# Unified methods for computing compressible and incompressible flows on structured and unstructured staggered grids

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# *Contents*

Two themes: a. Unified methods for (in)compressible flows
b. Irregular grids

1. Unification of governing equations
2. Extension of incompressible scheme
3. Accuracy of staggered schemes on rough curvilinear grids
4. Unified scheme for (in)compressible flow with arbitrary equation of state
5. HEM model for hydrodynamic cavitation
6. Unified method on unstructured grid

# *Need for unified methods*

Objective: Mach-uniform accuracy and efficiency

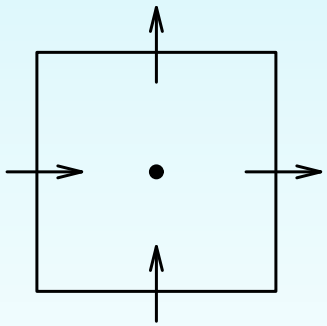
Non-standard example: Cyclic sheet cavitation: $0.001 <$ Mach $< 25$

Approach to Mach-uniformity:

Generalize incompressible scheme to compressible case

Incompressible scheme: MAC scheme (staggered)
Harlow & Welch 1965

Harlow & Welch 1965
On Cartesian grids this scheme is best.



# *Unification of governing equations*

| Compressible | Incompressible |
|---|---|
| $\rho_t + (u\rho)_x = 0$ | $\rho_t + (u\rho)_x = 0$ |
| $m_t + (um + p)_x = 0$ | $m_t + (um + p)_x = 0$ |
| $(\rho E)_t + (\rho u H)_x = 0$ | $\mathrm{div} u = 0$ |
| $U = \begin{bmatrix} \rho \\ m \\ \rho E \end{bmatrix}$ | $U = \begin{bmatrix} \rho \\ m \\ p \end{bmatrix}$ |
| $m = \rho u, \quad E = e + \frac{1}{2}u^2$ <br> $H = \gamma e + \frac{1}{2}u^2, \quad p = p(\rho, e)$ | Discr. in space $\Rightarrow$ DAE ! <br> - Projection methods <br> - Distributive iteration methods <br> - Regularization (Rhie-Chow) |

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# *Unification: pressure-based energy equation*

$$U = \begin{bmatrix} \rho \\ m \\ p \end{bmatrix}$$

Pressure-based energy equation:

$$[p + (\gamma - 1)\rho q]_t + [u(\gamma p + (\gamma - 1)\rho q)]_x = 0, \quad q \equiv \tfrac{1}{2}u^2$$

Units: $u_r, \quad \rho_r, \quad T_r$
Low Mach asymptotics, no acoustics: $\epsilon = \gamma \mathrm{M}_r^2$

$$p = p_0(t) + \epsilon p_1(t, x) + \mathcal{O}(\epsilon^2)$$

$p_1(t, x)$ is the quantity to be computed.

# *Unification: pressure-based energy equation*

This motivates the following choice for the dimensionless pressure:

$$\tilde{p} = \frac{p - p_r}{\rho_r u_r^2}, \quad p_r = \mathrm{R}\rho_r T_r$$

Mach-unified equations:

$$\begin{aligned} \rho_t + (u\rho)_x &= 0 \\ m_t + (um + p)_x &= 0 \\ \mathrm{M}_r^2 \left[ (p + (\gamma - 1)\rho q)_t + (u(\gamma p + (\gamma - 1)\rho q))_x \right] + \mathrm{div} u &= 0 \end{aligned}$$

# *Unification: pressure-based energy equation*

Full equations are used; no use of asymptotics.
As $\mathrm{M}_r \downarrow 0$, incompressible form of equations is recovered.
This makes incompressible solution methods (pressure correction, distributive iteration, ...) applicable.

As $\mathrm{M}_r \downarrow 0$, accuracy and efficiency of incompressible case is obtained.

