfies that $T \subseteq \alpha_{2n}$.*

Proof. Suppose that M is generated by $x_1, \ldots, x_m$ over R. Set $I = \mathrm{anni}(e)$. Then, I is an ideal of R and $I \cap (1+T) = \emptyset$. Let $\overline{T} = \{t + I \in R/I \mid t \in T\}$. From $I \cap (1 + T) = \emptyset$, one can show that $\overline{T}$ is a preordering of level n on R/I. By Proposition 2.4 in [6], R/I admits an ordering $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{2n}\}$ of level n such that $\overline{T} \subseteq \overline{\alpha}_{2n}$. Denote by π the canonical ring homomorphism from R into R/I and set $\alpha_i = \pi^{-1}(\overline{\alpha}_i)$ for $i = 1, \ldots, 2n$. By Lemma 2, $\alpha = \{\alpha_1, \ldots, \alpha_{2n}\}$ is an ordering of level n on R. Clearly, $T \subseteq \alpha_{2n}$. Denote by $\wp$ the support of α, then $\wp$ is a real prime ideal of R such that $\mathrm{anni}(e) \subseteq \wp$ and $(1 + T) \cap \wp = \emptyset$.

Note that $R_\wp$ is a local ring with a maximal ideal $\wp R_\wp$ and $M_\wp$ is a finitely generated $R_\wp$-module. We claim that $\wp M_\wp \neq M_\wp$. Indeed, if otherwise, then we have $\wp R_\wp M_\wp = \wp M_\wp = M_\wp$. Thus, $M_\wp = 0$ by Nakayama lemma and therefore $\frac{x_i}{1} = 0$ for $i = 1, \ldots, m$. Hence, there exists $s_i \in R \setminus \wp$ such that $s_i x_i = 0$ for $i = 1, \ldots, m$. Set $s = s_1 s_2 \cdots s_m$, then $s \in R \setminus \wp$ and $sx_i = 0$ for $i = 1, \ldots, m$. Thus, $s \in (0 : M) \subseteq \mathrm{anni}(e) \subseteq \wp$, a contradiction. So, $\wp M_\wp \neq M_\wp$. Set $V = M_\wp / \wp M_\wp$ and $F = R_\wp / \wp R_\wp$, then F is a field and V is a nonzero F-vector space. Note that F can be also viewed as the quotient field $k(\wp)$ of $R/\wp$. Hence, α induces uniquely an ordering $\alpha' = \{\alpha'_1, \ldots, \alpha'_{2n}\}$ of level n on F.

Since $V \neq 0$, we can decompose V into a direct sum: $F \cdot \xi \oplus W$ where ξ is a non-zero element of V and W is a subspace of V. This decomposition of V yields a family of subsets of V as follows:

$$Q'_i = \{v \in V \mid v = a\xi + w \text{ where } a \in \alpha'_i \text{ and } w \in W\} \quad \text{for } i = 1, \ldots, 2n.$$

Denote by φ the composition of two canonical homomorphisms $M \to M_\wp$ and $M_\wp \to V$ as defined in Theorem 1 and set $Q_i = \{x \in M \mid \varphi(x) \in Q'_i\}$ for $i = 1, \ldots, 2n$. By an argument similar to the one used in the proof for the sufficiency of Theorem 1, we know that $\{Q_1, \ldots, Q_{2n}\}$ is an ordering of level n on M with the associated ordering α on R. $\square$

From Proposition 5, we naturally get

Theorem 3 *Let M be a finitely generated R-module. Then, M admits an ordering of level n if and only if M is semireal.*

Proof. The necessity is proved by Proposition 4. Suppose that M is a semireal R-module, then there exists $e \in M$ such that $0 \notin (1 + T_R^{2n})e$ where the symbol T_R^{2n} is of same meaning as the one introduced in the proof of Proposition 4. Clearly, we have $0 \notin 1 + T_R^{2n}$ and then $-1 \notin T_R^{2n}$, which implies that T_R^{2n} is a preordering of level n on R. The sufficiency then follows from Proposition 5. $\square$

4 An Intersection Theorem for Higher Level Orderings

In Theorem 6 of [7], R. Berr gave a strict intersection theorem for higher level orderings on commutative rings. In this section, we establish a similar theorem for higher level orderings on modules.

