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An Efficient Representation for Computing Geodesics Between n-Dimensional Elastic Shapes

Shantanu H. Joshi
Department of Electrical Engineering,
Florida State University, Tallahassee, FL
joshi@eng.fsu.edu

Eric Klassen
Department of Mathematics,
Florida State University, Tallahassee, FL
klassen@math.fsu.edu

Anuj Srivastava
Department of Statistics,
Florida State University, Tallahassee, FL
anuj@stat.fsu.edu

Ian Jermyn
ARIANA Group, INRIA,
Sophia Antipolis, France
ian.jermyn@sophia.inria.fr

Abstract

We propose an efficient representation for studying shapes of closed curves in $\mathbb{R}^n$. This paper combines the strengths of two important ideas - elastic shape metric and path-straightening methods - and results in a very fast algorithm for finding geodesics in shape spaces. The elastic metric allows for optimal matching of features between the two curves while path-straightening ensures that the algorithm results in geodesic paths. For the novel representation proposed here, the elastic metric becomes the simple $\mathbb{L}^2$ metric, in contrast to the past usage where more complex forms were used. We present the step-by-step algorithms for computing geodesics and demonstrate them with 2-D as well as 3-D examples.

1. Introduction

Over the last few years, a number of mathematical representations and metrics have been proposed to analyze shapes of planar, closed curves. Despite the multitudes of metrics proposed, there is an emerging consensus on the suitability of the elastic metric for curve-shape analysis. This metric uses a combination of bending and stretching/compression to find optimal deformations from one shape to another. On pre-defined spaces of shapes, these deformations are computed as the shortest paths, or geodesics, under this chosen metric. This metric was first suggested by Younes [9] and subsequently utilized by Mio et al. [5] who developed an algorithm to compute geodesic paths between arbitrary shapes. Several other authors, including Michor and Mumford [4], have also highlighted the advantages of this H^1 metric. Shah [8] has also proposed a similar H^2 metric and computed geodesics in the space of planar curves.

In view of the past ideas on representations and metrics, is there a need or scope for yet another representation, or a new shape analysis method in this area? We argue that the computational evaluations of different approaches are yet to be performed. The elastic metric is widely accepted for shape analysis as it is the only metric that remains invariant to arbitrary re-parameterizations of curves. It is also known in the computer vision community as the Fisher-Rao metric. We consider the question: Under what representation of curves is this metric most efficient computationally. Different authors have used different representations for parameterized curves: normal vector fields, coordinate functions, angle functions, curvature functions, speed functions, etc. For example, Mio et al. have used a pair of functions: an angle function and a speed function, to represent closed curves. For a parameterized curve β in $\mathbb{R}^2$, one represents the velocity vector $\dot{\beta}(s)$ as a complex scalar $r(s)e^{i\theta(s)}$. Here $r(s)$ is the instantaneous speed and $\theta(s)$ is the angle made by $\dot{\beta}(s)$ with the positive X axis. Mio et al. used the pair (ϕ, θ), with $\phi = \log(r)$, to represent and analyze shape of α. In their case, the Riemannian metric that translates into elastic deformations of shapes is then given by:

$$\langle (h_1, g_1), (h_2, g_2) \rangle_{(\phi,\theta)} = \int h_1(s)h_2(s)e^{\phi(s)}ds + \int g_1(s)g_2(s)e^{\phi(s)}ds \ . \quad (1)$$

Similarly, other researchers have used r directly, or its integral form $\int r(s)ds$ as representatives of speeds of curves. This gives rise to various difficulties. Firstly, the elastic metrics under these representations, owing to speed-invariance, assumes complicated forms. Secondly, they may not be computationally efficient. As an example, the elastic metric under the log-speed (ϕ, θ) representation (Eqn. 1) varies from point to point on the shape manifold, and is thereby complicated to implement.

We propose a new representation using the square-root speed ($\sqrt{r(s)}$) of the curve, that addresses the above issues and provides the following advantages:

- It uses a single function, $\sqrt{r(s)}e^{i\theta(s)}$, instead of a pair, to represent the curve.
- It is the only representation, where the elastic metric reduces to a simple $\mathbb{L}^2$ metric. Not only is the metric same at all points, but also much simpler to implement and study. With this representation, the pre-shape space is actually a subset of a unit sphere inside a Hilbert space. The use of geometry of the sphere helps simplify computations to a large extent.
- This paper combines the strengths of the elastic metric and the path straightening method for finding geodesics.
- Furthermore, there are convenient, isometric mappings from the proposed representation to other forms used previously.
- The new representation makes re-parameterization of curves by diffeomorphisms an action by isometries.
- This approach is general and works for curves in n dimensions. We are not aware of any past paper on elastic shape analysis for curves in three or higher dimensions.

This paper is organized as follows. Section 2 introduces the proposed representation of curves for shape analysis. Section 3 defines the pre-shape space of open as well as closed curves in $\mathbb{R}^n$. A Riemannian structure is imposed on this pre-shape space in Sec. 4 followed by the computation of geodesics in Sec. 5. We also provide step-by-step procedures for implementations of the ideas presented in the paper.