# *Colocated scheme*

Hyperbolic system: $U_t + f(U)_x = 0$
Colocated scheme:

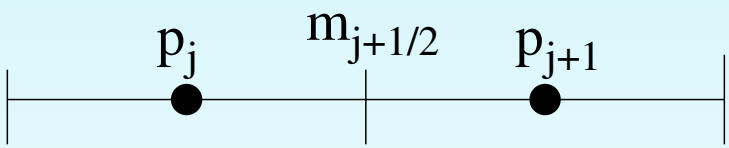
$$\frac{dU_j}{dt} + \frac{1}{h}(F_{j+1/2} - F_{j-1/2}) = 0$$
$$F_{j+1/2} = f(U_{j-p}, \cdots, U_{j+q})$$

Scalar case: solution of monotone conservative scheme converges to entropy solution.
Systems, colocated, flux splitting: (“upwind")

$$F_{j+1/2} = F_j^+ + F_{j+1}^-$$

# *Staggered scheme*

Staggered scheme for compressible flows:
Harlow & Amsden 1968, 1971
Issa, Gosman & Watkins 1986
Karki & Patankar 1989
Shyy & Braaten 1988
Shyy, Chen & Sun 1992
Shuen, Chen & Choi 1993
Bijl & W JCP **141** 1998




Mach-uniform:

$$\lim_{M_{\downarrow 0}} \{\text{scheme}\} = \text{classical incompressible scheme}$$

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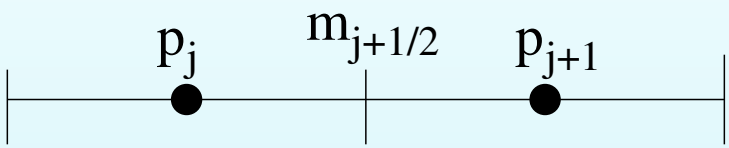
# *Unified scheme for (in)compressible flow*




Flux splitting not possible.
Upwind bias for “monotonicity"?
Convergence for compressible flow?

First test case: barotropic Euler equations
Arbitrary $\rho = \rho(p)$ (as in HEM model for cavitation)

$$
\begin{aligned}
m_t + (um + p)_x &= 0, \qquad u \equiv m/\rho \\
\rho_t + (u\rho)_x &= 0
\end{aligned}
$$

# *Unified scheme for (in)compressible flow*

1D staggered grid: $\cdots m_{j-1/2} \quad \rho_j \quad m_{j+1/2} \quad \rho_{j+1} \cdots$

$$(m^{n+1} - m^n)_{j+1/2} + \lambda (u^n m^{n+1})|_{j-1/2}^{j+1/2} + \lambda p^{n+1}|_j^{j+1} = d_{j+1/2}^n, \quad \lambda = \delta t / \delta x$$

Deferred correction: $d_{j+1/2} = \lambda [(um)|_{j-1/2}^{j+1/2} - (\widetilde{um})|_j^{j+1}]$

$$\begin{aligned} \tilde{\phi}_j &= \tilde{\phi}_{j-1/2} + \tfrac{1}{2}\psi(r_{j-1/2})(\tilde{\phi}_{j+1/2} - \tilde{\phi}_{j-1/2}) \\ r_{j-1/2} &= \frac{\phi_{j-1/2} - \phi_{j-3/2}}{\phi_{j+1/2} - \phi_{j-1/2}} \end{aligned}$$

Matrix-free scheme.

Flux Jacobian or approximate Riemann solver not used

# *Unified scheme for (in)compressible flow*

$$\rho_j^{n+1} - \rho_j^n + \lambda(u_{j+1/2}^{\beta}\rho_j^{n+1} - u_{j-1/2}^{\beta}\rho_{j-1}^{n+1}) = d_j^n$$

$$u^{\beta} = \beta u^{n+1} + (1-\beta)u^n$$

Upwind to satisfy entropy condition
$\beta$ is tuned for stability:

$$\Psi \leq 0 \Rightarrow \beta = 1/2$$

$$\Psi > 0 \begin{cases} 0 < \mathrm{M} \leq 1/2: & \beta = 1 \\ 1/2 < \mathrm{M} \leq 8: & \beta = 1/2 \\ \mathrm{M} > 8: & \beta = 1/8 \end{cases}$$

# *Unified scheme for (in)compressible flow*

Incompressible limit ($\mathrm{M} \downarrow 0$):

$$\begin{aligned} \mathbf{m}_t + (u_\alpha \mathbf{m})_{,\alpha} + \nabla p &= 0 \\ \operatorname{div} \mathbf{m} &= 0 \end{aligned}$$

Differential algebraic system (after semi-discretization)
$\Rightarrow$ Advantage of staggered grid

