SYMBOLIC INTEGRATION TUTORIAL

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Contents

An *elementary function* of a variable x is a function that can be obtained from the rational functions in x by repeatedly adjoining a finite number of nested logarithms, exponentials, and algebraic numbers or functions. Since $\sqrt{-1}$ is elementary, the trigonometric functions and their inverses are also elementary (when they are rewritten using complex exponentials and logarithms) as well as all the "usual" functions of calculus. For example,

$$\sin(x + \tan(x^3 - \sqrt{x^3 - x + 1})) \tag{1}$$

is elementary when rewritten as

$$\frac{\sqrt{-1}}{2}(e^{t-x\sqrt{-1}} - e^{x\sqrt{-1}-t}) \quad \text{where} \quad t = \frac{1 - e^{2\sqrt{-1}(x^3-\sqrt{x^3-x+1})}}{1 + e^{2\sqrt{-1}(x^3-\sqrt{x^3-x+1})}}\,.$$

This tutorial describes recent algorithmic solutions to the *problem of integration in finite terms*: to decide in a finite number of steps whether a given elementary function has an elementary indefinite integral, and to compute it explicitly if it exists. While this problem was studied extensively by Abel and Liouville during the last century, the difficulties posed by algebraic functions caused Hardy (1916) to state that "there is reason to suppose that no such method can be given". This conjecture was eventually disproved by Risch (1970), who described an algorithm for this problem in a series of reports [12, 13, 14, 15]. In the past 30 years, this procedure has been repeatedly improved, extended and refined, yielding practical algorithms that are now becoming standard and are implemented in most of the major computer algebra systems. In this tutorial, we outline the above algorithms for various classes of elementary functions, starting with rational functions and progressively increasing the class of functions up to general elementary functions. Proofs of correctness of the algorithms presented here can be found in several of the references, and are generally too long and too detailed to be described in this tutorial.

Notations: we write x for the variable of integration, and $'$ for the derivation d/dx. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote respectively the integers, rational, real and complex numbers. All fields are commutative and, except when mentioned explicitly otherwise, have characteristic 0. If K is a field, then $\overline{K}$ denotes its algebraic closure. For a polynomial p, $\mathrm{pp}(p)$ denotes the primitive part of p, *i.e.p* divided by the gcd of its coefficients.

1 Rational Functions

By a *rational function*, we mean a quotient of polynomials in the integration variable x. This means that other functions can appear in the integrand, provided they do not involve x, hence that the coefficients of our polynomials in x lie in an arbitrary field K satisfying: $\forall a \in K, a' = 0$.

1.1 The full partial-fraction algorithm

This method, which dates back to Newton, Leibniz and Bernoulli, should not be used in practice, yet it remains the method found in most calculus texts and is often taught. Its major drawback is the factorization of the denominator of the integrand over the real or complex numbers. We outline it because it provides the theoretical foundations for all the subsequent algorithms. Let $f \in \mathbb{R}(x)$ be our integrand, and write $f = P + A/D$ where $P, A, D \in \mathbb{R}[x]$, $\gcd(A, D) = 1$, and $\deg(A) < \deg(D)$. Let

$$D = c \prod_{i=1}^{n} (x - a_i)^{e_i} \prod_{j=1}^{m} (x^2 + b_j x + c_j)^{f_j}$$

be the irreducible factorization of D over $\mathbb{R}$, where c, the a_i's, b_j's and c_j's are in $\mathbb{R}$ and the e_i's and f_j's are positive integers. Computing the partial fraction decomposition of f, we get

$$f = P + \sum_{i=1}^{n} \sum_{k=1}^{e_i} \frac{A_{ik}}{(x - a_i)^k} + \sum_{j=1}^{m} \sum_{k=1}^{f_j} \frac{B_{jk} x + C_{jk}}{(x^2 + b_j x + c_j)^k}$$

where the A_{ik}'s, B_{jk}'s and C_{jk}'s are in $\mathbb{R}$. Hence,

$$\int f = \int P + \sum_{i=1}^{n} \sum_{k=1}^{e_i} \int \frac{A_{ik}}{(x - a_i)^k} + \sum_{j=1}^{m} \sum_{k=1}^{f_j} \int \frac{B_{jk} x + C_{jk}}{(x^2 + b_j x + c_j)^k} \,.$$

Computing $\int P$ poses no problem (it will for any other class of functions), and for the other terms we have

$$\int \frac{A_{ik}}{(x - a_i)^k} = \begin{cases} A_{ik}(x - a_i)^{1-k}/(1-k) & \text{if } k > 1 \\ A_{i1} \log(x - a_i) & \text{if } k = 1 \end{cases} \tag{2}$$

and, noting that $b_j^2 - 4c_j < 0$ since $x^2 + b_j x + c_j$ is irreducible in $\mathbb{R}[x]$,

$$\int \frac{B_{j1} x + C_{j1}}{(x^2 + b_j x + c_j)} = \frac{B_{j1}}{2} \log(x^2 + b_j x + c_j) + \frac{2C_{j1} - b_j B_{j1}}{\sqrt{4c_j - b_j^2}} \arctan\left(\frac{2x + b_j}{\sqrt{4c_j - b_j^2}}\right)$$

and for $k > 1$,

$$\begin{aligned} \int \frac{B_{jk} x + C_{jk}}{(x^2 + b_j x + c_j)^k} &= \frac{(2C_{jk} - b_j B_{jk})x + b_j C_{jk} - 2c_j B_{jk}}{(k-1)(4c_j - b_j^2)(x^2 + b_j x + c_j)^{k-1}} \\ &\quad + \int \frac{(2k-3)(2C_{jk} - b_j B_{jk})}{(k-1)(4c_j - b_j^2)(x^2 + b_j x + c_j)^{k-1}} \,. \end{aligned}$$

This last formula is then used recursively until $k = 1$.

An alternative is to factor D linearly over $\mathbb{C}$: $D = \prod_{i=1}^{q} (x - \alpha_i)^{e_i}$, and then use (2) on each term of

$$f = P + \sum_{i=1}^{q} \sum_{j=1}^{e_i} \frac{A_{ij}}{(x - \alpha_i)^j} \,. \tag{3}$$

Note that this alternative is applicable to coefficients in any field K, if we factor D linearly over its algebraic closure $\overline{K}$, and is equivalent to expanding f into its Laurent series at all its finite poles, since that series at $x = \alpha_i \in \overline{K}$ is

$$f = \frac{A_{ie_i}}{(x - \alpha_i)^{e_i}} + \cdots + \frac{A_{i2}}{(x - \alpha_i)^2} + \frac{A_{i1}}{(x - \alpha_i)} + \cdots$$

where the A_{ij}'s are the same as those in (3). Thus, this approach can be seen as expanding the integrand into series around all its poles (including ∞), then integrating the series termwise, and then interpolating for the answer, by summing all the polar terms, obtaining the integral of (3). In addition, this alternative shows that any rational function $f \in K(x)$ has an elementary integral in the form

$$\int f = v + c_1 \log(u_1) + \cdots + c_m \log(u_m) \tag{4}$$

where $v, u_1, \ldots, u_m \in \overline{K}(x)$ are rational functions, and $c_1, \ldots, c_m \in \overline{K}$ are constants. The original Risch algorithm is essentially a generalization of this approach that searches for integrals of arbitrary elementary functions in a form similar to (4).

1.2 The Hermite reduction

The major computational inconvenience of the full partial fraction approach is the need to factor polynomials over $\mathbb{R}$, $\mathbb{C}$ or $\overline{K}$, thereby introducing algebraic numbers even if the integrand and its integral are both in $\mathbb{Q}(x)$. On the other hand, introducing algebraic numbers may be necessary, for example it is proven in [14] that any field containing an integral of $1/(x^2 + 2)$ must also contain $\sqrt{2}$. Modern research has yielded so-called "rational" algorithms that

- compute as much of the integral as possible with all calculations being done in $K(x)$, and
- compute the minimal algebraic extension of K necessary to express the integral.

The first rational algorithms for integration date back to the 19$^{\text{th}}$ century, when both Hermite [6] and Ostrogradsky [11] invented methods for computing the v of (4) entirely within $K(x)$. We describe here only Hermite's method, since it is the one that has been generalized to arbitrary elementary functions. The basic idea is that if an irreducible $p \in K[x]$ appears with multiplicity $k > 1$ in the factorization of the denominator of the integrand, then (2) implies that it appears with multiplicity $k-1$ in the denominator of the integral. Furthermore, it is possible to compute the product of all such irreducibles for each k without factoring the denominator into irreducibles by computing its *squarefree factorization*, *i.e.* a factorization $D = D_1 D_2^2 \cdots D_m^m$, where each D_i is squarefree and $\gcd(D_i, D_j) = 1$ for $i \neq j$. A straightforward way to compute it is as follows: let $R = \gcd(D, D')$, then $R = D_2 D_3^2 \cdots D_m^{m-1}$, so $D/R = D_1 D_2 \cdots D_m$ and $\gcd(R, D/R) = D_2 \cdots D_m$, which implies finally that

$$D_1 = \frac{D/R}{\gcd(R, D/R)}\,.$$

Computing recursively a squarefree factorization of R completes the one for D. Note that [23] presents a more efficient method for this decomposition. Let now $f \in K(x)$ be our integrand, and write $f = P + A/D$ where $P, A, D \in K[x]$, $\gcd(A, D) = 1$, and $\deg(A) < \deg(D)$. Let $D = D_1 D_2^2 \cdots D_m^m$ be a squarefree factorization of D and suppose that $m \geq 2$ (otherwise D is already squarefree). Let then $V = D_m$ and $U = D/V^m$. Since $\gcd(UV', V) = 1$, we can use the extended Euclidean algorithm to find $B, C \in K[x]$ such that

$$\frac{A}{1-m} = BUV' + CV$$

and $\deg(B) < \deg(V)$. Multiplying both sides by $(1-m)/(UV^m)$ gives

$$\frac{A}{UV^m} = \frac{(1-m)BV'}{V^m} + \frac{(1-m)C}{UV^{m-1}}$$

so, adding and subtracting B'/V^{m-1} to the right hand side, we get

$$\frac{A}{UV^m} = \left(\frac{B'}{V^{m-1}} - \frac{(m-1)BV'}{V^m}\right) + \frac{(1-m)C - UB'}{UV^{m-1}}$$

and integrating both sides yields

$$\int \frac{A}{UV^m} = \frac{B}{V^{m-1}} + \int \frac{(1-m)C - UB'}{UV^{m-1}}$$

so the integrand is reduced to one with a smaller power of V in the denominator. This process is repeated until the denominator is squarefree, yielding $g, h \in K(x)$ such that $f = g' + h$ and h has a squarefree denominator.