2. Curve Representation

For an unit interval $I \equiv [0, 2\pi]$, let $\beta : I \to \mathbb{R}^n$ be an $\mathbb{L}_1^2(I)$ curve. Any function f is said to be an $\mathbb{L}_1^2(I)$ function, if both f and its derivative f' are $\mathbb{L}^2(I)$ functions. In order for the curve to stretch, shrink and bend freely, we represent the shape of the elastic curve β by the function $q : I \to \mathbb{R}^n$ as follows,

$$q(s) = \frac{\dot{\beta}(s)}{\sqrt{||\dot{\beta}(s)||}} \in \mathbb{R}^n \ . \qquad (2)$$

Here, $s \in I$, $||\cdot|| \equiv \sqrt{(\cdot,\cdot)_{\mathbb{R}^n}}$, and $(\cdot,\cdot)_{\mathbb{R}^n}$ is taken to be the standard Euclidean inner product in $\mathbb{R}^n$. The quantity $||q(s)||$ represents the square-root of the instantaneous speed of the curve β, whereas the ratio $\frac{q(s)}{||q(s)||}$ is the direction function for each $s \in [0, 2\pi)$ along the curve. Thus the curve β can be recovered using $\beta(s) = \int_0^s ||q(t)||\ q(t)\ dt$.

3. Pre-Shape Space of Curves

Let $\mathcal{Q} \equiv \{q = (q_1, q_2, \ldots, q_n) | q(s) : I \to \mathbb{R}^n\}$ be the space of all vector valued functions representing all elastic curves described above. This is an infinite-dimensional vector space of all functions in $\mathbb{L}^2(\mathbb{R}^n)$. Each element of this set represents a simple elastic curve (not necessarily closed) on $\mathbb{R}^n$. Similar to the idea by Kendall [2], we would like to study shapes of curves as equivalences under rigid motions, uniform scaling and other such "shape-preserving" transformations. In the following subsections, we identify spaces of such curves where one or more indeterminacies in the shape representation are removed. We refer to such spaces as pre-shape spaces of elastic curves in $\mathbb{R}^n$.

3.1. Open curves

We denote $\mathcal{B} \equiv \{q : I \to \mathbb{R}^n | \int_0^{2\pi} (q(s), q(s))_{\mathbb{R}^n} ds = 1\}$ as the space of all unit-length, elastic curves. The space $\mathcal{B}$ is in fact an infinite-dimensional unit-sphere and represents the pre-shape space of all open elastic curves invariant to translation and uniform scaling. The tangent space of $\mathcal{B}$ is easy to define and is given as $T_q(\mathcal{B}) = \{w = (w_1, w_2, \ldots, w_n) | w(s) : I \to \mathbb{R}^n\ \forall s \in [0, 2\pi) \mid \int_0^{2\pi} (w(s), q(s))_{\mathbb{R}^n}\ ds = 0\}$.

Geodesics on a sphere are great circles and can be specified analytically. The geodesic on $\mathcal{B}$ between the two points $x_1, x_2 \in \mathcal{B}$ along a unit direction $f \in T_{x_1}(\mathcal{B})$ towards x_2 for time t is given as,

$$\chi_t(x_1; f) = \cos\left(t \cos^{-1} \int_0^{2\pi} (x_1, x_2)_{\mathbb{R}^n}\ ds\right) x_1 + \sin\left(t \cos^{-1} \int_0^{2\pi} (x_1, x_2)_{\mathbb{R}^n}\ ds\right) f \qquad (3)$$

Any tangent vector transported along this geodesic preserves its length as well as its angle w.r.t the geodesic. For any two points x_1 and x_2 on this unit sphere, the map $\pi : T_{x_1}(\mathcal{B}) \to T_{x_2}(\mathcal{B})$ parallel-transports a tangent vector a

from x_1 to x_2 and is given by,

$$\pi(a; x_1, x_2) = a - 2\frac{(x_1 + x_2)\int_0^{2\pi}(a, x_2)_{\mathbb{R}^n}\,ds}{\int_0^{2\pi}(x_1 + x_2, x_1 + x_2)_{\mathbb{R}^n}ds} \tag{4}$$

3.2. Closed curves ($\mathcal{C}$)

Although, matching of open curves has important applications involving 2-D or 3-D anatomical or biological curves, it is a relatively easier problem than comparing closed curves. This is due to the fixed extremities, and the fact that geodesics between open curves are straight lines in the appropriate space. In contrast, the choice of origin is completely arbitrary in the case of closed curves. Additionally, the deformation between any two closed curves is expected to yield closed intermediate curves along the geodesic. The closure condition for a curve β requires that $\int_0^{2\pi}\dot{\beta}(t)dt = 0$. For our shape representation scheme, this translates to $\int_0^{2\pi}||q(s)||q(s)\,ds = 0$. We define a mapping $\mathcal{G} \equiv (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n)$ as $\mathcal{G}_1 = \int_0^{2\pi} q_1(s)\,||q(s)||ds$, $\mathcal{G}_2 = \int_0^{2\pi} q_2(s)\,||q(s)||ds$, ..., $\mathcal{G}_n = \int_0^{2\pi} q_n(s)\,||q(s)||ds$. The space obtained by the inverse image $\mathcal{A} = \mathcal{G}^{-1}(\underbrace{0, 0, \ldots 0}_{n})$ is the space of all closed, elastic (arbitrary speed parameterizations) curves. Then the subset $\mathcal{C} = \mathcal{A} \cap \mathcal{B} \subset \mathcal{Q}$ is the space of all closed, elastic curves, invariant to translation and scaling. For the remainder of this paper, we concentrate on the pre-shape space of closed curves ($\mathcal{C}$) and study its properties under an appropriate Riemannian metric on this space.