Pressure-correction method for efficiency as $\mathrm{M} \downarrow 0$

$$\mathbf{m}^* - \mathbf{m}^n + \tau (u_\alpha^n \mathbf{m}^*)_{,\alpha} + \tau \nabla p^n = 0$$

$$\mathbf{m}^{n+1} - \mathbf{m}^* + \tau \nabla \delta p = 0, \quad \delta p = p^{n+1} - p^n$$

Mass conservation equation gives pressure-correction equation:

$$\rho(p^n + \delta p) - \rho(p^n) + \tau \operatorname{div} (\mathbf{m}^* - \tau \nabla \delta p) = 0$$

# *Unified scheme for (in)compressible flow*

Handling of nonlinearity:

$$A\delta p + B\rho(p^n + \delta p) = b$$

Combination of nonlinear Gauss-Seidel and ILU-GMRES for robustness and efficiency:

$$\mathrm{GS} \Rightarrow \delta p^{1/2} \qquad \delta p = \delta p^{1/2} + \delta p^1$$

$$\left[A + B\left(\frac{d\rho}{dp}\right)_{p^n+\delta p^{1/2}}\right]\delta p_1 = -A\delta p^{1/2} - B\rho(p^n + \delta p^{1/2})$$

10 GMRES steps suffice

Convergence in 10 iterations

# *Validation: Riemann problem with nonconvex equation of state*

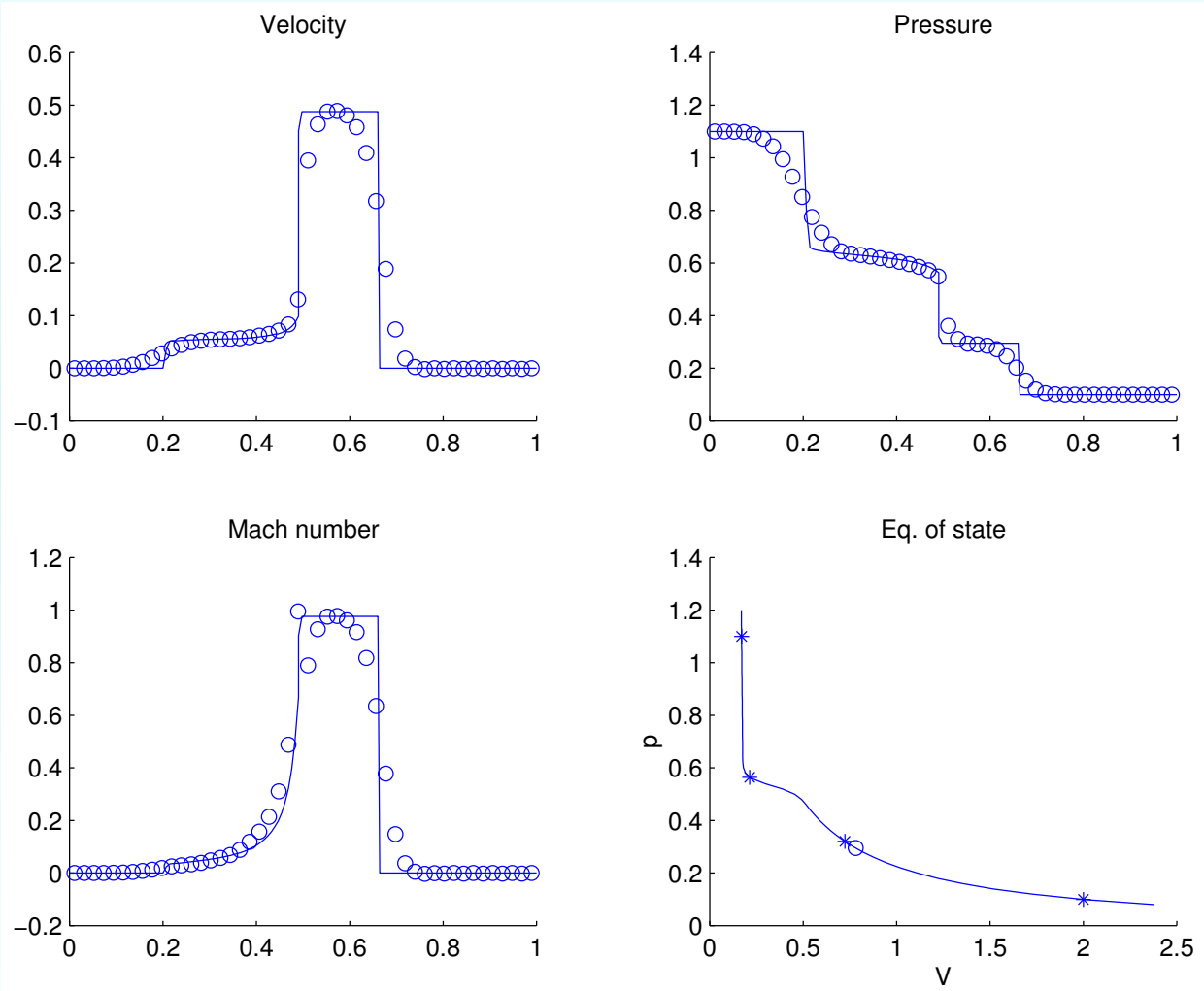
Barotropic Euler in Lagrangian coordinates:

$$\begin{aligned} V_t - u_y &= 0, \quad V = 1/\rho \\ u_t + p(V)_y &= 0 \end{aligned}$$

This is the well-known p-system.

Riemann problem can be solved for arbitrary $p(V)$ using Oleinik's entropy condition

# *Validation: Riemann problem with nonconvex equation of state*




Note that we have an expansion shock

# *Homogeneous equilibrium model*

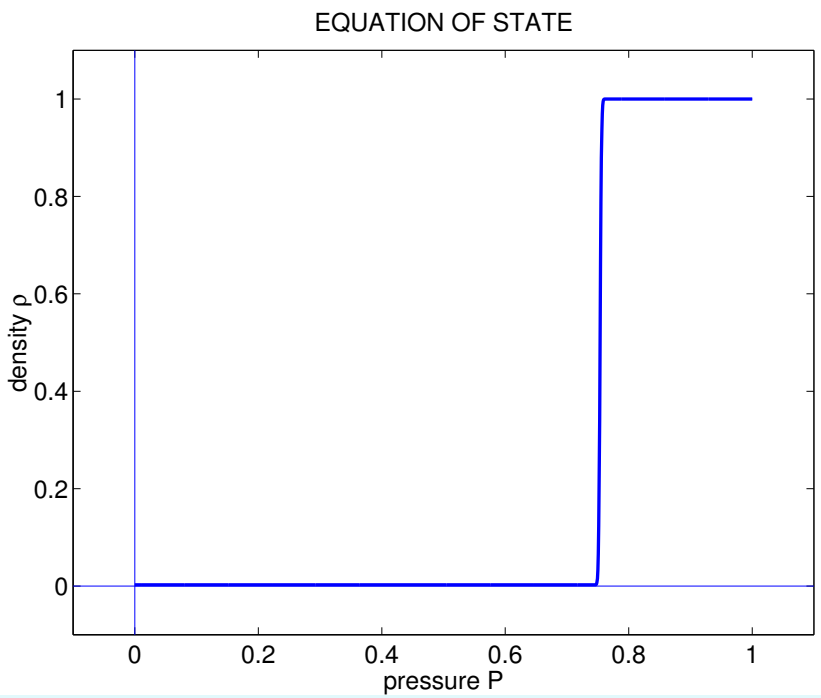
HEM model for hydrodynamic cavitation:
Barotropic Euler equations:

$$\rho_t + \operatorname{div} \mathbf{m} = 0, \quad \rho = \rho(p)$$

$$\mathbf{m}_t + (u_\alpha \mathbf{m})_{,\alpha} + \nabla p = 0, \quad \mathbf{u} = \mathbf{m}/\rho$$

Difficulty: strongly nonlinear nonconvex equation of state $\rho = \rho(p)$.