1.3 The Rothstein–Trager and Lazard–Rioboo–Trager algorithms

Following the Hermite reduction, we only have to integrate fractions of the form $f = A/D$ with $\deg(A) < \deg(D)$ and D squarefree. It follows from (2) that

$$\int f = \sum_{i=1}^{n} a_i \log(x - \alpha_i)$$

where the α_i's are the zeros of D in $\overline{K}$, and the a_i's are the residues of f at the α_i's. The problem is then to compute those residues without splitting D. Rothstein [18] and Trager [19] independently proved that the a_i's are exactly the zeroes of

$$R = \text{resultant}_x(D, A - tD') \quad \in K[t] \tag{5}$$

and that the splitting field of R over K is indeed the minimal algebraic extension of K necessary to express the integral in the form (4). The integral is then given by

$$\int \frac{A}{D} = \sum_{i=1}^{m} \sum_{a|R_i(a)=0} a \log(\gcd(D, A - aD')) \tag{6}$$

where $R = \prod_{i=1}^{m} R_i^{e_i}$ is the irreducible factorization of R over K. Note that this algorithm requires factoring R into irreducibles over K, and computing greatest common divisors in $(K[t]/(R_i))[x]$, hence computing with algebraic numbers. Trager and Lazard & Rioboo [7] independently discovered that those computations can be avoided, if one uses the subresultant PRS algorithm to compute the resultant of (5): let $((R_0, R_1, \ldots, R_k \neq 0, 0, \ldots)$ be the subresultant PRS with respect to x of D and $A - tD'$ and $R = Q_1 Q_2^2 \ldots Q_m^m$ be a *squarefree* factorization of their resultant. Then,

$$\sum_{a|Q_i(a)=0} a \log(\gcd(D, A - aD')) =$$

$$\begin{cases} \sum_{a|Q_i(a)=0} a \log(D) & \text{if } i = \deg(D) \\ \sum_{a|Q_i(a)=0} a \log(\text{pp}_x(R_{k_i})(a, x)) & \text{where } \deg(R_{k_i}) = i, 1 \leq k_i \leq n \\ & \text{if } i < \deg(D) \end{cases}$$

Evaluating $\text{pp}_x(R_{k_i})$ at $t = a$ where a is a root of Q_i is equivalent to reducing each coefficient with respect to x of $\text{pp}_x(R_{k_i})$ modulo Q_i, hence computing in the algebraic extension $K[t]/(Q_i)$. Even this step can be avoided: it is in fact sufficient to ensure that Q_i and the leading coefficient with respect to x of R_{k_i} do not have a nontrivial common factor, which implies then that the remainder by Q_i is nonzero, see [10] for details and other alternatives for computing $\text{pp}_x(R_{k_i})(a, x)$.

2 Algebraic Functions

By an *algebraic function*, we mean an element of a finitely generated algebraic extension E of the rational function field $K(x)$. This includes nested radicals and implicit algebraic functions, not all of which can be expressed by radicals. It turns out that the algorithms we used for rational functions can be extended to algebraic functions, but with several difficulties, the first one being to define the proper analogues of polynomials, numerators and denominators. Since E is algebraic over $K(x)$, for any $\alpha \in E$, there exists a polynomial $p \in K[x][y]$ such that $p(x, \alpha) = 0$. We say that $\alpha \in E$ is *integral over* $K[x]$ if there is a polynomial $p \in K[x][y]$, *monic in* y, such that $p(x, \alpha) = 0$. Integral elements are analogous to polynomials in that their value is defined for any $x \in \overline{K}$ (unlike non-integral elements, which must have at least one pole in $\overline{K}$). The set

$$\mathcal{O}_{K[x]} = \{\alpha \in E \text{ such that } \alpha \text{ is integral over } K[x]\}$$

is called the *integral closure of* $K[x]$ *in* E. It is a ring and a finitely generated $K[x]$-module. Let $\alpha \in E^*$ be any element and $p = \sum_{i=0}^m a_i y^i \in K[x][y]$ be such that $p(x, \alpha) = 0$ and $a_m \neq 0$. Then, $q(x, a_m y) = 0$ where $q = y^m + \sum_{i=0}^{m-1} a_i a_m^{m-i-1} y^i$ is monic in y, so $a_m y \in \mathcal{O}_{K[x]}$. We need a canonical representation for algebraic functions similar to quotients of polynomials for rational functions. Expressions as quotients of integral functions are not unique, for example $\sqrt{x}/x = x/\sqrt{x}$. However, E is a finite-dimensional vector space over $K(x)$, so let $n = [E : K(x)]$ and $w = (w_1, \ldots, w_n)$ be any basis for E over $K(x)$. By the above remark, there are $a_1, \ldots, a_n \in K(x)^*$ such that $a_i w_i \in \mathcal{O}_{K[x]}$ for each i. Since $(a_1 w_1, \ldots, a_n w_n)$ is also a basis for E over $K(x)$, we can assume without loss of generality that the basis w is composed of integral elements. Any $\alpha \in E$ can be written uniquely as $\alpha = \sum_{i=1}^n f_i w_i$ for $f_1, \ldots, f_n \in K(x)$, and putting the f_i's over a monic common denominator $D \in K[x]$, we get an expression

$$\alpha = \frac{A_1 w_1 + \ldots + A_n w_n}{D}$$

where $A_1, \ldots, A_n \in K[x]$ and $\gcd(D, A_1, \ldots, A_n) = 1$. We call $\sum_{i=1}^n A_i w_i \in \mathcal{O}_{K[x]}$ and $D \in K[x]$ respectively the *numerator* and *denominator* of α with respect to w. They are defined uniquely once the basis w is fixed.

2.1 The Hermite reduction

Now that we have numerators and denominators for algebraic functions, we can attempt to generalize the Hermite reduction of the previous section, so let $f \in E$ be our integrand, $w = (w_1, \ldots, w_n) \in {\mathcal{O}_{K[x]}}^n$ be a basis for E over $K(x)$ and let $\sum_{i=1}^m A_i w_i \in \mathcal{O}_{K[x]}$ and $D \in K[x]$ be the numerator and denominator of f with respect to w, Let $D = D_1 D_2^2 \cdots D_m^m$ be a squarefree factorization of D and suppose that $m \geq 2$. Let then $V = D_m$ and $U = D/V^m$, and we ask whether we can compute $B = \sum_{i=1}^n B_i w_i \in \mathcal{O}_{K[x]}$ and $h \in E$ such that $\deg(B_i) < \deg(V)$ for each i,

$$\int \frac{\sum_{i=1}^n A_i w_i}{U V^m} = \frac{B}{V^{m-1}} + \int h \tag{7}$$

and the denominator of h with respect to w has no factor of order m or higher. This turns out to reduce to solving the following linear system

$$f_1 S_1 + \ldots + f_n S_n = A_1 w_1 + \ldots + A_n w_n \tag{8}$$

for $f_1, \ldots, f_n \in K(x)$, where

$$S_i = U V^m \left(\frac{w_i}{V^{m-1}}\right)' \quad \text{for } 1 \leq i \leq n\,. \tag{9}$$

Indeed, suppose that (8) has a solution $f_1, \ldots, f_n \in K(x)$, and write $f_i = T_i/Q$, where $Q, T_1, \ldots, T_n \in K[x]$ and $\gcd(Q, T_1, \ldots, T_n) = 1$. Suppose further that $\gcd(Q, V) = 1$. Then, we can use the extended Euclidean algorithm to find $A, R \in K[x]$ such that $AV + RQ = 1$, and Euclidean division to find $Q_i, B_i \in K[x]$ such that $\deg(B_i) < \deg(V)$ when $B_i \neq 0$ and $RT_i = VQ_i + B_i$ for each i. We then have

$$
\begin{aligned}
h &= f - \left(\frac{\sum_{i=1}^n B_i w_i}{V^{m-1}}\right)' \\
&= \frac{\sum_{i=1}^n A_i w_i}{UV^m} - \frac{\sum_{i=1}^n B_i' w_i}{V^{m-1}} - \sum_{i=1}^n (RT_i - VQ_i)\left(\frac{w_i}{V^{m-1}}\right)' \\
&= \frac{\sum_{i=1}^n A_i w_i}{UV^m} - \frac{R\sum_{i=1}^n T_i S_i}{UV^m} + V\sum_{i=1}^n Q_i\left(\frac{w_i}{V^{m-1}}\right)' - \frac{\sum_{i=1}^n B_i' w_i}{V^{m-1}} \\
&= \frac{(1-RQ)\sum_{i=1}^n A_i w_i}{UV^m} + \frac{\sum_{i=1}^n Q_i w_i'}{V^{m-2}} - (m-1)V'\frac{\sum_{i=1}^n Q_i w_i}{V^{m-1}} - \frac{\sum_{i=1}^n B_i' w_i}{V^{m-1}} \\
&= \frac{\sum_{i=1}^n AA_i w_i}{UV^{m-1}} - \frac{\sum_{i=1}^n ((m-1)V'Q_i + B_i')w_i}{V^{m-1}} + \frac{\sum_{i=1}^n Q_i w_i'}{V^{m-2}}\,.
\end{aligned}
$$

Hence, if in addition the denominator of h has no factor of order m or higher, then $B = \sum_{i=1}^n B_i w_i \in \mathcal{O}_{K[x]}$ and h solve (7) and we have reduced the integrand. Unfortunately, it can happen that the denominator of h has a factor of order m or higher, or that (8) has no solution in $K(x)$ whose denominator is coprime with V, as the following example shows.