4. Riemannian Geometry of $\mathcal{C}$

The length of a geodesic or the "shortest path" between two points on a manifold depends on the Riemannian metric associated with an inner product defined on the tangent space of the manifold. Thus we would like to construct a tangent space $T_q(\mathcal{C})$ at each point q . We observe that the tangent space of $\mathcal{Q}$ is the space $\mathcal{Q}$ itself. Any tangent vector w of $\mathcal{Q}$, where $w = (w_1, w_2, \ldots, w_n)|$ $w(s) : I \to \mathbb{R}^n\ \forall s \in [0, 2\pi)$ of $\mathcal{Q}$ has the property that $||w(s)|| \in \mathbb{L}^2, \forall s$. We define an inner-product on $\mathcal{Q}$ as follows.

Definition 1. Given a curve $q \in \mathcal{Q}$, and the first order perturbations of q given by $u, v \in T_q(\mathcal{Q})$, respectively, the inner product between the tangent vectors u, v to $\mathcal{Q}$ at q is defined as,

$$\langle u, v\rangle = \int_0^{2\pi}(u(s), v(s))_{\mathbb{R}^n}ds. \tag{5}$$

The inner product given in Eqn. 5 induces a Riemannian metric on $\mathcal{C}$.

4.1. Tangent Space of $\mathcal{C}$

In order to define the space of tangent vectors to $\mathcal{C}$, we derive a notion of the normal space of $\mathcal{C}$ at q at first. The directional derivative of the map $\mathcal{G}$ at a point q in the direction of $w \in T_q(\mathcal{Q})$ is given by

$$\begin{aligned}
d\mathcal{G}_1(w(s)) &= \int_0^{2\pi}\left(w(s), \frac{q_1(s)}{||q(s)||}q(s) + ||q(s)||\mathbf{e}^1\right)_{\mathbb{R}^n} ds,\\
&\vdots\\
d\mathcal{G}_n(w(s)) &= \int_0^{2\pi}\left(w(s), \frac{q_n(s)}{||q(s)||}q(s) + ||q(s)||\mathbf{e}^n\right)_{\mathbb{R}^n} ds\\
\therefore\quad d\mathcal{G}_1(w(s)) &= \left\langle w, \frac{q_1(s)}{||q(s)||}q(s) + ||q(s)||\mathbf{e}^1\right\rangle,\\
&\vdots\\
d\mathcal{G}_n(w(s)) &= \left\langle w, \frac{q_n(s)}{||q(s)||}q(s) + ||q(s)||\mathbf{e}^n\right\rangle
\end{aligned}$$

where $\mathbf{e}^i$ is the i^{th} column of I_n, an identity matrix. The normal space of $\mathcal{A}$ is now the span of the gradient vectors of $\mathcal{G}$ as follows,

$$\begin{aligned}
N_q(\mathcal{A}) = \text{span}\,\{\nabla\mathcal{G}_1(s) = \frac{q_1(s)}{||q(s)||}q(s) + ||q(s)||\mathbf{e}^1,\\
\ldots, \nabla\mathcal{G}_n(s) = \frac{q_n(s)}{||q(s)||}q(s) + ||q(s)||\mathbf{e}^n\}, \forall s \in [0, 2\pi)
\end{aligned} \tag{6}$$

Remark 1. Given a curve $q \in \mathcal{Q}$, and the tangent vector w to $\mathcal{Q}$ at q, the tangent space of $\mathcal{C}$ at q is defined as $T_q(\mathcal{C}) = \{w : I \to \mathbb{R}^n | w \in T_q(\mathcal{B}), w \perp N_q(\mathcal{A})\}$.

The inner product given by Def. 1 is a symmetric, bilinear positive-definite form on $T_q(\mathcal{C})$ and results in $\mathcal{C}$ being a Riemannian manifold. A useful tool in constructing geodesics under this Riemannian metric is the projection of a curve $q \in \mathcal{Q}$ in the space of closed curves $\mathcal{C}$. This is achieved by projecting the curve q to $\mathcal{A}$ by an iterative method and further projecting it to $\mathcal{C}$. The idea is to define a residual vector $l(q) = -\mathcal{G}(q), l \in \mathbb{R}^n$ and evolve q in the direction normal to the level set of $\mathcal{G}$ so as to move the residual l quickly to the origin **0**. Algorithm 1 describes the procedure to project an open curve $q \in \mathcal{Q}$, to $\mathcal{C}$.