# *Homogeneous equilibrium model*




Nonconvex barotropic equation of state

$$p < p_1 : \quad \text{vapor}; \quad p > p_2 : \quad \text{liquid}.$$

# *Hydrodynamic cavitation*

Homogeneous equilibrium model:

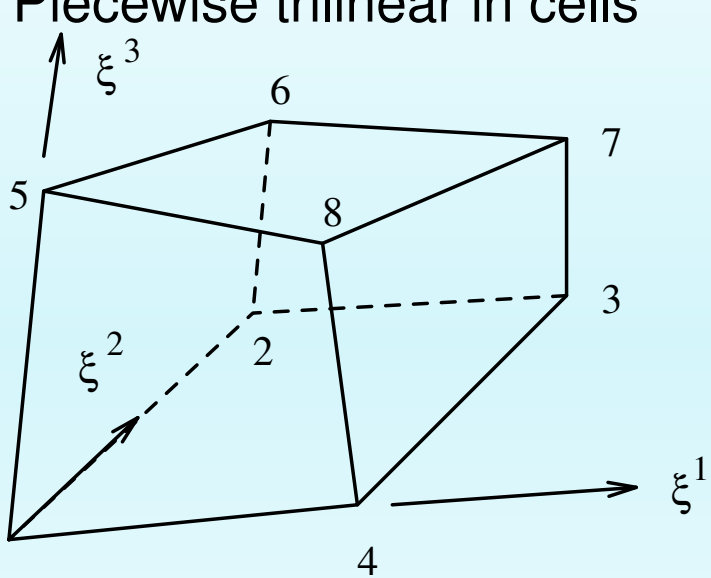
Delannoy (1989)
Merkle, Feng & Buelow (1998)
Shin & Ikohagi (1998)
Song & He (1998)
Ventikos & Tzabiras (2000)

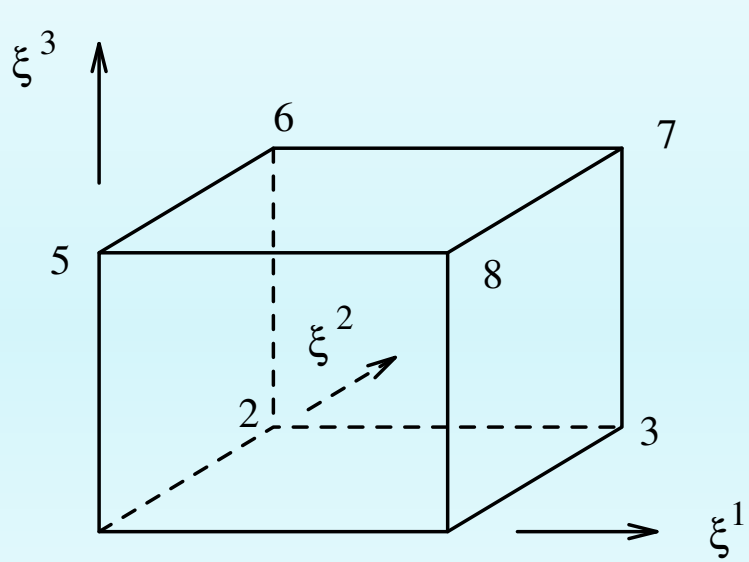
Difficulties: $10^{-3} < \mathrm{M} < 25$
Strong nonlinearity of $\rho = \rho(p)$

Cyclic sheet cavitation

# *Accuracy of staggered schemes on rough curvilinear grids*

Boundary-fitted coordinate mapping $\mathbf{x_j} = \mathbf{x_j}(\xi_\mathbf{j})$
Piecewise trilinear in cells




Cell faces are doubly ruled. *Exact formula for cell volume:*

$$|\Omega_j| = \frac{1}{3}\{\mathbf{b}_1 \cdot (\mathbf{s}_{1265} + \mathbf{s}_{4378}) + \mathbf{b}_2 \cdot (\mathbf{s}_{1584} + \mathbf{s}_{2673}) + \mathbf{b}_3 \cdot (\mathbf{s}_{1432} + \mathbf{s}_{8765})\}$$

# *Accuracy of staggered schemes on rough curvilinear grids*

Coordinate-invariant scheme required. Covariant derivative:

$$U^{\alpha}_{,\beta} = \frac{\partial U^{\alpha}}{\partial \xi^{\beta}} + \left\{ {\alpha \atop \beta\gamma} \right\} U^{\gamma} \,, \qquad \left\{ {\alpha \atop \beta\gamma} \right\} = \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial \xi^{\beta} \partial \xi^{\gamma}} \,,$$

In general, coordinate mapping $\mathbf{x}(\xi)$ not twice differentiable $\Rightarrow$ schemes using Christoffel symbols $\{\}$ are inaccurate on rough grids.