Example 1 *Let $E = K(x)[y]/(y^4 + (x^2+x)y - x^2)$ with basis $w = (1, y, y^2, y^3)$ over $K(x)$ and consider the integrand*

$$
f = \frac{y^3}{x^2} = \frac{w_4}{x^2} \in E\,.
$$

We have $D = x^2$, so $U = 1$, $V = x$ and $m = 2$. Then, $S_1 = x^2(1/x)' = -1$,

$$
S_2 = x^2\left(\frac{y}{x}\right)' = \frac{24(1-x^2)y^3+32x(1-x)y^2-(9x^4+45x^3+209x^2+63x+18)y-18x(x^3+x^2-x-1)}{27x^4+108x^3+418x^2+108x+27}\,,
$$

$$
S_3 = x^2\left(\frac{y^2}{x}\right)' = \frac{64x(1-x)y^3+9(x^4+2x^3-2x-1)y^2+12x(x^3+x^2-x-1)y+48x^2(1-x^2)}{27x^4+108x^3+418x^2+108x+27}
$$

and

$$
S_4 = x^2\left(\frac{y^3}{x}\right)' = \frac{(27x^4+81x^3+209x^2+27x)y^3+18x(x^3+x^2-x-1)y^2+24x^2(x^2-1)y+96x^3(1-x)}{27x^4+108x^3+418x^2+108x+27}
$$

so (8) becomes

$$
M\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (10)
$$

where

$$M = \begin{pmatrix} -1 & \frac{-18x(x^3+x^2-x-1)}{F} & \frac{48x^2(1-x^2)}{F} & \frac{96x^3(1-x)}{F} \\ 0 & \frac{-(9x^4+45x^3+209x^2+63x+18)}{F} & \frac{12x(x^3+x^2-x-1)}{F} & \frac{24x^2(x^2-1)}{F} \\ 0 & \frac{32x(1-x)}{F} & \frac{9(x^4+2x^3-2x-1)}{F} & \frac{18x(x^3+x^2-x-1)}{F} \\ 0 & \frac{24(1-x^2)}{F} & \frac{64x(1-x)}{F} & \frac{(27x^4+81x^3+209x^2+27x)}{F} \end{pmatrix}$$

and $F = 27x^4 + 108x^3 + 418x^2 + 108x + 27$. *The system (10) admits the unique solution* $f_1 = f_2 = 0, f_3 = -2$ *and* $f_4 = (x+1)/x$, *whose denominator is not coprime with* V, *so the Hermite reduction is not applicable.*

The above problem was first solved by Trager [20], who proved that if w is an *integral basis*, *i.e.* its elements generate $\mathcal{O}_{K[x]}$ over $K[x]$, then the system (8) always has a unique solution in $K(x)$ when $m > 1$, and that solution always has a denominator coprime with V. Furthermore, the denominator of each w_i' must be squarefree, implying that the denominator of h is a factor of FUV^{m-1} where $F \in K[x]$ is squarefree and coprime with UV. He also described an algorithm for computing an integral basis, a necessary preprocessing for his Hermite reduction. The main problem with that approach is that computing the integral basis, whether by the method of [20] or the local alternative [21], can be in general more expensive than the rest of the reduction process. We describe here the lazy Hermite reduction [5], which avoids the precomputation of an integral basis. It is based on the observation that if $m > 1$ and (8) does not have a solution allowing us to perform the reduction, then either

- the S_i's are linearly dependent over $K(x)$, or
- (8) has a unique solution in $K(x)$ whose denominator has a nontrivial common factor with V, or
- the denominator of some w_i is not squarefree.

In all of the above cases, we can replace our basis w by a new one, also made up of integral elements, so that the $K[x]$-module generated by the new basis strictly contains the one generated by w:

Theorem 1 ([5]) *Suppose that* $m \geq 2$ *and that* $\{S_1, \dots, S_n\}$ *as given by (9) are linearly dependent over* $K(x)$, *and let* $T_1, \dots, T_n \in K[x]$ *be not all* 0 *and such that* $\sum_{i=1}^n T_i S_i = 0$. *Then,*

$$w_0 = \frac{U}{V} \sum_{i=1}^n T_i w_i \in \mathcal{O}_{K[x]} \,.$$

Furthermore, if $\gcd(T_1, \dots, T_n) = 1$, *then* $w_0 \notin K[x]w_1 + \cdots + K[x]w_n$.

Theorem 2 ([5]) *Suppose that* $m \geq 2$ *and that* $\{S_1, \dots, S_n\}$ *as given by (9) are linearly independent over* $K(x)$, *and let* $Q, T_1, \dots, T_n \in K[x]$ *be such that*

$$\sum_{i=1}^n A_i w_i = \frac{1}{Q} \sum_{i=1}^n T_i S_i \,.$$

Then,

$$w_0 = \frac{U(V/\gcd(V,Q))}{\gcd(V,Q)} \sum_{i=1}^{n} T_i w_i \in \mathcal{O}_{K[x]}\,.$$

Furthermore, if $\gcd(Q, T_1, \ldots, T_n) = 1$ *and* $\deg(\gcd(V,Q)) \geq 1$*, then* $w_0 \notin K[x]w_1 + \cdots + K[x]w_n$*.*

Theorem 3 ([5]) *Suppose that the denominator* F *of some* w_i *is not squarefree, and let* $F = F_1 F_2^2 \cdots F_k^k$ *be its squarefree factorization. Then,*

$$w_0 = F_1 \cdots F_k w_i' \in \mathcal{O}_{K[x]} \setminus (K[x]w_1 + \cdots + K[x]w_n)\,.$$

The lazy Hermite reduction proceeds by solving the system (8) in $K(x)$. Either the reduction will succeed, or one of the above theorems produces an element $w_0 \in \mathcal{O}_{K[x]} \setminus (K[x]w_1 + \cdots + K[x]w_n)$. Let then $\sum_{i=1}^{n} C_i w_i$ and F be the numerator and denominator of w_0 with respect to w. Using Hermitian row reduction, we can zero out the last row of

$$\begin{pmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F \\ C_1 & C_2 & \cdots & C_n \end{pmatrix}$$

obtaining a matrix of the form

$$\begin{pmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & & \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $C_{ij} \in K[x]$. Let $\overline{w}_i = (\sum_{j=1}^{n} C_{ij} w_j)/F$ for $1 \leq i \leq n$. Then, $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_n)$ is a basis for E over K and

$$K[x]\overline{w}_1 + \cdots + K[x]\overline{w}_n = K[x]w_1 + \cdots + K[x]w_n + K[x]w_0$$

is a submodule of $\mathcal{O}_{K[x]}$, which strictly contains $K[x]w_1 + \cdots + K[x]w_n$, since it contains w_0. Any stricly increasing chain of submodules of $\mathcal{O}_{K[x]}$ must stabilize after a finite number of steps, which means that this process produces a basis for which either the Hermite reduction can be carried out, or for which f has a squarefree denominator.

Example 2 *Continuing example 1 for which the Hermite reduction failed, Theorem 2 implies that*

$$w_0 = \frac{1}{x}(-2xw_3 + (x+1)w_4) = \left(-2xy^2 + (x+1)y^3\right)x \in \mathcal{O}_{K[x]}\,.$$

Performing a Hermitian row reduction on

$$\begin{pmatrix} x & & & \\ & x & & \\ & & x & \\ & & & x \\ 0 & 0 & -2x & x+1 \end{pmatrix}$$

yields

$$\begin{pmatrix} x & & & \\ & x & & \\ & & x & \\ & & & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the new basis is $\overline{w} = (1, y, y^2, y^3/x)$*, and the denominator of* f *with respect to* $\overline{w}$ *is* x*, which is squarefree.*

2.2 Simple radical extensions

The integration algorithm becomes easier when E is a simple radical extension of $K(x)$,*i.e.* $E = K(x)[y]/(y^n - a)$ for some $a \in K(x)$. Write $a = A/D$ where $A, D \in K[x]$, and let $AD^{n-1} = A_1 A_2^2 \cdots A_k^k$ be a squarefree factorization of AD^{n-1}. Writing $i = nq_i + r_i$ for $1 \le i \le k$, where $0 \le r_i < n$, let $F = A_1^{q_1} \cdots A_k^{q_k}$, $H = A_1^{r_1} \cdots A_k^{r_k}$ and $z = yD/F$. Then,

$$z^n = \left(y\frac{D}{F}\right)^n = \frac{y^n D^n}{F^n} = \frac{AD^{n-1}}{F} = A_1^{r_1} \ldots A_k^{r_k} = H\,.$$

Since $r_i < n$ for each i, the squarefree factorization of H is of the form $H = H_1 H_2^2 \cdots H_m^m$ with $m < n$. An integral basis is then $w = (w_1, \ldots, w_n)$ where

$$w_i = \frac{z^{i-1}}{\prod_{j=1}^m H_j^{\lfloor (i-1)j/n \rfloor}} \qquad \text{for } 1 \le i \le n \tag{11}$$

and the Hermite reduction with respect to the above basis is always guaranteed to succeed. Furthermore, when using that basis, the system (8) becomes diagonal and its solution can be written explicitly: writing $D_i = \prod_{j=1}^m H_j^{\lfloor ij/n \rfloor}$ we have

$$\begin{aligned} S_i &= UV^m\left(\frac{w_i}{V^{m-1}}\right)' = UV^m\left(\frac{z^{i-1}}{D_{i-1}V^{m-1}}\right)' \\ &= UV^m\left(\frac{i-1}{n}\frac{H'}{H} - \frac{D_{i-1}{}'}{D_{i-1}} - (m-1)\frac{V'}{V}\right)\left(\frac{z^{i-1}}{D_{i-1}V^{m-1}}\right) \\ &= U\left(V\left(\frac{i-1}{n}\frac{H'}{H} - \frac{D_{i-1}{}'}{D_{i-1}}\right) - (m-1)V'\right)w_i \end{aligned}$$

so the unique solution of (8) in $K(x)$ is

$$f_i = \frac{A_i}{U\left(V\left(\frac{i-1}{n}\frac{H'}{H} - \frac{D_{i-1}'}{D_{i-1}}\right) - (m-1)V'\right)} \qquad \text{for } 1 \le i \le n \tag{12}$$

and it can be shown that the denominator of each f_i is coprime with V when $m \ge 2$.

Example 3 *Consider*

$$\int \frac{(2x^8+1)\sqrt{x^8+1}}{x^{17}+2x^9+x}dx\,.$$

The integrand is

$$f = \frac{(2x^8+1)y}{x^{17}+2x^9+x} \in E = \mathbb{Q}(x)[y]/(y^2-x^8-1)$$

so $H = x^8+1$ *which is squarefree, implying that the integral basis (11) is* $(w_1, w_2) = (1, y)$*. The squarefree factorization of* $x^{17}+2x^9+x$ *is* $x(x^8+1)^2$ *so* $U = x$, $V = x^8+1$, $m = 2$, *and the solution (12) of (8) is*

$$f_1 = 0, \quad f_2 = \frac{2x^8+1}{x\left((x^8+1)\frac{1}{2}\frac{8x^7}{x^8+1} - 8x^7\right)} = -\frac{(2x^8+1)/4}{x^8}\,.$$

We have $Q = x^8$*, so* $V - Q = 1$, $A = 1$, $R = -1$ *and* $RQf_2 = V/2 - 1/4$*, implying that*

$$B = -\frac{y}{4} \quad \text{and} \quad h = f - \left(\frac{B}{V}\right)' = \frac{y}{x(x^8+1)}$$

solve (7),i.e.

$$\int \frac{(2x^8+1)\sqrt{x^8+1}}{x^{17}+2x^9+x}dx = -\frac{\sqrt{x^8+1}}{4(x^8+1)} + \int \frac{\sqrt{x^8+1}}{x(x^8+1)}dx$$

and the remaining integrand has a squarefree denominator.