Figure 1 shows examples of projecting 2-D and 3-D open curves $q \in \mathcal{Q}$ onto $\mathcal{C}$ using Algorithm 1.

Another important tool in constructing geodesic paths is the projection of a tangent vector $w \in T_q(\mathcal{Q})$ into $T_q(\mathcal{C})$. Algorithm 2 outlines the procedure.

5. Geodesics using Path straightening Flows

There have been two prominent approaches for computing geodesic paths between shapes of closed curves. One

Algorithm 1 Projection of $q \in \mathcal{Q}$ to $\mathcal{C}$

Initialize $l(q)_i = \mathbf{1}_n$. Let $\epsilon > 0$.
while $||l(q)|| > \epsilon$ **do**
  Compute $l(q)_i = -\mathcal{G}_i(q), \;\; i = 1, \ldots, n$.
  Calculate the Jacobian matrix, $J_{i,j} = \langle \nabla \mathcal{G}_i(q), \nabla \mathcal{G}_j(q) \rangle$ as follows,

$$J_{ij} = 3 \int_0^{2\pi} q_i(s) q_j(s) ds, \;\; i = 1, \ldots, n$$

  .
  Solve the equation $J(q)x^T = l^T(q)$ for x.
  Update $q = q + \sum_{i=1}^{n} x_i \nabla \mathcal{G}_i(q)\delta, \;\; \delta > 0$.
  $q = \frac{q}{\sqrt{\langle q,q \rangle_q}}$
end while

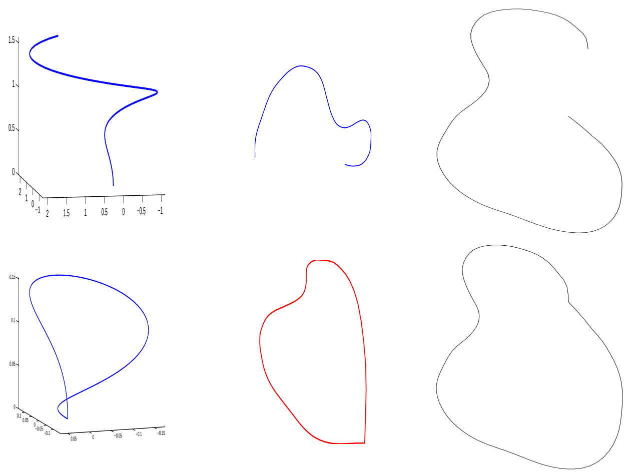

Figure 1. Algorithm 1 applied to project open curves (top row) on $\mathcal{C}$ (bottom row).

Algorithm 2 Projection of $w \in T_q(\mathcal{Q})$ into $T_q(\mathcal{C})$

Start by projecting w into $T_q(\mathcal{B})$ by

$$\tilde{w} \equiv w - \langle w, q \rangle \, q \tag{7}$$

Compute an orthonormal basis $\left\{ \mathbf{e}^i_{\mathcal{G}(q)} \right\}$ for $\left\{ \nabla \mathcal{G}_i(s) \right\}, \;\; i = 1, \ldots, n.$ w.r.t. the inner product given in Eqn. 5.
Then the projection of $\tilde{w}$ into $T_q(\mathcal{C})$ is given as,

$$w_{proj} \equiv \tilde{w} - \sum_{i=1}^{n} \left\langle \tilde{w}, \mathbf{e}^i_{\mathcal{G}(q)} \right\rangle \mathbf{e}^i_{\mathcal{G}(q)}. \tag{8}$$

approach uses the shooting method as demonstrated by [3, 5]. Here, given a pair of shapes, one finds a tangent direction, such that its image under the exponential map at the first shape yields the target shape. The search for the tangent direction uses a shooting method which successively refines the tangent vector estimate based upon some external criteria.

We will use another, more stable approach that uses path-straightening flows to find a geodesic between two shapes. In this approach, the given pair of shapes is connected by an initial arbitrary path that is iteratively "straightened" so as to minimize its length. The curve-straightening method is supposed to overcome some of the limitations in the earlier approach as suggested by [7] who compare geodesics between the two approaches.

Given two curves q_0 and q_1, our goal is to find a geodesic between them. Let $\alpha : [0,1] \rightarrow \mathcal{C}$ be any path connecting $q_0, q_1 \in \mathcal{C}$. Then the critical points of the energy

$$E[\alpha] = \frac{1}{2} \int_0^1 \langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle \;\, dt \tag{9}$$

are geodesics in $\mathcal{C}$. In order to minimize the integral in Eqn. 9, we need to find the gradient of the energy $E[\alpha]$ in the space of all paths on $\mathcal{C}$. For this purpose, we define $\mathcal{F}$ as the collection of all paths in $\mathcal{C}$, and $\mathcal{F}_0 \subset \mathcal{F}$ as the collection of all paths going from q_0 to q_1. Since each element along the path α is a curve in $\mathcal{C}$, the tangent space $T_\alpha(\mathcal{F})$ is written as $T_\alpha(\mathcal{F}) = \{w | w(t) \in T_{\alpha(t)}(\mathcal{C}) \; \forall t \in [0,1]\}$. We adopt the Palais metric [6] on $T_\alpha(\mathcal{F})$ to impose a Riemannian structure on the space of all paths $\mathcal{F}$. For $u_1, u_2 \in T_\alpha(\mathcal{F})$, the Palais metric is given by the inner product,

$$\langle\langle u_1, u_2 \rangle\rangle = \langle u_1(0), u_2(0) \rangle + \int_0^1 \left\langle \frac{Du_1}{dt}(t), \frac{Du_2}{dt}(t) \right\rangle dt \tag{10}$$