Staggered schemes can be accurate, if lack of smoothness of $\mathbf{x}(\xi)$ is taken into account properly.
Coordinate-invariant staggered scheme without Christoffel symbols:

PW: Principles of CFD, Springer 2001

# *Coordinate-invariant scheme without Christoffel symbols*

Define coordinate mapping $\mathbf{x} = \mathbf{x}(\xi)$ uniquely by trilinear interpolation.
Take nonsmoothness of $\mathbf{x} = \mathbf{x}(\xi)$ carefully into account

Basic idea for avoiding $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$: first step: transform only coordinates ,

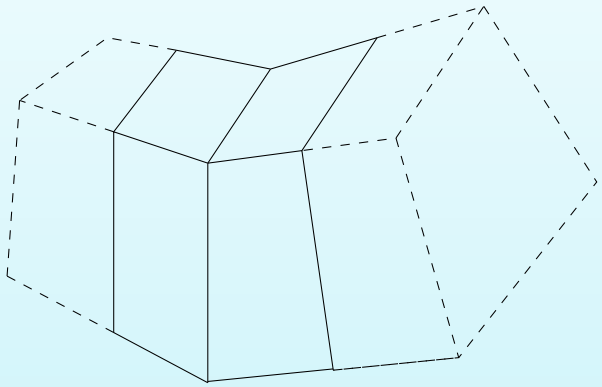
not components. Example: inertia term:

$$N(\rho, \mathbf{u}) \equiv \frac{\partial u^\alpha \mathbf{m}}{\partial x^\alpha} = \frac{1}{\sqrt{g}} \frac{\partial V^\alpha \mathbf{m}}{\partial \xi^\alpha}, \quad \sqrt{g} \equiv \det\left(\frac{\partial \mathbf{x}}{\partial \xi}\right)$$

$V^\alpha d\xi^\beta d\xi^\gamma$ is volume flux through $d\xi^\beta d\xi^\gamma$: contravariant flux.
Primitive staggered velocity variables: $V^\alpha$

Finite volume integration over staggered volumes.

# *Coordinate-invariant scheme without Christoffel symbols*



$$\int_{\Omega_{j+e_1}} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^1} (\mathbf{u} V^1) d\Omega = \int_{G_{j+e_1}} \frac{\partial}{\partial \xi^1} (\mathbf{u} V^1) d\xi^1 d\xi^2 d\xi^3 \cong \Delta\xi^2 \Delta\xi^3 (\mathbf{u} V^1)|_j^{j+2e_1}$$

Interpolation inside cell ($\mathbf{x}(\xi)$ smooth!):

$$V_j^\alpha \cong \tfrac{1}{2}(V_{j-e_\alpha}^\alpha + V_{j+e_\alpha}^\alpha), \quad \mathbf{u}_j = \left( \frac{\partial \mathbf{x}}{\partial \xi^\alpha} V^\alpha / \sqrt{g} \right)_j$$

# *Coordinate-invariant scheme without Christoffel symbols*

Accuracy requirement on interpolations: $\mathbf{u} \Rightarrow V^\alpha \Rightarrow \mathbf{u}$ exactly for constant $\mathbf{u}$ on arbitrary nonsmooth grids
Other terms similarly
Last step: take inner product with normal face vector:

$$(\sqrt{g}\mathbf{a}^{(\alpha)})_{j+e_\alpha} \cdot \int_{\Omega_{j+e_\alpha}} N(\mathbf{u}, p) d\Omega = 0 \quad \text{(no summation)}$$

$$\text{Result: equation for } \frac{dV^\alpha_{j+e_\alpha}}{dt} \quad \text{(no summation)}$$

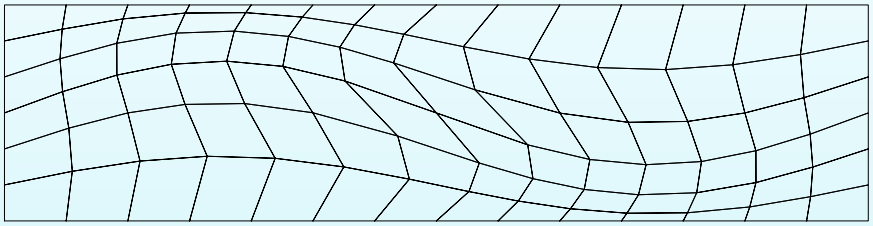
Scheme is coordinate invariant.