2.3 Liouville's Theorem

Up to this point, the algorithms we have presented never fail, yet it can happen that an algebraic function does not have an elementary integral, for example

$$\int \frac{xdx}{\sqrt{1-x^3}}$$

which is not an elementary function of x. So we need a way to recognize such functions before completing the integration algorithm. Liouville was the first to state and prove a precise theorem from Laplace's observation that we can restrict the elementary integration problem by allowing only new logarithms to appear linearly in the integral, all the other terms appearing in the integral being already in the integrand.

Theorem 4 (Liouville [8, 9]) *Let E be an algebraic extension of the rational function field $K(x)$, and $f \in E$. If f has an elementary integral, then there exist $v \in E$, constants $c_1, \dots, c_k \in \overline{K}$ and $u_1, \dots, u_k \in E(c_1, \dots, c_k)^*$ such that*

$$f = v' + c_1 \frac{u_1'}{u_1} + \cdots + c_k \frac{u_k'}{u_k}. \tag{13}$$

The above is a restriction to algebraic functions of the strong Liouville Theorem, whose proof can be found in [4, 14]. An elegant and elementary algebraic proof of a slightly weaker version can be found in [17]. As a consequence, we can look for an integral in the form (4), Liouville's Theorem guaranteeing that there is no elementary integral if we cannot find one in that form. Note that the above theorem does not say that every integral must have the above form, and in fact that form is not always the most convenient one, for example

$$\int \frac{dx}{1+x^2} = \arctan(x) = \frac{\sqrt{-1}}{2} \log\left(\frac{\sqrt{-1}+x}{\sqrt{-1}-x}\right).$$

2.4 The integral part

Following the Hermite reduction, we can assume that we have a basis $w = (w_1, \dots, w_n)$ of E over $K(x)$ made of integral elements such that our integrand is of the form $f = \sum_{i=1}^n A_i w_i / D$ where $D \in K[x]$ is squarefree. Given Liouville's Theorem, we now have to solve equation (13) for $v, u_1, \dots, u_k$ and the constants $c_1, \dots, c_k$. Since D is squarefree, it can be shown that $v \in \mathcal{O}_{K[x]}$ for any solution, and in fact v corresponds to the polynomial part of the integral of rational functions. It is however more difficult to compute than the integral of polynomials, so Trager [20] gave a change of variable that guarantees that either $v' = 0$ or f has no elementary integral. In order to describe it, we need to define the analogue for algebraic functions of having a nontrivial polynomial part: we say that $\alpha \in E$ is *integral at infinity* if there is a polynomial $p = \sum_{i=1}^m a_i y^i \in K[x][y]$ such that $p(x, \alpha) = 0$ and $\deg(a_m) \ge \deg(a_i)$ for each i. Note that a rational function $A/D \in K(x)$ is integral at infinity if and only if $\deg(A) \le \deg(D)$ since it is a zero of $Dy - A$. When $\alpha \in E$ is not integral at infinity, we say that it has a *pole at infinity.* Let

$$\mathcal{O}_\infty = \{\alpha \in E \text{ such that } \alpha \text{ is integral at infinity}\}.$$

A set $(b_1, \dots, b_n) \in E^n$ is called *normal at infinity* if there are $r_1, \dots, r_n \in K(x)$ such that every $\alpha \in \mathcal{O}_\infty$ can be written as $\alpha = \sum_{i=1}^n B_i r_i b_i / C$ where $C, B_1, \dots, B_n \in K[x]$ and $\deg(C) \ge \deg(B_i)$ for each i. We say that the differential αdx is integral at infinity if $\alpha x^{1+1/r} \in \mathcal{O}_\infty$ where r is the smallest ramification index at infinity. Trager [20] described an algorithm that converts an arbitrary integral basis $w_1, \dots, w_n$ into one that is also normal at infinity, so the first part of his integration algorithm is as follows:

1. Pick any basis $b = (b_1, \dots, b_n)$ of E over $K(x)$ that is composed of integral elements.

2. Pick an integer $N \in \mathbb{Z}$ that is not zero of the denominator of f with respect to b, nor of the discriminant of E over $K(x)$, and perform the change of variable $x = N + 1/z$, $dx = -dz/z^2$ on the integrand.

3. Compute an integral basis w for E over $K(z)$ and make it normal at infinity.

4. Perform the Hermite reduction on f using w, this yields $g, h \in E$ such that $\int f dz = g + \int h dz$ and h has a squarefree denominator with respect to w.

5. If hz^2 has a pole at infinity, then $\int f dz$ and $\int h dz$ are not elementary functions.

6. Otherwise, $\int h dz$ is elementary if and only if there are constants $c_1, \dots, c_k \in \overline{K}$ and $u_1, \dots, u_k \in E(c_1, \dots, c_k)^*$ such that

$$h = \frac{c_1}{u_1}\frac{du_1}{dz} + \cdots + \frac{c_k}{u_k}\frac{du_k}{dz} \tag{14}$$

The condition that N is not a zero of the denominator of f with respect to b implies that the $f dz$ is integral at infinity after the change of variable, and Trager proved that if hdz is not integral at infinity after the Hermite reduction, then $\int h dz$ and $\int f dz$ are not elementary functions. The condition that N is not a zero of the discriminant of E over $K(x)$ implies that the ramification indices at infinity are all equal to 1 after the change of variable, hence that hdz is integral at infinity if and only if $hz^2 \in \mathcal{O}_\infty$. That second condition on N can be disregarded, in which case we must replace hz^2 in step 5 by $hz^{1+1/r}$ where r is the smallest ramification index at infinity. Note that $hz^2 \in \mathcal{O}_\infty$ implies that $hz^{1+1/r} \in \mathcal{O}_\infty$, but not conversely. Finally, we remark that for simple radical extensions, the integral basis (11) is already normal at infinity.

Alternatively, we can use the lazy Hermite reduction in the above algorithm: in step 3, we pick any basis made of integral elements, then perform the lazy Hermite reduction in step 4. If $h \in K(z)$ after the Hermite reduction, then we can complete the integral without computing an integral basis. Otherwise, we compute an integral basis and make it normal at infinity between steps 4 and 5. This lazy variant can compute $\int f dx$ whenever it is an element of E without computing an integral basis.

2.5 The logarithmic part

Following the previous sections, we are left with solving equation (14) for the constants $c_1, \dots, c_k$ and for $u_1, \dots, u_k$. We must make at this point the following additional assumptions:

- we have an integral primitive element for E over $K(z)$, *i.e.* $y \in \mathcal{O}_{K[z]}$ such that $E = K(z)(y)$,

- $[E : K(z)] = [E : \overline{K}(z)]$, *i.e.* the minimal polynomial for y over $K[z]$ is absolutely irreducible, and
- we have an integral basis $w = (w_1, \dots, w_n)$ for E over $K(z)$, and w is normal at infinity.

A primitive element can be computed by considering linear combinations of the generators of E over $K(x)$ with random coefficients in $K(x)$, and Trager [20] describes an absolute factorization algorithm, so the above assumptions can be ensured, although those steps can be computationally very expensive, except in the case of simple radical extensions. Before describing the second part of Trager's integration algorithm, we need to define some concepts from the theory of algebraic curves. Given a finite algebraic extension $E = K(z)(y)$ of $K(z)$, a *place* P of E is a proper local subring of E containing K, and a *divisor* is a formal sum $\sum n_P P$ with finite support, where the n_P's are integers and the P's are places. Let P be a place, then its maximal ideal μ_P is principal, so let $p \in E$ be a generator of μ_P. The *order at* P is the function $\nu_P : E^* \to \mathbb{Z}$ which maps $f \in E^*$ to the largest $k \in \mathbb{Z}$ such that $f \in p^k P$. Given $f \in E^*$, the *divisor of* f is $(f) = \sum \nu_P(f) P$ where the sum is taken over all the places. It has finite support since $\nu_P(f) \neq 0$ if and only if P is a pole or zero of f. Finally, we say that a divisor $\delta = \sum n_P P$ is *principal* if $\delta = (f)$ for some $f \in E^*$. Note that if δ is principal, then $\sum n_P = 0$, but the converse is not generally true, except if $E = K(z)$. Trager's algorithm proceeds essentially by constructing candidate divisors for the u_i's of (14):

1. Let $\sum_{i=1}^n A_i w_i$ be the numerator of h with respect to w, and D be its (squarefree) denominator.
2. Write $\sum_{i=1}^n A_i w_i = G/H$, where $G \in K[z, y]$ and $H \in K[z]$.
3. Let $F \in K[z, y]$ be the (monic) minimum polynomial for y over $K(z)$, t be a new indeterminate and compute
$$R(t) = \text{resultant}_z\left(\text{pp}_t\left(\text{resultant}_y\left(G - tH\frac{dD}{dz}, F\right)\right), D\right) \in K[t]\,.$$
4. Let $\alpha_1, \dots, \alpha_s \in \overline{K}$ be the distinct nonzero roots of R, $(q_1, \dots, q_k)$ be a basis for the vector space that they generate over $\mathbb{Q}$, write $\alpha_i = r_{i1} q_1 + \cdots + r_{ik} q_k$ for each i, where $r_{ij} \in \mathbb{Q}$ and let $m > 0$ be a common denominator for all the r_{ij}'s.
5. For $1 \le j \le k$, let $\delta_j = \sum_{i=1}^s m r_{ij} \sum_l r_l P_l$ where r_l is the ramification index of P_l and P_l runs over all the places at which $h dz$ has residue $r_l \alpha_i$.
6. If there are nonzero integers $n_1, \dots, n_k$ such that $n_j \delta_j$ is principal for each j, then let
$$u = h - \frac{1}{m} \sum_{j=1}^k \frac{q_j}{n_j u_j} \frac{du_j}{dz}$$

where $u_j \in E(\alpha_1, \ldots, \alpha_s)^*$ is such that $n_j \delta_j = (u_j)$. If $u = 0$, then $\int h dz = \sum_{j=1}^{k} q_j \log(u_j)/(m n_j)$, otherwise if either $u \neq 0$ or there is no such integer n_j for at least one j, then hdz has no elementary integral.

