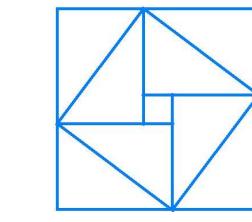


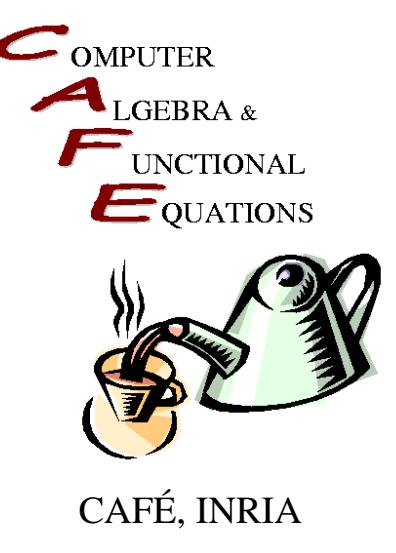
Factoring Finite-rank Linear Functional Systems

Min Wu

email: mwu@mmrc.iss.ac.cn



AMSS, Academia Sinica,
Beijing 100080, China



Sophia Antipolis 06560, France

1 Problem and goal

Finite-rank (linear functional) system S:

- a set of linear homogeneous differential and difference equations
- $\text{sol}(S)$ is finite-dim over some suitable constant field

Goal: an algorithm for factoring a finite-rank system S , i.e., finding “subsystems” L s.t

$$\text{sol}(L) \subsetneq \text{sol}(S).$$

Result of this poster: a factorization algorithm applicable for

- Systems of linear differential equations
- Reflexive systems of linear difference equations
- Reflexive systems of linear differential-difference equations

2 Example

$$L = \left\{ \partial_x^3 + \frac{x^3 - 3x}{x^2 - 1} \partial_x^2, \partial_y \partial_x + \frac{3yx^2}{x^2 - 1} \partial_x^2 + 3yx \partial_x + x \partial_y, \right. \\ \left. \partial_y^2 + \frac{9y^2 x + 3x + y}{x^2 - 1} \partial_x^2 + 3y \partial_y + (3 + 9y^2) \partial_x \right\}.$$

We find

$$\text{sol}(L) = \text{sol}(\{\partial_x + x, \partial_y^2 + 3y \partial_y + y\}) \oplus \text{sol}(\{\partial_x, \partial_y\}) \\ \oplus \text{sol}(\{\partial_x - 2/(2x - 3y^2), \partial_y + 6y/(2x - 3y^2)\}).$$

3 Previous work



Systems	Factoring
LODE's	Beke (1894), Schwarz (89) Bronstein (94), van Hoeij (97) Giesbrecht & Zhang (03), Cluzeau (03)
LOΔE's	Bronstein & Petkovsek (96) van Hoeij (99) (first order factors) Abramov & Paule & Petkovsek (98)
LPDE's	Li & Schwarz & Tsarev (02, 03)
LP{D + Δ}E's	Labahn & Li (04) (first order factors)

4 Basic definitions

$\mathcal{F} = \mathcal{C}(x_1, \dots, x_n)$; for $i = 1, \dots, n$, let σ_i be an automorphism of \mathcal{F} and let δ_i be an additive mapping of \mathcal{F} with the property:

$$\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b \quad \forall a, b \in \mathcal{F}.$$

Assume $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$, $\sigma_i \circ \delta_j = \delta_j \circ \sigma_i$, $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, $\forall i, j$.

An *Ore polynomial ring* [Chyzak & Salvy, 98] is defined to be $\mathcal{A} = \mathcal{F}[\partial_1, \dots, \partial_n]$ with multiplicative rules:

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i a = \sigma_i(a) \partial_i + \delta_i(a), \quad \forall a \in \mathcal{F}, \forall i, j.$$

$\mathcal{A} = \mathcal{F}[\partial_1, \dots, \partial_n]$ is *orthogonal* [Labahn & Li, 04] if,

$$\begin{array}{lll} \sigma_i = 0, & \delta_i = \frac{\partial}{\partial x_i}, & \text{for } i = 1, \dots, p, \\ \sigma_j : x_j \mapsto x_j + 1, & \delta_j = 0, & \text{for } j = p + 1, \dots, n. \end{array}$$

For a (left) ideal I of \mathcal{A} , $\text{rank}(I) := \dim_{\mathcal{F}}(\mathcal{A}/I)$.