Accurate staggered schemes on rough grids: He & Salcudean (1994), Karki & Patankar (1988), Melaaen (1992), Wesseling, Segal & Kassels (1999).

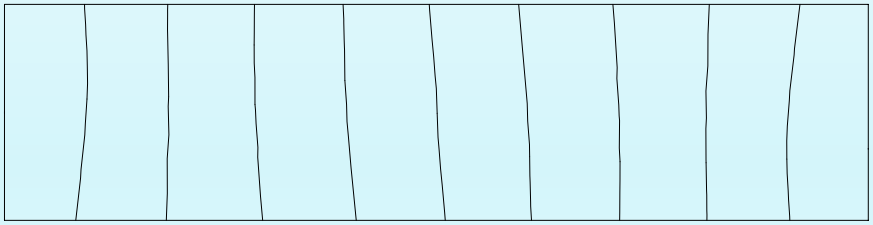
# *Coordinate-invariant scheme without Christoffel symbols*

Detailed inspection shows that discrete approximations to finite volume integrals of Christoffel symbols can be recognized.

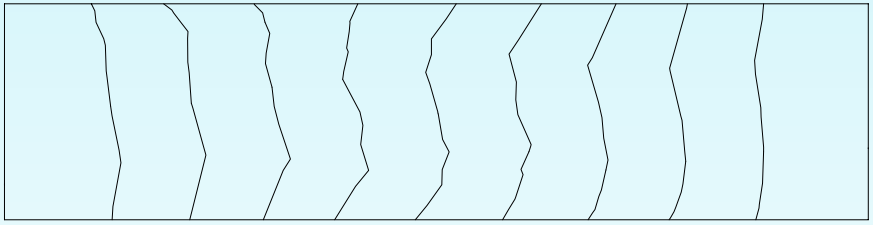
# *Accuracy of staggered schemes on rough curvilinear grids*