Note that this algorithm expresses the integral, when it is elementary, with the smallest possible number of logarithms. Steps 3 to 6 requires computing in the splitting field K_0 of R over K, but it can be proven that, as in the case of rational functions, K_0 is the minimal algebraic extension of K necessary to express the integral in the form (4). Trager [20] describes a representation of divisors as fractional ideals and gives algorithms for the arithmetic of divisors and for testing whether a given divisor is principal. In order to determine whether there exists an integer N such that $N\delta$ is principal, we need to reduce the algebraic extension to one over a finite field $\mathbb{F}_{p^q}$ for some "good" prime $p \in \mathbb{Z}$. Over $\mathbb{F}_{p^q}$, it is known that for every divisor $\delta = \sum n_P P$ such that $\sum n_P = 0$, $M\delta$ is principal for some integer $1 \leq M \leq (1 + \sqrt{p^q})^{2g}$, where g is the genus of the curve [22], so we compute such an M by testing $M = 1, 2, 3, \ldots$ until we find it. It can then be shown that for almost all primes p, if $M\delta$ is not principal in characteristic 0, then $N\delta$ is not principal for any integer $N \neq 0$. Since we can test whether the prime p is "good" by testing whether the image in $\mathbb{F}_{p^q}$ of the discriminant of the discriminant of the minimal polynomial for y over $K[z]$ is 0, this yields a complete algorithm. In the special case of hyperelliptic extensions, *i.e.* simple radical extensions of degree 2, Bertrand [1] describes a simpler representation of divisors for which the arithmetic and principality test are more efficient than the general methods.

Example 4 *Continuing example 3, we were left with the integrand*

$$\frac{\sqrt{x^8+1}}{x(x^8+1)} = \frac{w_2}{x(x^8+1)} \in E = \mathbb{Q}(x)[y]/(y^2 - x^8 - 1)$$

where $(w_1, w_2) = (1, y)$ is an integral basis normal at infinity, and the denominator $D = x(x^8+1)$ of the integrand is squarefree. Its numerator is $w_2 = y$, so the resultant of step 3 is

$$\textit{resultant}_x(\textit{pp}_t(\textit{resultant}_y(y - t(9x^8+1), y^2 - x^8 - 1)), x(x^8+1)) = ct^{16}(t^2-1)$$

where c is a large nonzero integer. Its nonzero roots are ± 1, and the integrand has residue 1 at the place P corresponding to the point $(x, y) = (0, 1)$ and -1 at the place Q corresponding to the point $(x, y) = (0, -1)$, so the divisor δ_1 of step 5 is $\delta_1 = P - Q$. It turns out that $\delta_1, 2\delta_1$ and $3\delta_1$ are not principal, but that

$$4\delta_1 = \left(\frac{x^4}{1+y}\right) \quad \textit{and} \quad \frac{w_2}{x(x^8+1)} - \frac{1}{4}\frac{(x^4/(1+y))'}{x^4/(1+y)} = 0$$

which implies that

$$\int \frac{\sqrt{x^8+1}}{x(x^8+1)} dx = \frac{1}{4} \log\left(\frac{x^4}{1+\sqrt{x^8+1}}\right).$$

Example 5 *Consider*

$$\int \frac{xdx}{\sqrt{1-x^3}}\,.$$

The integrand is

$$f = \frac{xy}{1-x^3} \in E = \mathbb{Q}(x)[y]/(y^2+x^3-1)$$

where $(w_1, w_2) = (1, y)$ *is an integral basis normal at infinity, and the denominator* $D = 1 - x^3$ *of the integrand is squarefree. Its numerator is* $xw_2 = xy$*, so the resultant of step 3 is*

$$\textit{resultant}_x(pp_t(\textit{resultant}_y(xy + 3tx^2, y^2 + x^3 - 1)), 1 - x^3) = 729t^6$$

whose only root is 0*. Since* $f \neq 0$*, we conclude from step 6 that* $\int f dx$ *is not an elementary function.*

Example 6

$$\int \frac{dx}{x\sqrt{1-x^3}}\,.$$

The integrand is

$$f = \frac{y}{x-x^4} \in E = \mathbb{Q}(x)[y]/(y^2+x^3-1)$$

where $(w_1, w_2) = (1, y)$ *is an integral basis normal at infinity, and the denominator* $D = x - x^4$ *of the integrand is squarefree. Its numerator is* $w_2 = y$*, so the resultant of step 3 is*

$$\textit{resultant}_x(pp_t(\textit{resultant}_y(y + t(4x^3-1), y^2 + x^3 - 1)), x - x^4) = 729t^6(t^2-1)\,.$$

Its nonzero roots are ± 1*, and the integrand has residue* 1 *at the place* P *corresponding to the point* $(x, y) = (0, 1)$ *and* -1 *at the place* Q *corresponding to the point* $(x, y) = (0, -1)$*, so the divisor* δ_1 *of step 5 is* $\delta_1 = P - Q$*. It turns out that* δ_1 *and* $2\delta_1$ *are not principal, but that*

$$3\delta_1 = \left(\frac{y-1}{y+1}\right) \quad \textit{and} \quad \frac{y}{x-x^4} - \frac{1}{3}\frac{((y-1)/(y+1))'}{(y-1)/(y+1)} = 0$$

which implies that

$$\int \frac{dx}{x\sqrt{1-x^3}} = \frac{1}{3}\log\left(\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right).$$

3 Elementary Functions

Let f be an arbitrary elementary function. In order to generalize the algorithms of the previous sections, we need to build an algebraic model in which f behaves in some sense like a rational or algebraic function. For that purpose, we need to formally define differential fields and elementary functions.

3.1 Differential algebra

A *differential field* $(K, ')$ is a field K with a given map $a \to a'$ from K into K, satisfying $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$. Such a map is called a *derivation* on K. An element $a \in K$ which satisfies $a' = 0$ is called a *constant*, and the set $\text{Const}(K) = \{a \in K \text{ such that } a' = 0\}$ of all the constants of K is a subfield of K.
A differential field $(E, ')$ is a *differential extension of* $(K, ')$ if $K \subseteq E$ and the derivation on E extends the one on K. In that case, an element $t \in E$ is a *monomial* over K if t is transcendental over K and $t' \in K[t]$, which implies that both $K[t]$ and $K(t)$ are closed under $'$. An element $t \in E$ is *elementary* over K, if either

- $t' = b'/b$ for some $b \in K^*$, in which case we say that t is a *logarithm* over K, and write $t = \log(b)$, or
- $t' = b't$ some $b \in K^*$, in which case we say that t is an *exponential* over K, and write $t = e^b$, or
- t is algebraic over K.

A differential extension $(E, ')$ of $(K, ')$ is *elementary* over K, if there exist $t_1, \dots, t_m$ in E such that $E = K(t_1, \dots, t_m)$ and each t_i is elementary over $K(t_1, \dots, t_{i-1})$. We say that $f \in K$ *has an elementary integral* over K if there exists an elementary extension $(F, ')$ of $(K, ')$ and $g \in F$ such that $g' = f$. An *elementary function* of the variable x is an element of an elementary extension of the rational function field $(C(x), d/dx)$, where $C = \text{Const}(C(x))$.
Elementary extensions are useful for modeling any function as a rational or algebraic function of one main variable over the other terms present in the function: given an elementary integrand $f(x)dx$, the integration algorithm first constructs a field C containing all the constants appearing in f, then the rational function field $(C(x), d/dx)$, then an elementary tower $E = C(x)(t_1, \dots, t_k)$ containing f. Note that such a tower is not unique, and in addition, adjoining a logarithm could in fact adjoin only a new constant, and an exponential could in fact be algebraic, for example $\mathbb{Q}(x)(\log(x), \log(2x)) = \mathbb{Q}(\log(2))(x)(\log(x))$ and $\mathbb{Q}(x)(e^{\log(x)/2}) = \mathbb{Q}(x)(\sqrt{x})$. There are however algorithms that detect all such occurences and modify the tower accordingly [16], so we can assume that all the logarithms and exponentials appearing in E are monomials, and that $\text{Const}(E) = C$. Let now k_0 be the largest index such that t_{k_0} is transcendental over $K = C(x)(t_1, \dots, t_{k_0-1})$ and $t = t_{k_0}$. Then E is a finitely generated algebraic extension of $K(t)$, and in the special case $k_0 = k$, $E = K(t)$. Thus, $f \in E$ can be seen as a univariate rational or algebraic function over K, the major difference with the pure rational or algebraic cases being that K is not constant with respect to the derivation. It turns out that the algorithms of the previous sections can be generalized to such towers, new methods being required only for the polynomial (or integral) part. We note that Liouville's Theorem remains valid when E is an arbitrary differential field, so the integration algorithms work by attempting to solve equation (13) as previously.

Example 7 *The function (1) is the element* $f = (t - t^{-1})\sqrt{-1}/2$ *of* $E = K(t)$ *where* $K = \mathbb{Q}(\sqrt{-1})(x)(t_1, t_2)$ *with*

$$t_1 = \sqrt{x^3 - x + 1}, \quad t_2 = e^{2\sqrt{-1}(x^3 - t_1)}, \quad \text{and} \quad t = e^{((1-t_2)/(1+t_2)) - x\sqrt{-1}}$$

which is transcendental over K*. Alternatively, it can also be written as the element* $f = 2\theta/(1+\theta^2)$ *of* $F = K(\theta)$ *where* $K == \mathbb{Q}(x)(\theta_1, \theta_2)$ *with*

$$\theta_1 = \sqrt{x^3 - x + 1}, \quad \theta_2 = \tan(x^3 - \theta_1), \quad \text{and} \quad \theta = \tan\left(\frac{x + \theta_2}{2}\right)$$

which is a transcendental monomial over K*. It turns out that both towers can be used in order to integrate* f*.*

The algorithms of the previous sections relied extensively on squarefree factorization and on the concept of squarefree polynomials. The appropriate analogue in monomial extensions is the notion of *normal* polynomials: let t be a monomial over K, we say that $p \in K[t]$ is *normal* (with respect to $'$) if $\gcd(p, p') = 1$, and that p is *special* if $\gcd(p, p') = p$,*i.e.* $p \mid p'$ in $K[t]$. For $p \in K[t]$ squarefree, let $p_s = \gcd(p, p')$ and $p_n = p/p_s$. Then, $p = p_s p_n$ while p_s is special and p_n is normal. Therefore, squarefree factorization can be used to write any $q \in K[t]$ as a product $q = q_s q_n$, where $\gcd(q_s, q_n) = 1$, q_s is special and all the squarefree factors of q_n are normal. We call q_s the *special part* of q and q_n its *normal part.*

3.2 The Hermite reduction

The Hermite reductions we presented for rational and algebraic functions work in exactly the same way algebraic extensions of monomial extensions of K, as long as we apply them only to the normal part of the denominator of the integrand. Thus, if D is the denominator of the integrand, we let S be the special part of D, $D_1 D_2^2 \cdots D_m^m$ be a squarefree factorization of the *normal* part of D, $V = D_m$, $U = D/V^m$ and the rational and algebraic Hermite reductions proceed normally, eventually yielding an integrand whose denominator has a squarefree normal part.