The gradient of $E[\alpha]$ is a vector field in the tangent space of $\mathcal{F}_0$, where $T_\alpha(\mathcal{F}_0) = \{w \in T_\alpha(\mathcal{F}) | w(0) = w(1) = 0\}$. Here $w(t)$ is a tangent vector field on the curve $\alpha(t) \in \mathcal{C}$. Before deriving the energy minimization framework in the space $\mathcal{F}$, we review some definitions below.

Definition 2. Covariant derivative [1]: For a path $\alpha \in \mathcal{C}$, the covariant derivative of a vector field $w \in T_\alpha(\mathcal{F})$ is defined as the orthogonal projection of the derivative $\frac{dw}{dt}$ on the tangent space $T_{\alpha(t)}(\mathcal{C})$ for all t and is denoted as $\frac{Dw}{dt}$.

Similarly the *covariant integral* of w along α is given by the vector field $u \in T_\alpha \mathcal{F}$ such that $\frac{Du}{dt} = w$. Algorithm 3 describes the procedure for computing the covariant integration of the velocity vector field $\frac{d\alpha}{dt}$ along α. To derive the gradient vector field of $E[\alpha]$ on $T_\alpha(\mathcal{F})$, we state the following theorem without proof.

Theorem 1. *The gradient vector field of E in $T_\alpha(\mathcal{F})$ is given by v such that $\frac{Dv}{dt} = \dot{\alpha}$, and $v(0) = 0$.*

Theorem 1 implies that the gradient of E in $T_\alpha(\mathcal{F})$ is given by covariant integration of the velocity vector field along the curve α. For this purpose, we need to compute

Algorithm 3 Covariant integration of $\frac{d\alpha}{dt}$

Let $w(0) = 0$.
for $\tau = 1$ to k **do**
  $w(\frac{\tau}{k}) = \Pi\left(w(\frac{\tau-1}{k}); \alpha(\frac{\tau-1}{k}), \alpha(\frac{\tau}{k})\right) + \frac{1}{k}\frac{d\alpha}{dt}(\frac{\tau}{k})$
  Project $w(\frac{\tau}{k})$ into $T_{\alpha(\frac{\tau}{k})}(\mathcal{C})$ using Algorithm 2.
end for

the path velocity $\frac{d\alpha}{dt}$. Since we are dealing with discretized curves in computer implementations, we will compute an approximation to the velocity vector field for discrete intervals along the path by computing the derivative of $\alpha(\tau)$ on the sphere $\mathcal{B}$, and projecting it on $\mathcal{C}$. Algorithm 4 outlines the procedure to compute $\frac{d\alpha}{dt}$.

Algorithm 4 Velocity vector field $\frac{d\alpha}{dt}$ for a path α

Let $\frac{d\alpha}{dt}(0) = \mathbf{0}$.
for $\tau = 1$ to k **do**
  $\theta = \cos^{-1}\left\langle \alpha\left(\frac{\tau-1}{k}\right), \alpha\left(\frac{\tau}{k}\right)\right\rangle$
  $f = -\alpha\left(\frac{\tau-1}{k}\right) + \alpha\left(\frac{\tau}{k}\right)\cos(\theta)$
  $\frac{d\alpha}{d\tau}\left(\frac{\tau}{k}\right) = \frac{kf\theta}{\sqrt{\langle f, f\rangle}}$
  Project $\frac{d\alpha}{d\tau}\left(\frac{\tau}{k}\right)$ into $T_{\alpha(\frac{\tau}{k})}(\mathcal{C})$ using Algorithm 2
end for

Definition 3. Parallel Transport: Let $w_0 \in T_{\alpha(0)}(\mathcal{C})$ be a vector field along a curve $\alpha : [0, 1] \to \mathcal{C}$. Then there exists a unique parallel vector field $w(t)$ such that $\frac{Dw(t)}{dt} = 0$ and $w(0) = w_0$. Furthermore $w(t_1)$ ($\tilde{w}(t_1) = w(1 - t_1)$) is the forward (backward) parallel transport of w_0 along α at t_1.

Algorithm 5 outlines the procedure for the parallel transport of a tangent vector field $w \in T_{\alpha(\tau)}\mathcal{F}$ to $w^{\|} \in T_{\alpha(\tau+1)}\mathcal{F}$. It is noted that the same algorithm can perform forward as well as a backward parallel transport.