Let $Q = \{Q_1, \ldots, Q_{2n}\}$ be an ordering of level n on an R-module M. Set $Q^+ := Q^*_{2n}$. We call Q^+ the set of *strictly positive elements* of M with respect to the ordering Q. Note that Q^+ may be empty.

Let M be an R-module. An element $e \in M$ is called *normal* if $\frac{e}{1} \notin \wp M_\wp$ for each real prime ideal $\wp$ of R containing $\mathrm{anni}(e)$. We observe that if a set Γ is a basis of a free R-module M then each element of Γ is normal.

Now, we establish the following

Theorem 4 *Let M be an R-module and let T be a preordering of level n on R. Suppose that e is a normal element of M and $x \in Re$. Then, the following two statements are equivalent:*

(i) *For any ordering $Q = \{Q_1, \dots, Q_{2n}\}$ of level n on M satisfying $e \in Q^+$ and $T \subseteq (Q_{2n} : Q_{2n})$, we have $x \in Q^+$;*

(ii) *There exist elements t and t' of T such that $tx = (1 + t')e$.*

Proof. Suppose that (ii) holds. Let $Q = \{Q_1, \dots, Q_{2n}\}$ be any ordering of level n on M satisfying $e \in Q^+$ and $T \subseteq (Q_{2n} : Q_{2n})$, and let $\alpha = \{\alpha_1, \dots, \alpha_{2n}\}$ be the associated ordering on R with Q. By Proposition 3 (ii), $(Q_{2n} : Q_{2n}) = \alpha_{2n}$. Suppose that $x \in Q_j$ for some $j \in \{1, \dots, 2n\}$, then $tx \in \alpha_{2n} \cdot Q_j = Q_j$. By (ii), we have $tx = (1 + t')e \in \alpha_{2n}^* \cdot Q_{2n}^* = Q_{2n}^*$. So, $j = 2n$ and $x \in Q_{2n}$. If $x \in \mathrm{supp}(Q)$, then $-(1 + t')e \in \mathrm{supp}(Q)$. Since $e \notin \mathrm{supp}(Q)$, we have $-(1+t') \in (\mathrm{supp}(Q) : M) \subseteq (Q_{2n} : Q_{2n}) = \alpha_{2n}$. This implies that $-1 \in t' + \alpha_{2n} \subseteq \alpha_{2n}$, a contradiction. So, $x \in Q^+$ and (i) holds.

Now suppose that (i) holds but (ii) is false. By supposition, there exists $a \in R$ such that $x = ae$. We claim that $(1 + T - aT) \cap I = \emptyset$ where $I := \mathrm{anni}(e)$. Indeed, if otherwise, there exist $t_1, t_2 \in T$ such that $(1 + t_1 - a\,t_2)\,e = 0$. Then $t_2 x = (at_2)e = (1 + t_1)e \in (1 + T)e$, which is a contradiction to our supposition that (ii) is false.

Set $\overline{T} = \{t + I \in R/I \mid t \in T\}$. From $(1 + T - aT) \cap I = \emptyset$, we deduce that $\overline{T}$ is a preordering of level n on R/I and $\overline{a}\overline{T} \cap (1 + \overline{T}) = \emptyset$ where $\overline{a} = a + I \in R/I$. By Theorem 6 in [7], R/I admits an ordering $\overline{\alpha} = \{\overline{\alpha}_1, \dots, \overline{\alpha}_{2n}\}$ of level n such that $\overline{T} \subseteq \overline{\alpha}_{2n}$ but $\overline{a} \notin \overline{\alpha}_{2n}^*$. Denote by π the canonical ring homomorphism from R to R/I and set $\alpha_i = \pi^{-1}(\overline{\alpha}_i)$ for $i = 1, \dots, 2n$. By Lemma 2, $\alpha = \{\alpha_1, \dots, \alpha_{2n}\}$ is an ordering of level n on R. Clearly, $T \subseteq \alpha_{2n}$. Write $\wp$ for the support of α, then $\wp$ is a real prime ideal of R satisfying $\mathrm{anni}(e) \subseteq \wp$ and $(1 + T) \cap \wp = \emptyset$. Since e is normal, we have $\frac{e}{1} \notin \wp M_\wp$. Hence, $\wp M_\wp \neq M_\wp$. Set $F = R_\wp/\wp R_\wp$, then F is a field and $V = M_\wp/\wp M_\wp$ is a non-zero F-vector space. View F as the quotient field of $R/\wp$, then α induces uniquely an ordering of level n on F.