Example 8 *Consider*

$$\int \frac{x - \tan(x)}{\tan(x)^2} dx\,.$$

The integrand is

$$f = \frac{x - t}{t^2} \in K(t) \quad \text{where } K = \mathbb{Q}(x) \text{ and } t' = t^2 + 1\,.$$

Its denominator is $D = t^2$*, and* $\gcd(t, t') = 1$ *implying that* t *is normal, so* $m = 2$*,* $V = t$*,* $U = D/t^2 = 1$*, and the extended Euclidean algorithm yields*

$$\frac{A}{1-m} = t - x = -x(t^2 + 1) + (xt + 1)t = -xUV' + (xt + 1)V$$

implying that

$$\int \frac{x - \tan(x)}{\tan(x)^2} dx = -\frac{x}{\tan(x)} - \int x dx$$

and the remaining integrand has a squarefree denominator.

Example 9 *Consider*

$$\int \frac{\log(x)^2 + 2x\log(x) + x^2 + (x+1)\sqrt{x + \log(x)}}{x\log(x)^2 + 2x^2\log(x) + x^3} dx\,.$$

The integrand is

$$f = \frac{t^2 + 2xt + x^2 + (x+1)y}{xt^2 + 2x^2t + x^3} \in E = K(t)[y]/(y^2 - x - t)$$

where $K = \mathbb{Q}(x)$ and $t = \log(x)$. The denominator of f with respect to the basis $w = (1, y)$ is $D = xt^2 + 2x^2t + x^3$ whose squarefree factorization is $x(t+x)^2$. Both x and $t + x$ are normal, so $m = 2$, $V = t + x$, $U = D/V^2 = x$, and the solution (12) of (8) is

$$f_1 = \frac{t^2 + 2xt + x^2}{x(-(t'+1))} = -\frac{t^2 + 2xt + x^2}{x+1}, f_2 = \frac{x+1}{x\left((t+x)\frac{1}{2}\frac{t'+1}{t+x} - (t'+1)\right)} = -2\,.$$

We have $Q = 1$, so $0V + 1Q = 1$, $A = 0$, $R = 1$, $RQf_1 = f_1 = -V^2/(x+1)$ and $RQf_2 = f_2 = 0V - 2$, so $B = -2y$ and

$$h = f - \left(\frac{B}{V}\right)' = \frac{1}{x}$$

implying that

$$\int \frac{\log(x)^2 + 2x\log(x) + x^2 + (x+1)\sqrt{x + \log(x)}}{x\log(x)^2 + 2x^2\log(x) + x^3} dx = -\frac{2}{\sqrt{x + \log(x)}} + \int \frac{dx}{x}$$

and the remaining integrand has a squarefree denominator.

3.3 The polynomial reduction

In the transcendental case $E = K(t)$ and when t is a monomial satisfying $\deg_t(t') \geq 2$, then it is possible to reduce the degree of the polynomial part of the integrand until it is smaller than $\deg_t(t')$. In the case when $t = \tan(b)$ for some $b \in K$, then it is possible either to prove that the integral is not elementary, or to reduce the polynomial part of the integrand to be in K. Let $f \in K(t)$ be our integrand and write $f = P + A/D$ where $P, A, D \in K[t]$ and $\deg(A) < \deg(D)$. Write $P = \sum_{i=0}^{e} p_i t^i$ and $t' = \sum_{i=0}^{d} c_i t^i$ where $p_0, \ldots, p_e, c_0, \ldots, c_d \in K$, $d \geq 2$, $p_e \neq 0$ and $c_d \neq 0$. It is easy to verify that if $e \geq d$, then

$$P = \left(\frac{a_e}{(e-d+1)c_d} t^{e-d+1}\right)' + \overline{P} \tag{15}$$

where $\overline{P} \in K[t]$ is such that $\overline{P} = 0$ or $\deg_t(\overline{P}) < e$. Repeating the above transformation we obtain $Q, R \in K[t]$ such that $R = 0$ or $\deg_t(R) < d$ and $P = Q' + R$. Write then $R = \sum_{i=0}^{d-1} r_i t^i$ where $r_0, \ldots, r_{d-1} \in K$. Again, it is easy to verify that for any *special* $S \in K[t]$ with $\deg_t(S) > 0$, we have

$$R = \frac{1}{\deg_t(S)} \frac{r_{d-1}}{c_d} \frac{S'}{S} + \overline{R}$$

where $\overline{R} \in K[t]$ is such that $\overline{R} = 0$ or $\deg_t(\overline{R}) < e - 1$. Furthermore, it can be proven [4] that if $R + A/D$ has an elementary integral over $K(t)$, then r_{d-1}/c_d is a constant, which implies that

$$\int R = \frac{1}{\deg_t(S)} \frac{r_{d-1}}{c_d} \log(S) + \int \left(\overline{R} + \frac{A}{D} \right)$$

so we are left with an integrand whose polynomial part has degree at most $\deg_t(t') - 2$. In the case $t = \tan(b)$ for $b \in K$, then $t' = b't^2 + b'$, so $\overline{R} \in K$.

Example 10 *Consider*

$$\int (1 + x\tan(x) + \tan(x)^2) dx \,.$$

The integrand is

$$f = 1 + xt + t^2 \in K(t) \quad \textit{where } K = \mathbb{Q}(x) \textit{ and } t' = t^2 + 1 \,.$$

Using (15), we get $\overline{P} = f - t' = f - (t^2 + 1) = xt$ *so*

$$\int (1 + x\tan(x) + \tan(x)^2) dx = \tan(x) + \int x \tan(x) dx$$

ans since $x' \neq 0$*, the above criterion implies that the remaining integral is not an elementary function.*

3.4 The residue criterion

Similarly to the Hermite reduction, the Rothstein–Trager and Lazard–Rioboo–Trager algorithms are easy to generalize to the transcendental case $E = K(t)$ for arbitrary monomials t: let $f \in K(t)$ be our integrand and write $f = P + A/D + B/S$ where $P, A, D, B, S \in K[t]$, $\deg(A) < \deg(D)$, S is special and, following the Hermite reduction, D is normal. Let then z be a new indeterminate, $\kappa : K[z] \to K[z]$ be given by $\kappa\left(\sum_i a_i z^i\right) = \sum_i a_i' z^i$,

$$R = \text{resultant}_t(D, A - zD') \quad \in K[z]$$

be the Rothstein–Trager resultant, $R = R_1 R_2^2 \ldots R_k^k$ be its squarefree factorization, $Q_i = \gcd_z(R_i, \kappa(R_i))$ for each i, and

$$g = \sum_{i=1}^{k} \sum_{a | Q_i(a) = 0} a \log(\gcd{}_t(D, A - aD')) \,.$$

Note that the the roots of each Q_i must all be constants, and that the arguments of the logarithms can be obtained directly from the subresultant PRS of D and $A - zD'$ as in the rational function case. It can then be proven [4] that

- $f - g'$ is always "simpler" than f,
- the splitting field of $Q_1 \dots Q_k$ over K is the minimal algebraic extension of K needed in order to express $\int f$ in the form (4),
- if f has an elementary integral over $K(t)$, then $R \mid \kappa(R)$ in $K[z]$ and the denominator of $f - g'$ is special.

Thus, while in the pure rational function case the remaining integrand is a polynomial, in this case the remaining integrand has a special denominator. In that case we have additionally that if its integral is elementary, then (13) has a solution such that $v \in K(t)$ has a special denominator, and each $u_i \in K(c_1, \dots, c_k)[t]$ is special.

Example 11 *Consider*

$$\int \frac{2\log(x)^2 - \log(x) - x^2}{\log(x)^3 - x^2 \log(x)} dx \,.$$

The integrand is

$$f = \frac{2t^2 - t - x^2}{t^3 - x^2 t} \in K(t) \qquad \textit{where } K = \mathbb{Q}(x) \textit{ and } t = \log(x)\,.$$

Its denominator is $D = t^3 - x^2 t$, which is normal, and the resultant is

$$\begin{aligned} R &= \textit{resultant}_t \left((t^3 - x^2 t, \frac{2x - 3z}{x} t^2 + (2xz - 1)t + x(z - x) \right) \\ &= 4x^3(1 - x^2) \left(z^3 - xz^2 - \frac{1}{4} z + \frac{x}{4} \right) \end{aligned}$$

which is squarefree in $K[z]$. We have

$$\kappa(R) = -x^2(4(5x^2 + 3)z^3 + 8x(3x^2 - 2)z^2 + (5x^2 - 3)z - 2x(3x^2 - 2))$$

so

$$Q_1 = \gcd{}_z(R, \kappa R) = x^2 \left(z^2 - \frac{1}{4} \right)$$

and

$$\gcd{}_t \left(t^3 + x^2 t, \frac{2x - 3a}{x} t^2 + (2xa - 1)t + x(a - x) \right) = t + 2ax$$

where $a^2 - 1/4 = 0$, whence

$$g = \sum_{a \mid a^2 - 1/4 = 0} a \log(t + 2ax) = \frac{1}{2} \log(t + x) - \frac{1}{2} \log(t - x)\,.$$

Computing $f - g'$ we find

$$\int \frac{2\log(x)^2 - \log(x) - x^2}{\log(x)^3 - x^2\log(x)}dx = \frac{1}{2}\log\left(\frac{\log(x)+x}{\log(x)-x}\right) + \int \frac{dx}{\log(x)}$$

and since $\deg_z(Q_1) < \deg_z(R)$, it follows that the remaining integral is not an elementary function (it is in fact the logarithmic integral $\mathrm{Li}(x)$).

In the most general case, when $E = K(t)(y)$ is algebraic over $K(t)$ and y is integral over $K[t]$, the criterion part of the above result remains valid: let $w = (w_1, \ldots, w_n)$ be an integral basis for E over $K(t)$ and write the integrand $f \in E$ as $f = \sum_{i=1}^{n} A_i w_i / D + \sum_{i=1}^{n} B_i w_i / S$ where S is special and, following the Hermite reduction, D is normal. Write $\sum_{i=1}^{n} A_i w_i = G/H$, where $G \in K[t,y]$ and $H \in K[t]$, let $F \in K[t,y]$ be the (monic) minimum polynomial for y over $K(t)$, z be a new indeterminate and compute

$$R(z) = \mathrm{resultant}_t(\mathrm{pp}_z(\mathrm{resultant}_y(G - tHD', F)), D) \in K[t]\,. \qquad (16)$$

It can then be proven [2] that if f has an elementary integral over E, then $R \mid \kappa(R)$ in $K[z]$.

Example 12 *Consider*

$$\int \frac{\log(1+e^x)^{(1/3)}}{1+\log(1+e^x)}dx\,. \qquad (17)$$

The integrand is

$$f = f = \frac{y}{t+1} \in E = K(t)[y]/(y^3 - t)$$

where $K = \mathbb{Q}(x)(t_1)$, $t_1 = e^x$ and $t = \log(1+t_1)$. Its denominator with respect to the integral basis $w = (1, y, y^2)$ is $D = t+1$, which is normal, and the resultant is

$$R = \mathit{resultant}_t(\mathit{pp}_z(\mathit{resultant}_y(y - zt_1/(1+t_1), y^3 - t)), t+1) = -\frac{t_1^3}{(1+t_1)^3}z^3 - 1\,.$$

We have

$$\kappa(R) = -\frac{3t_1^3}{(1+t_1)^4}z^3$$

which is coprime with R in $K[z]$, implying that the integral (17) is not an elementary function.