Algorithm 5 Parallel transport of tangent vector field w from $\alpha(\frac{\tau}{k})$ to $\alpha(\frac{\tau+1}{k})$ denoted as $w^{\|} = \Pi(w; \alpha(\frac{\tau}{k}), \alpha(\frac{\tau+1}{k}))$

Let $l_w = \langle w, w\rangle$.
$w^{\|} = \pi(w; \alpha(\frac{\tau-1}{k}), \alpha(\frac{\tau}{k}))$
Project $w^{\|}$ into $T_{(\alpha(\frac{\tau}{k})}(\mathcal{C})$ using Algorithm 2 and call it $w^{\|}_{proj}$.
Rescale the length as $w^{\|}_{proj} = \frac{l_w w^{\|}_{proj}}{\langle w^{\|}_{proj}, w^{\|}_{proj}\rangle}$

Definition 4. Geodesic: A path $\alpha : [0, 1] \to \mathcal{F}$ is a geodesic if the covariant derivative of it's velocity vector field is identically zero at all $t \in [0, 1]$, i.e. $\frac{D}{dt}(\frac{d\alpha}{dt}) = 0, \forall t \in [0, 1]$.

Lemma 1. The orthogonal complement of the tangent space $T_\alpha(\mathcal{F}_0)$ is given by $T^{\perp}_\alpha(\mathcal{F}_0) \equiv \{w \in T_\alpha(\mathcal{F}) \mid \frac{D}{dt}\left(\frac{Dw}{dt}\right) = 0\}$.

Proof. Let $w \in T_\alpha(\mathcal{F})$ be a vector field such that $\frac{D}{dt}\left(\frac{Dw}{dt}\right) = 0$. In this case $w(t)$ is a covariantly linear vector field. Let $u \in T_\alpha(\mathcal{F}_0)$ be an arbitrary vector field. Then

$$\begin{aligned}\langle\langle u, w\rangle\rangle_\alpha &= \int_0^1 \left\langle \frac{Du}{dt}, \frac{Dw}{dt}\right\rangle dt \\ &= \left\langle u(t), \frac{Dw}{dt}\right\rangle|_0^1 - \int_0^1 \left\langle u(t), \frac{D}{dt}\left(\frac{Dw}{dt}\right)\right\rangle = 0\end{aligned}$$

□

Using Lemma 1, a tangent vector field $v \in T_\alpha(\mathcal{F})$ can be projected onto $T_\alpha(\mathcal{F})$ by subtracting a covariantly linear vector field given by $t\tilde{v}(t)$, where $\tilde{v}(t)$ is a backward parallel transport of the vector field $v(1)$ along α. Algorithm 6 describes the procedure for backward parallel transport of the gradient vector field $w(1)$ along $\tilde{\alpha} = \alpha(1 - \tau)$. The verification that $t\tilde{v}(t)$ is a covariantly linear vector field is straightforward. Algorithm 7 describes the procedure for projecting the gradient vector field $w \in T_\alpha(\mathcal{F})$ to $v \in T_\alpha(\mathcal{F}_0)$. After obtaining the gradient of the energy $E[\alpha]$ in $\mathcal{F}_0$, we can update the path α in the direction of the gradient field v. Algorithm 8 describes the simple procedure for updating the path α.

Algorithm 6 Backward parallel transport of $w(1)$ along $\tilde{\alpha} = \alpha(1 - \tau)$

Let $\tilde{w}(1) = w(1)$
for $\tau = k - 1$ to 0 **do**
  $\tilde{w}(\frac{\tau}{k}) = \Pi\left(\tilde{w}(\frac{\tau+1}{k}); \alpha(\frac{\tau+1}{k}), \alpha(\frac{\tau}{k})\right)$
end for

Algorithm 7 Project the gradient vector field $w \in T_\alpha(\mathcal{F})$ to $v \in T_\alpha(\mathcal{F}_0)$

for $\tau = 0$ to k **do**
  $v(\frac{\tau}{k}) = w(\frac{\tau}{k}) - \frac{\tau}{k}\tilde{w}(\frac{\tau}{k})$
end for

Algorithm 8 Gradient update for α in the direction v

for $\tau = 0$ to k **do**
  $\alpha(\frac{\tau}{k}) = \chi_1\left(\alpha(\frac{\tau}{k}); -v(\frac{\tau}{k})\right)$
  Project $\alpha(\frac{\tau}{k})$ into $\mathcal{C}$ using Algorithm 1
end for

5.1. Computing geodesics between q_0 and q_1 on $\mathcal{C}$

In this subsection, we combine all the algorithms described above and use them to compute geodesics in the pre-shape space $\mathcal{C}$. In practice, we deal with discretized versions of the curves and tangent spaces. The first step is the initialization of a path α on $\mathcal{C}$ and is described in Algorithm 9.

Algorithm 9 Initialization of a path α on $\mathcal{C}$ between $q_0, q_1 \in \mathcal{C}$.

Let $\alpha(0) = q_0$. Let k be the number of steps along the discretized path.
$f = q_1 - \langle q_1, q_0\rangle q_0$, $f = \frac{f}{\langle f,f\rangle}$
for all $\tau = 1$ to k **do**
  $\alpha(\frac{\tau}{k}) = \chi_{\frac{\tau}{k}}(q_0, f)$
  Project $\alpha(\frac{\tau}{k})$ into $\mathcal{C}$ using Algorithm 1.
end for

Using the initialized path α, Algorithm 10 summarizes various steps using the path-straightening approach in computing the geodesic. The geodesic distance between the two curves is then given by $\int_0^1 \sqrt{\left\langle \dot{\hat{\alpha}}(t), \dot{\hat{\alpha}}(t)\right\rangle}\, dt$, where $\hat{\alpha}$ is the resulting geodesic path.