By the same argument as the one used in the proof for the sufficiency of Theorem 1, M admits an ordering $Q = \{Q_1, \dots, Q_{2n}\}$ of level n with the associated ordering α on R such that $\mathrm{supp}(Q) = \{y \in M \mid \frac{y}{1} \in \wp M_\wp\}$ and $T \subseteq \alpha_{2n}$. From $\overline{a} \notin \overline{\alpha}_{2n}^*$, we have $a \notin \alpha_{2n}^*$. If $a \in \wp$, then $x = ae \in \mathrm{supp}(Q)$, a contradiction to the assumption (i). Thus, we must have $a \notin \wp$ and then $a \in \alpha_j^*$ for some $j \in \{1, \dots, 2n\}$ with $j \neq 2n$. From the structure of $\mathrm{supp}(Q)$, we have $e \notin \mathrm{supp}(Q)$ and then $e \in Q_k^*$ for some $k \in \{1, \dots, 2n\}$. Hence, $x = ae \in \alpha_j^* \cdot Q_k^* \subseteq Q_{j+k}^*$ and so $x \notin Q_k^*$. By Remark 1 (1), we know that $Q_{(k)} = \{Q_{k+1}, \dots, Q_{2n}, Q_1, \dots, Q_k\}$ is also an ordering on M with the associated ordering α on R. Note that $T \subseteq \alpha_{2n} = (Q_k : Q_k)$ and $e \in Q_{(k)}^+$

but $x \notin Q^+_{(k)}$, which contradicts to the assumption (i). So, (ii) holds. $\square$

Since a ring R can be viewed as a free R-module with a basis $\{1\}$, then 1 is a normal element of R. By Remark 1 (2), a family of subsets $Q = \{Q_1, \ldots, Q_{2n}\}$ of R is an ordering of level n on R as a ring if and only if Q is an ordering of level n on R as an R-module such that $1 \in Q^+$. So, Theorem 4 generalizes Theorem 6 in [7].

Moreover, Theorem 4 induces

Corollary 2 *Let M be an R-module and let e be a semireal and normal element of M. Then, M admits an ordering Q of level n satisfying $e \in Q^+$.*

Proof. By supposition, we have $-e \in Re$ and $T_R^{2n}(-e) \cap (1 + T_R^{2n})e = \emptyset$ where T_R^{2n} is of same meaning as the one introduced in the proof of Proposition 4. This implies that T_R^{2n} is a preordering of level n on R. By the proof of "(i) implying (ii)" in Theorem 4, M admits an ordering Q of level n satisfying $e \in Q^+$ (but $-e \notin Q^+$). $\square$

At the end of this paper, we point out that, both in Theorem 4 and in Corollary 2, the hypothesis that e is normal is not superfluous even if the module M under discussion is finitely generated. Here is an example to illustrate this point.

Example 2 *Let $R = \mathbb{Q}[t]$ where t is an indeterminate over the field $\mathbb{Q}$ of rational numbers and let $M = R/Rt^2$ be an R-module. Then, $e = t + Rt^2$ is a semireal element of M.*

We assert that $e \in \text{supp}(Q)$ for any ordering Q of level n on M. Indeed, otherwise, suppose that there exists an ordering Q of level n on M such that $e \notin \text{supp}(Q)$. Suppose that α is the associated ordering on R with Q. From $te = 0 \in \text{supp}(Q)$, we have $t \in (\text{supp}(Q) : M) = \text{supp}(\alpha)$ and hence $e = t(1 + Rt^2) \in \text{supp}(\alpha) \cdot M \subseteq \text{supp}(Q)$, a contradiction.

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