• A system $S \subset \mathcal{A}$ is *of finite rank* if $\text{rank}((S)) < \infty$.

• S is *reflexive* if

$$\partial_j f \in (S) \text{ with } f \in \mathcal{A} \text{ and } j \in \{p + 1, \dots, n\} \implies f \in (S).$$

• S is *reflexive* $\iff \text{rank}((S)) = \dim_{\mathcal{F}} \text{sol}(S)$.

M is a *∂-finite module* if M is an \mathcal{A} -module which is also a finite-dim \mathcal{F} -vector space (a generalization of differential modules [van der Put & Singer, 03]).

A ∂ -finite module M is *reflexive* if

$$\partial_j m = 0 \text{ with } m \in M \text{ and } j \in \{p + 1, \dots, n\} \implies m = 0.$$

5 Translation into module language

System $S \implies$ ideal $(S) \iff \mathcal{A}$ -module $\mathcal{A}/(S)$

$$S \text{ reflexive} \iff \mathcal{A}/(S) \text{ reflexive}$$

Advantages of module language:

- more **concise**: bases of vector spaces instead of bases of ideals
- more **general**: applicable to systems in several unknowns
- more **powerful**: convenient for using multi-linear algebra

Let e_1, \dots, e_s be a basis of M over \mathcal{F} and

$$\partial_i(e_1, \dots, e_s) = (e_1, \dots, e_s)A_i, \quad i = 1, \dots, n.$$

Fact: M reflexive $\iff A_{p+1}, \dots, A_n$ invertible.

6 Idea: use exterior power

- M is reflexive:

$$\begin{array}{ccc} M \text{ reflexive} & \xrightarrow{\wedge} & \wedge^d M \text{ reflexive} \\ \partial_i(e_1, \dots, e_s) = (e_1, \dots, e_s)A_i & \xrightarrow{\wedge} & \partial_i(f_1, \dots, f_t) = (f_1, \dots, f_t)B_i \end{array}$$

$$\begin{array}{ccc} d\text{-dim } N \subsetneq M & \xrightarrow{\wedge} & 1\text{-dim } \wedge^d N \subsetneq \wedge^d M \end{array}$$

$$\begin{array}{ccc} \mathcal{F}w_1 \oplus \dots \oplus \mathcal{F}w_d & \xleftarrow{[\text{Compoint \& Weil, 04}]} & \mathcal{F}w \text{ with } w = w_1 \wedge \dots \wedge w_d \\ \text{def} & & \end{array}$$

hyperexponential-scaled sols of integrable system of $\wedge^d M$:
 $\{\partial_i Z = -B_i Z, i = 1, \dots, p,$
 $i.e., \frac{\partial_i(h)}{h} \in \mathcal{F}, \text{ for } i = 1, \dots, n\}$

Hyperexp sols of

$$\begin{array}{c} \downarrow \\ H = hY \text{ with } Y \in \mathcal{F}^t \text{ and} \\ h \text{ hyperexponential over } \mathcal{F}, \\ i.e., \frac{\partial_i(h)}{h} \in \mathcal{F}, \text{ for } i = 1, \dots, n. \end{array}$$

Consider a map $\varphi_w : M \longrightarrow \wedge^{d+1} M, v \mapsto v \wedge w$.

Theorem: $w = w_1 \wedge \dots \wedge w_d \iff \dim_{\mathcal{F}}(\ker(\varphi_w)) = d$.

- M is not reflexive: computing 1-dim submodules directly.

7 Example

$\mathcal{F} = \mathcal{C}(x, k)$, $\mathcal{A} = \mathcal{F}[\partial_x, E_k]$. A rank-four ideal I generated by

$$\partial_x^2 - \frac{2((k-x)^2+k)}{x(k-x)} \partial_x + \frac{((k-x)^3-3xk+3k^2+2k)}{x^2(k-x)}, \quad E_k^2 - \frac{2x(x-k-1)}{x-k-2} E_k + \frac{x^2(x-k)}{x-k-2}.$$

All first order factors of I :

$$\left(\partial_x - \frac{\partial_x(h)}{h}, \quad E_k - \frac{E_k(h)}{h} \right)$$

where $h = \frac{c_1+c_2k+c_3x}{x-k} \exp(-x)x^{k+1}$.

8 Future Study



LPDE's	* improve efficiency
LP{D + Δ}E's	* find any order factors, i.e., find any dim submodules of non-reflexive modules * improve efficiency