Algorithm 10 Given $q_0, q_1 \in \mathcal{C}$, compute a geodesic between them

Initialize a path α between q_0 and q_1 using Algorithm 9.
repeat
  Compute the path velocity $\alpha_t \equiv \frac{d\alpha}{dt}$ along α using Algorithm 4.
  Calculate the covariant integral (w) of α_t using Algorithm 3.
  Parallel translate (backward) $w(1)$ along α as $\tilde{w}$ using Algorithm 6.
  Compute the gradient of the energy E and project it to $\mathcal{F}_0$ as v using Algorithm 7.
  Update the path α in the direction v using Algorithm 8.
  Compute path energy $E = \frac{1}{2k}\sum_0^k \langle\alpha_t(\tau), \alpha_t(\tau)\rangle$.
until $||\nabla E|| > \epsilon$

6. Experimental Results and Future Directions

Lastly, we present some experimental results for computing elastic geodesics by implementing the above algorithms in MATLAB®. Figure 2 shows pairwise geodesics between 2-D curves in $\mathcal{C}$. Intermediate shapes along the geodesics have tick-marks placed around the curve, that help identify parts of the curve traversed by non-uniform speed. Figure 3 shows two different views of a geodesic path computed between a pair of 3-D curves. It is emphasized that the intermediate curves along the geodesic do not cross each other.

In the previous sections, we have constructed geodesics in the pre-shape space of translation and scale invariant curves. In fact, the shape of a curve also remains unchanged by rigid rotation. Further, if we are dealing with closed curves, the shape also remains same every time a different starting point is chosen along the curve. Since we allow the curves to stretch, shrink and bend freely, its shape remains invariant to the speed of traversal along the curve. Then we can define the elastic shape space as the quotient space $\mathcal{S} = \mathcal{C}/(\mathbb{S}^1 \times SO(n) \times \mathcal{D})$. The problem of finding geodesics between two shapes in $\mathcal{S}$ can now be modified as finding the shortest path among all possible paths between the equivalence classes of the given pair of shapes. This is a consideration for future work.

7. Summary

We have presented a differential geometric approach for studying shapes of elastic curves in $\mathbb{R}^n$. The novelty in our approach is the representation of elastic curves by a single vector valued function that incorporates both stretching and bending along the curve. The Riemannian metric is a simple $\mathbb{L}^2$ metric that remains same at all points in the space. Geodesics between curves are obtained using a path-straightening approach. We have also provided detailed algorithms for computing these geodesics, along with examples.

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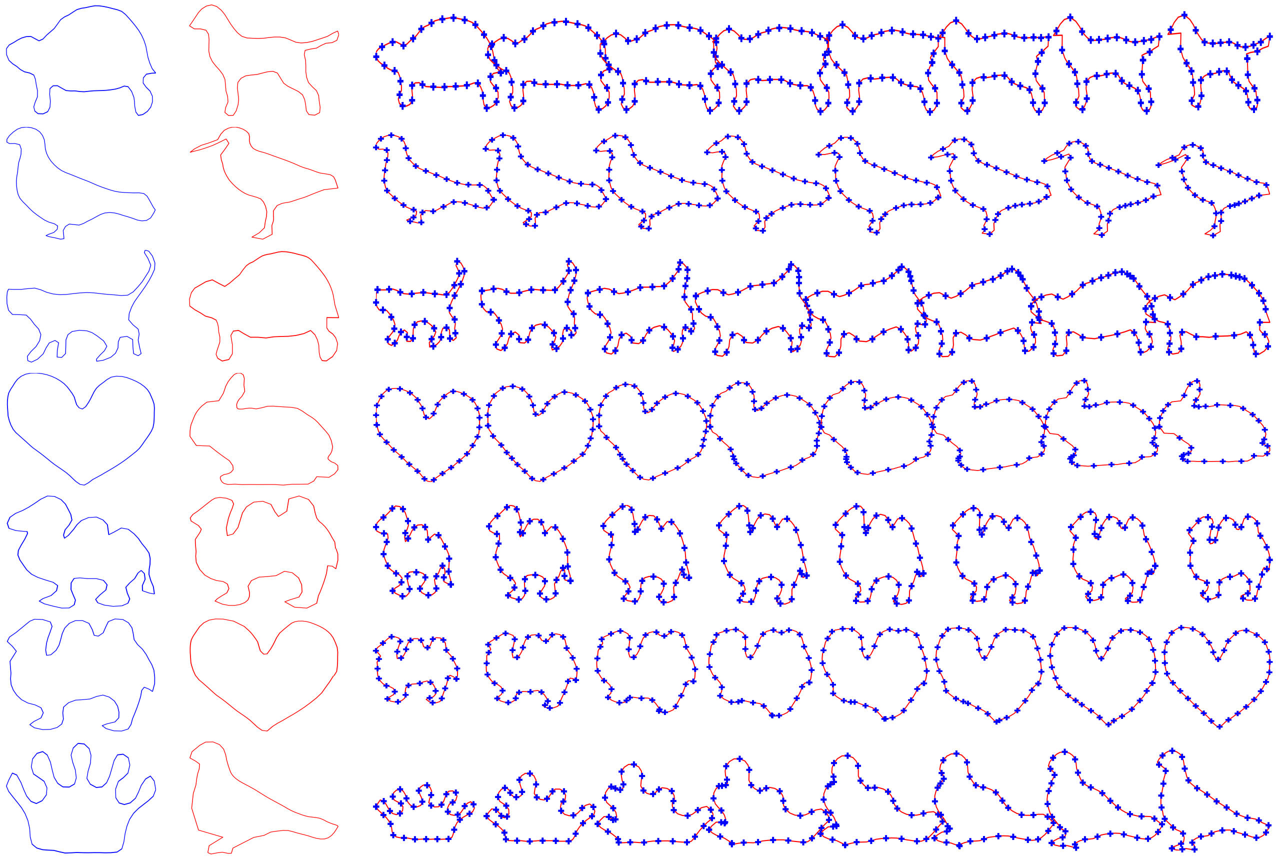