3.5 The transcendental logarithmic case

Suppose now that $t = \log(b)$ for some $b \in K^*$, and that $E = K(t)$. Then, every special polynomial must be in K, so, following the residue criterion, we must look for a solution $v \in K[t]$, $u_1, \ldots, u_k \in K(c_1, \ldots, c_n)^*$ of (13). Furthermore, the integrand f is also in $K[t]$, so write $f = \sum_{i=0}^{d} f_i t^i$ where $f_0, \ldots, f_d \in K$ and $f_d \neq 0$. We must have $\deg_t(v) \leq d+1$, so writing $v = \sum_{i=0}^{d+1} v_i t^i$, we get

$$\int f_d t^d + \ldots f_1 t + f_0 = v_{d+1} t^{d+1} + \ldots + v_1 t + v_0 + \sum_{i=1}^{k} c_i \log(u_i)\,.$$

If $d = 0$, then the above is simply an integration problem for $f_0 \in K$, which can be solved recursively. Otherwise, differentiating both sides and equating the coefficients of t^d, we get ${v_{d+1}}' = 0$ and

$$f_d = {v_d}' + (d+1) v_{d+1} \frac{b'}{b}\,. \tag{18}$$

Since $f_d \in K$, we can recursively apply the integration algorithm to f_d, either proving that (18) has no solution, in which case f has no elementary integral, or obtaining the constant v_{d+1}, and v_d up to an additive constant (in fact, we apply recursively a specialized version of the integration algorithm to equations of the form (18), see [4] for details). Write then $v_d = \overline{v_d} + c_d$ where $\overline{v_d} \in K$ is known and $c_d \in \mathrm{Const}(K)$ is undetermined. Equating the coefficients of t^{d-1} yields

$$f_{d-1} - d\overline{v_d}\frac{b'}{b} = {v_{d-1}}' + d c_d \frac{b'}{b}$$

which is an equation of the form (18), so we again recursively compute c_d and v_{d-1} up to an additive constant. We repeat this process until either one of the recursive integrations fails, in which case f has no elementary integral, or we reduce our integrand to an element of K, which is then integrated recursively. The algorithm of this section can also be applied to real arc-tangent extensions,*i.e.* $K(t)$ where t is a monomial satisfying $t' = b'/(1+b^2)$ for some $b \in K$.

3.6 The transcendental exponential case

Suppose now that $t = e^b$ for some $b \in K$, and that $E = K(t)$. Then, every nonzero special polynomial must be of the form at^m for $a \in K^*$ and $m \in \mathbb{N}$. Since

$$\frac{(at^m)'}{at^m} = \frac{a'}{a} + m\frac{t'}{t} = \frac{a'}{a} + mb'\,,$$

we must then look for a solution $v \in K[t, t^{-1}]$, $u_1, \ldots, u_k \in K(c_1, \ldots, c_n)^*$ of (13). Furthermore, the integrand f is also in $K[t, t^{-1}]$, so write $f = \sum_{i=e}^{d} f_i t^i$ where $f_e, \ldots, f_d \in K$ and $e, d \in \mathbb{Z}$. Since $(at^m)' = (a' + mb')t^m$ for any $m \in \mathbb{Z}$, we must have $v = Mb + \sum_{i=e}^{d} v_i t^i$ for some integer M, hence

$$\int \sum_{i=e}^{d} f_i t^i = Mb + \sum_{i=e}^{d} v_i t^i + \sum_{i=1}^{k} c_i \log(u_i)\,.$$

Differentiating both sides and equating the coefficients of each power of t^d, we get

$$f_0 = (v_0 + Mb)' + \sum_{i=1}^{k} c_i \frac{u_i'}{u_i}\,,$$

which is simply an integration problem for $f_0 \in K$, and

$$f_i = v_i' + ib'v_i \quad \text{for } e \le i \le d, i \ne 0\,.$$

The above problem is called a *Risch differential equation over* K. Although solving it seems more complicated than solving $g' = f$, it is actually simpler than an integration problem because we look for the solutions v_i in K only rather than in an extension of K. Bronstein [2, 3, 4] and Risch [12, 13, 14] describe algorithms for solving this type of equation when K is an elementary extension of the rational function field.

3.7 The transcendental tangent case

Suppose now that $t = \tan(b)$ for some $b \in K$,*i.e.* $t' = b'(1 + t^2)$, that $\sqrt{-1} \notin K$ and that $E = K(t)$. Then, every nonzero special polynomial must be of the form $a(t^2 + 1)^m$ for $a \in K^*$ and $m \in \mathbb{N}$. Since

$$\frac{(a(t^2+1)^m)'}{a(t^2+1)^m} = \frac{a'}{a} + m\frac{(t^2+1)'}{t^2+1} = \frac{a'}{a} + 2mb't$$

we must look for $v = V/(t^2 + 1)^m$ where $V \in K[t]$, $m_1, \ldots, m_k \in \mathbb{N}$, constants $c_1, \ldots, c_k \in \overline{K}$ and $u_1, \ldots, u_k \in K(c_1, \ldots, c_k)^*$ such that

$$f = v' + 2b't\sum_{i=1}^{k} c_i m_i + \sum_{i=1}^{k} c_i \frac{u_i'}{u_i}\,.$$

Furthermore, the integrand $f \in K(t)$ following the residue criterion must be of the form $f = A/(t^2 + 1)^M$ where $A \in K[t]$ and $M \ge 0$. If $M > 0$, it can be shown that $m = M$ and that

$$\begin{pmatrix} c' \\ d' \end{pmatrix} + \begin{pmatrix} 0 & -2mb' \\ 2mb' & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{19}$$

where $at+b$ and $ct+d$ are the remainders modulo t^2+1 of A and V respectively. The above is a coupled differential system, which can be solved by methods similar to the ones used for Risch differential equations [4]. If it has no solution, then the integral is not elementary, otherwise we reduce the integrand to $h \in K[t]$, at which point the polynomial reduction either proves that its integral is not elementary, or reduce the integrand to an element of K, which is integrated recursively.

Example 13 *Consider*

$$\int \frac{\sin(x)}{x} dx\,.$$

The integrand is

$$f = \frac{2t/x}{t^2+1} \in K(t) \qquad \text{where } K = \mathbb{Q}(x) \text{ and } t = \tan\left(\frac{x}{2}\right).$$

Its denominator is $D = t^2 + 1$, which is special, and the system (19) becomes

$$\begin{pmatrix} c' \\ d' \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2/x \\ 0 \end{pmatrix}$$

which has no solution in $\mathbb{Q}(x)$, implying that the integral is not an elementary function.

3.8 The algebraic logarithmic case

The transcendental logarithmic case method also generalizes to the case when $E = K(t)(y)$ is algebraic over $K(t)$, $t = \log(b)$ for $b \in K^*$ and y is integral over $K[t]$: following the residue criterion, we can assume that $R \mid \kappa(R)$ where R is given by (16), hence that all its roots in $\overline{K}$ are constants. The polynomial part of the integrand is replaced by a family of at most $[E : K(t)]$ Puiseux expansions at infinity, each of the form

$$a_{-m}\theta^{-m} + \ldots + a_{-1}\theta^{-1} + \sum_{i\geq 0} a_i\theta^i \tag{20}$$

where $\theta^r = t^{-1}$ for some positive integer r. Applying the integration algorithm recursively to $a_r \in \overline{K}$, we can test whether there exist $\rho \in \mathrm{Const}(\overline{K})$ and $v \in \overline{K}$ such that

$$a_r = v' + \rho\frac{b'}{b}\,.$$

If there are no such v and c for at least one of the series, then the integral is not elementary, otherwise ρ is uniquely determined by a_r, so let $\rho_1, \ldots, \rho_q$ where $q \leq [E : K(t)]$ be the distinct constants we obtain, $\alpha_1, \ldots, \alpha_s \in \overline{K}$ be the distinct nonzero roots of R, and $(q_1, \ldots, q_k)$ be a basis for the vector space generated by the ρ_i's and α_i's over $\mathbb{Q}$. Write $\alpha_i = r_{i1}q_1 + \cdots + r_{ik}q_k$ and $\rho_i = s_{i1}q_1 + \cdots + s_{ik}q_k$ for each i, where $r_{ij}, s_{ij} \in \mathbb{Q}$ and let $m > 0$ be a common denominator for all the r_{ij}'s and s_{ij}'s. For $1 \leq j \leq k$, let

$$\delta_j = \sum_{i=1}^{s} m r_{ij} \sum_l r_l P_l - \sum_{i=1}^{q} m s_{ij} \sum_l s_l Q_l$$

where r_l is the ramification index of P_l, s_l is the ramification index of Q_l, P_l runs over all the finite places at which $h dz$ has residue $r_l\alpha_i$ and Q_l runs over all the infinite places at which $\rho = \rho_i$. As in the pure algebraic case, if there is a

j for which $N\delta_j$ is not principal for any nonzero integer N, then the integral is not elementary, otherwise, let $n_1, \ldots, n_k$ be nonzero integers such that $n_j\delta_j$ is principal for each j, and

$$h = f - \frac{1}{m}\sum_{j=1}^{k} \frac{q_j}{n_j}\frac{u_j'}{u_j}$$

where f is the integrand and $u_j \in E(\alpha_1, \ldots, \alpha_s, \rho_1, \ldots, \rho_q)^*$ is such that $n_j\delta_j = (u_j)$. If the integral of h is elementary, then (13) must have a solution with $v \in \mathcal{O}_{K[x]}$ and $u_1, \ldots, u_k \in \overline{K}$, so we must solve

$$h = \frac{\sum_{i=1}^{n} A_i w_i}{D} = \sum_{i=1}^{n} v_i' w_i + \sum_{i=1}^{n} v_i w_i' + \sum_{i=1}^{k} c_i \frac{u_i'}{u_i} \tag{21}$$

for $v_1, \ldots, v_n \in K[t]$, constants $c_1, \ldots, c_n \in \overline{K}$ and $u_1, \ldots, u_k \in \overline{K}^*$ where $w = (w_1, \ldots, w_n)$ is an integral basis for E over $K(t)$.

If E is a simple radical extension of $K(t)$, and we use the basis (11) and the notation of that section, then $w_1 = 1$ and

$$w_i' = \left(\frac{i-1}{n}\frac{H'}{H} - \frac{D_{i-1}'}{D_{i-1}}\right) w_i \quad \text{for } 1 \leq i \leq n\,. \tag{22}$$

This implies that (21) becomes

$$\frac{A_1}{D} = v_1' + \sum_{i=1}^{k} c_i \frac{u_i'}{u_i} \tag{23}$$

which is simply an integration problem for $A_1/D \in K(t)$, and

$$\frac{A_i}{D} = v_i' + \left(\frac{i-1}{n}\frac{H'}{H} - \frac{D_{i-1}'}{D_{i-1}}\right) v_i \quad \text{for } 1 < i \leq n \tag{24}$$

which are Risch differential equations over $K(t)$.

Example 14 *Consider*

$$\int \frac{(x^2+2x+1)\sqrt{x+\log(x)} + (3x+1)\log(x) + 3x^2 + x}{(x\log(x)+x^2)\sqrt{x+\log(x)} + x^2\log(x) + x^3}\, dx\,.$$

The integrand is

$$f = \frac{((3x+1)t - x^3 + x^2)y - (2x^2 - x - 1)t - 2x^3 + x^2 + x}{xt^2 - (x^3 - 2x^2)t - x^4 + x^3} \in E = K(t)[y]/(F)$$

where $F = y^2 - x - t$, $K = \mathbb{Q}(x)$ *and* $t = \log(x)$. *Its denominator with respect to the integral basis* $w = (1, y)$ *is* $D = xt^2 - (x^3 - 2x^2)t - x^4 + x^3$, *which is normal, and the resultant is*

$$\begin{aligned} R &= \textit{resultant}_t(pp_z(\textit{resultant}_y(((3x+1)t - x^3 + x^2)y \\ &\qquad\qquad -(2x^2 - x - 1)t - 2x^3 + x^2 + x - zD', F)), D) \\ &= x^{12}(2x+1)^2(x+1)^2(x-1)^2 z^3 (z-2) \end{aligned}$$

We have

$$\kappa(R) = \frac{36x^3 + 16x^2 - 28x - 12}{x(2x+1)(x+1)(x-1)} R$$

so $R \mid \kappa(R)$ *in* $K[z]$. *Its only nonzero root is* 2, *and the integrand has residue* 2 *at the place* P *corresponding to the point* $(t, y) = (x^2 - x, -x)$. *There is only one place* Q *at infinity of ramification index* 2, *and the coefficient of* t^{-1} *in the Puiseux expansion of* f *at* Q *is*

$$a_2 = 1 - 2x + \frac{1}{x} = (x - x^2)' + \frac{x'}{x}$$

which implies that the corresponding ρ *is* 1. *Therefore, the divisor for the logand is* $\delta = 2P - 2Q$. *It turns out that* $\delta = (u)$ *where* $u = (x + y)^2 \in E^*$, *so the new integrand is*

$$h = f - \frac{u'}{u} = f - 2\frac{(x+y)'}{x+y} = \frac{(x+1)y}{xt + x^2}\,.$$

We have $y^2 = t + x$, *which is squarefree, so (23) becomes*

$$0 = v_1' + \sum_{i=1}^{k} c_i \frac{u_i'}{u_i}$$

whose solution is $v_1 = k = 0$, *and (24) becomes*

$$\frac{x+1}{xt + x^2} = v_2' + \frac{x+1}{2xt + 2x^2} v_2$$

whose solution is $v_2 = 2$, *implying that* $h = 2y'$, *hence that*

$$\int \frac{(x^2 + x + 1)\sqrt{x + \log(x)} + (3x+1)\log(x) + 3x^2 + x}{(x\log(x) + x^2)\sqrt{x + \log(x)} + x^2\log(x) + x^3}\,dx \quad = \quad 2\sqrt{x + \log(x)} + 2\log(x + \sqrt{x + \log(x)})\,.$$

In the general case when E is not a radical extension of $K(t)$, (21) is solved by bounding $\deg_t(v_i)$ and comparing the Puiseux expansions at infinity of $\sum_{i=1}^{n} v_i w_i$ with those of the form (20) of h, see [2, 12] for details.

3.9 The algebraic exponential case

The transcendental exponential case method also generalizes to the case when $E = K(t)(y)$ is algebraic over $K(t)$, $t = e^b$ for $b \in K$ and y is integral over $K[t]$: following the residue criterion, we can assume that $R \mid \kappa(R)$ where R is given by (16), hence that all its roots in $\overline{K}$ are constants. The denominator of the integrand must be of the form $D = t^m U$ where $\gcd(U, t) = 1$, U is squarefree, and $m \geq 0$.

If $m > 0$, E is a simple radical extension of $K(t)$, and we use the basis (11), then it is possible to reduce the power of t appearing in D by a process similar to the Hermite reduction: writing the integrand $f = \sum_{i=1}^n A_i w_i/(t^m U)$, we ask whether we can compute $b_1, \ldots, b_n \in K$ and $C_1, \ldots, C_n \in K[t]$ such that

$$\int \frac{\sum_{i=1}^n A_i w_i}{t^m U} = \frac{\sum_{i=1}^n b_i w_i}{t^m} + \int \frac{\sum_{i=1}^n C_i w_i}{t^{m-1} U}\,.$$

Differentiating both sides and multiplying through by t^m we get

$$\frac{\sum_{i=1}^n A_i w_i}{U} = \sum_{i=1}^n b_i' w_i + \sum_{i=1}^n b_i w_i' - mb' \sum_{i=1}^n b_i w_i + \frac{t \sum_{i=1}^n C_i w_i}{U}\,.$$

Using (22) and equating the coefficients of w_i on both sides, we get

$$\frac{A_i}{U} = b_i' + (\omega_i - mb')b_i + \frac{tC_i}{U} \quad \text{for } 1 \le i \le n \tag{25}$$

where

$$\omega_i = \frac{i-1}{n}\frac{H'}{H} - \frac{D_{i-1}'}{D_{i-1}} \in K(t)\,.$$

Since $t'/t = b' \in K$, it follows that the denominator of ω_i is not divisible by t in $K[t]$, hence, evaluating (25) at $t = 0$, we get

$$\frac{A_i(0)}{U(0)} = b_i' + (\omega_i(0) - mb')b_i \quad \text{for } 1 \le i \le n \tag{26}$$

which are Risch differential equations over $K(t)$. If any of them has no solution in $K(t)$, then the integral is not elementary, otherwise we repeat this process until the denominator of the integrand is normal. We then perform the change of variable $\bar{t} = t^{-1}$, which is also exponential over K since $\bar{t}' = -b'\bar{t}$, and repeat the above process in order to eliminate the power of $\bar{t}$ from the denominator of the integrand. It can be shown that after this process, any solution of (13) must have $v \in K$.

Example 15 *Consider*

$$\int \frac{3(x+e^x)^{(1/3)} + (2x^2+3x)e^x + 5x^2}{x(x+e^x)^{(1/3)}} dx\,.$$

The integrand is

$$f = \frac{((2x^2+3x)t + 5x^2)y^2 + 3t + 3x}{xt + x^2} \in E = K(t)[y]/(y^3 - t - x)$$

where $K = \mathbb{Q}(x)$ *and* $t = e^x$. *Its denominator with respect to the integral basis* $w = (1, y, y^2)$ *is* $D = xt + x^2$, *which is normal, and the resultant is*

$$\begin{aligned} R \;=\; & \mathit{resultant}_t(\mathit{pp}_z(\mathit{resultant}_y(((2x^2+3x)t + 5x^2)y^2 + 3t + 3x - zD', \\ & y^3 - t - x)), D) = x^8(1-x)^3 z^3\,. \end{aligned}$$

We have

$$\kappa(R) = \frac{11x-8}{x(x-1)} R$$

so $R \mid \kappa(R)$ *in* $K[z]$*, its only root being* 0*. Since* D *is not divisible by* t*, let* $\overline{t} = t^{-1}$ *and* $z = \overline{t}y$*. We have* $\overline{t}' = -\overline{t}$ *and* $z^3 - \overline{t}^2 - x\overline{t}^3 = 0$*, so the integral basis (11) is*

$$\overline{w} = (\overline{w}_1, \overline{w}_2, \overline{w}_3) = \left(1, z, \frac{z^2}{\overline{t}}\right).$$

Writing f *in terms of that basis gives*

$$f = \frac{3x\overline{t}^2 + 3\overline{t} + (5x^2\overline{t} + 2x^2 + 3x)\overline{w}_3}{x^2\overline{t}^2 + x\overline{t}}$$

whose denominator $\overline{D} = \overline{t}(x + x^2\overline{t})$ *is divisible by* $\overline{t}$*. We have* $H = \overline{t}^2(1 + x\overline{t})$*, so* $D_0 = D_1 = 1$ *and* $D_2 = \overline{t}$*, implying that*

$$\omega_1 = 0, \omega_2 = \frac{(1-3x)\overline{t} - 2}{3x\overline{t} + 3}, \text{ and } \omega_3 = \frac{(2-3x)\overline{t} - 1}{3x\overline{t} + 3}.$$

Therefore the equations (26) become

$$0 = b_1' + b_1, 0 = b_2' + \frac{1}{3}b_2, \text{ and } 2x + 3 = b_3' + \frac{2}{3}b_3$$

whose solutions are $b_1 = b_2 = 0$ *and* $b_3 = 3x$*, implying that the new integrand is*

$$h = f - \left(\frac{3x\overline{w}_3}{\overline{t}}\right)' = \frac{3}{x}$$

hence that

$$\int \frac{3(x+e^x)^{(1/3)} + (2x^2+3x)e^x + 5x^2}{x(x+e^x)^{(1/3)}} dx = 3x(x+e^x)^{(2/3)} + 3\int \frac{dx}{x}.$$

In the general case when E is not a radical extension of $K(t)$, following the Hermite reduction, any solution of (13) must have $v = \sum_{i=1}^n v_i w_i / t^m$ where $v_1, \ldots, v_m \in K[t]$. We can compute v by bounding $\deg_t(v_i)$ and comparing the Puiseux expansions at $t = 0$ and at infinity of $\sum_{i=1}^n v_i w_i / t^m$ with those of the form (20) of the integrand, see [2, 12] for details.

Once we are reduced to solving (13) for $v \in K$, constants $c_1, \ldots, c_k \in \overline{K}$ and $u_1, \ldots, u_k \in E(c_1, \ldots, c_k)^*$, constants $\rho_1, \ldots, \rho_s \in \overline{K}$ can be determined at all the places above $t = 0$ and at infinity in a manner similar to the algebraic logarithmic case, at which point the algorithm proceeds by constructing the divisors δ_j and the u_j's as in that case. Again, the details are quite technical and can be found in [2, 12, 13].

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