Figure 2. Row-wise geodesic paths in $\mathcal{C}$ between the pair of curves shown to the left.

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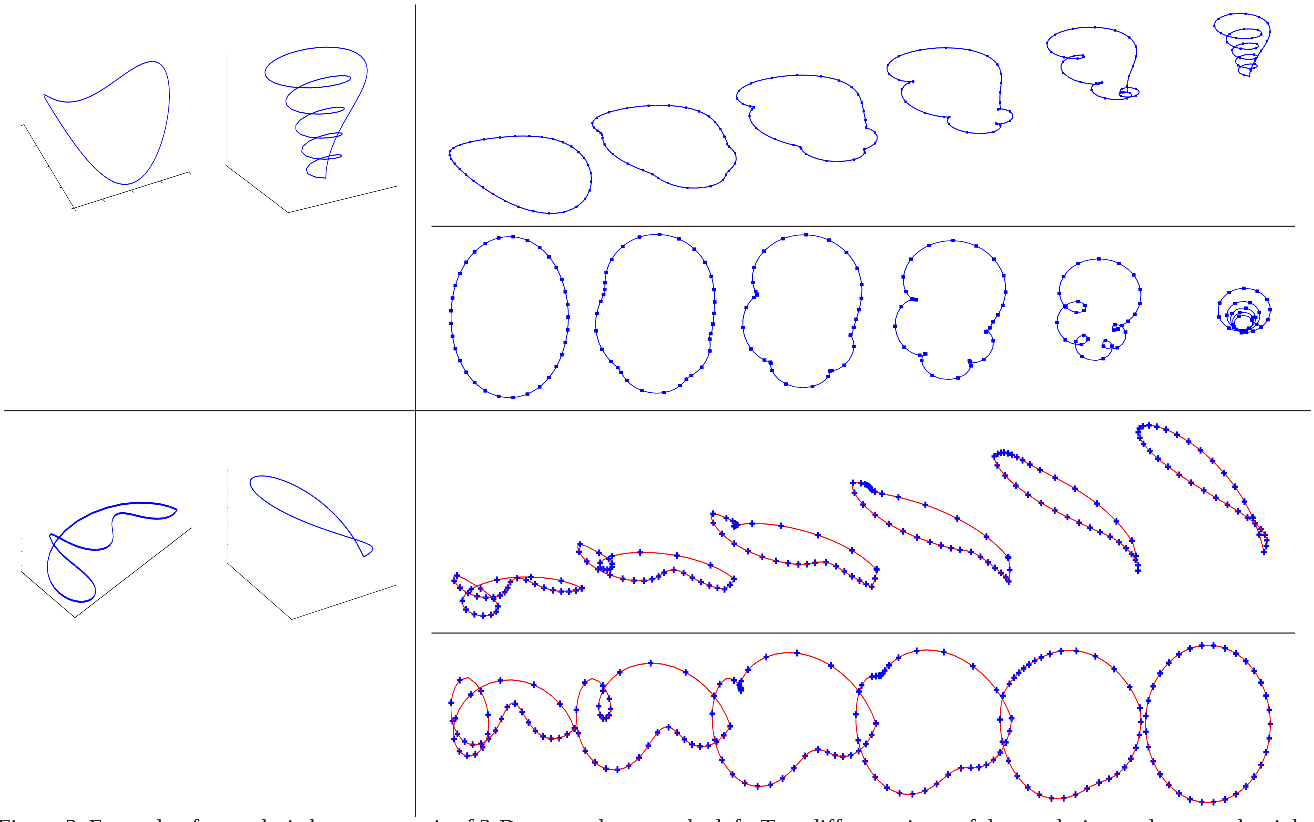

Figure 3. Example of a geodesic between a pair of 3-D curves shown to the left. Two different views of the geodesic are shown to the right.