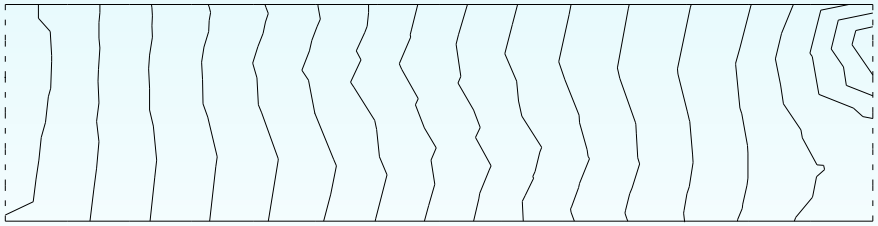
Grid for Poiseuille flow



Isobars for staggered scheme

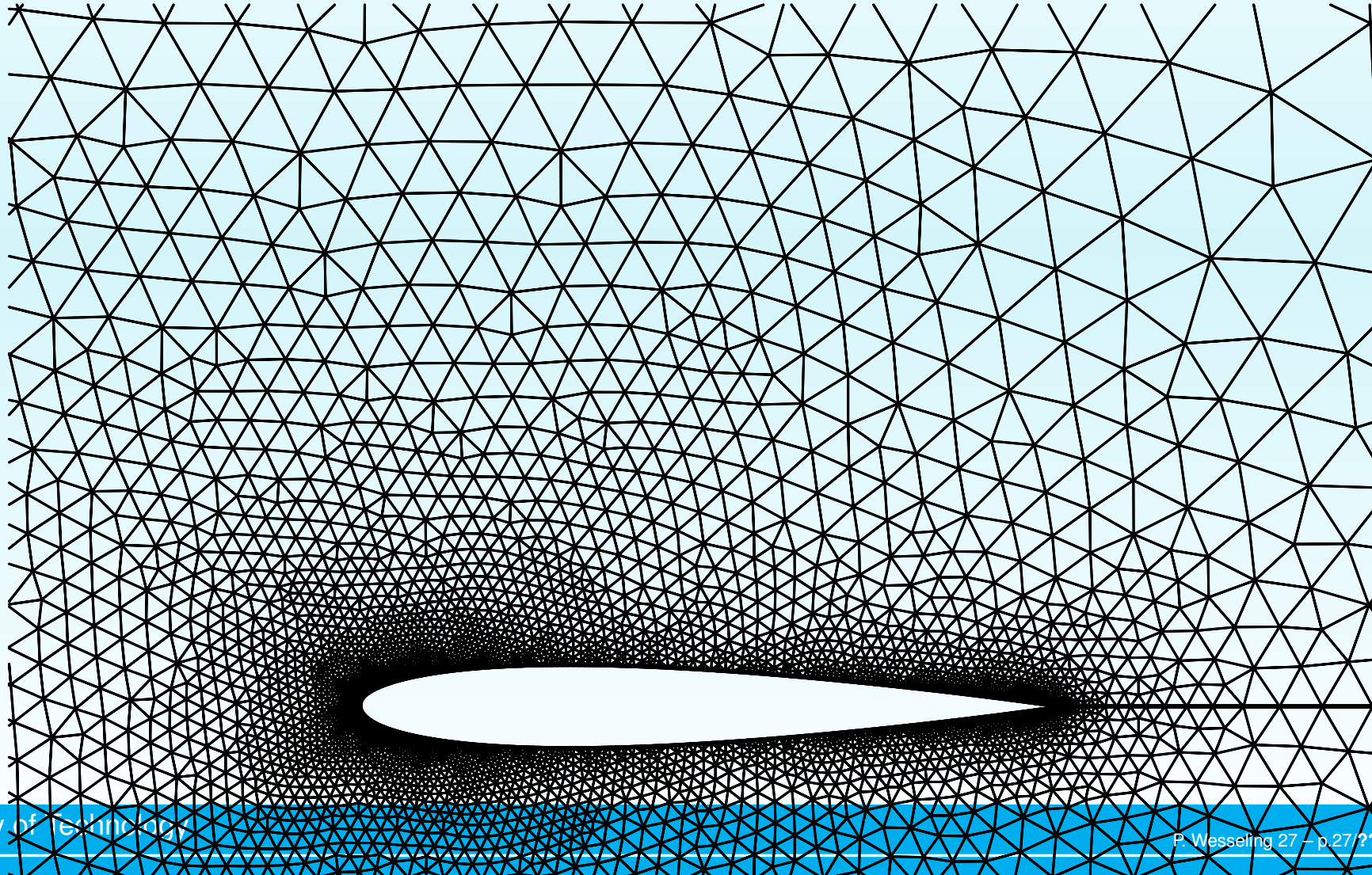


Isobars for finite element method

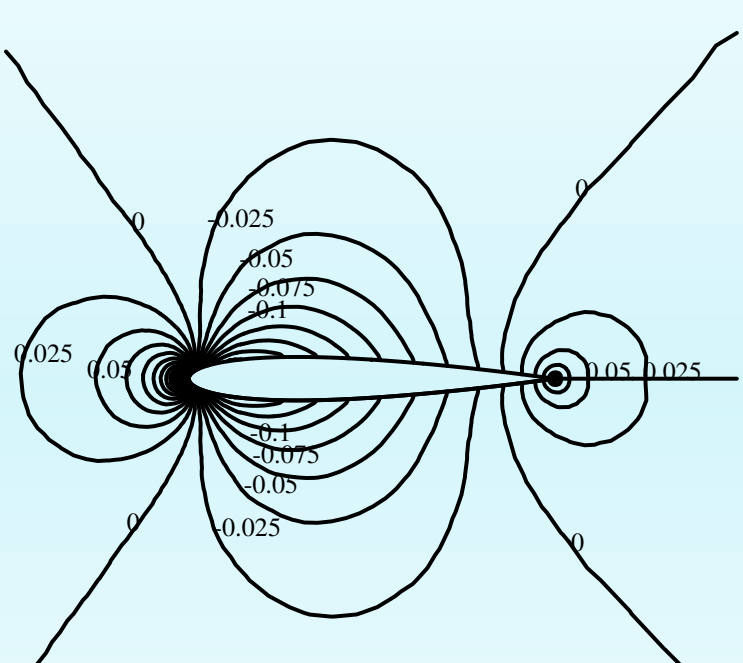


Isobars for commercial colocated scheme

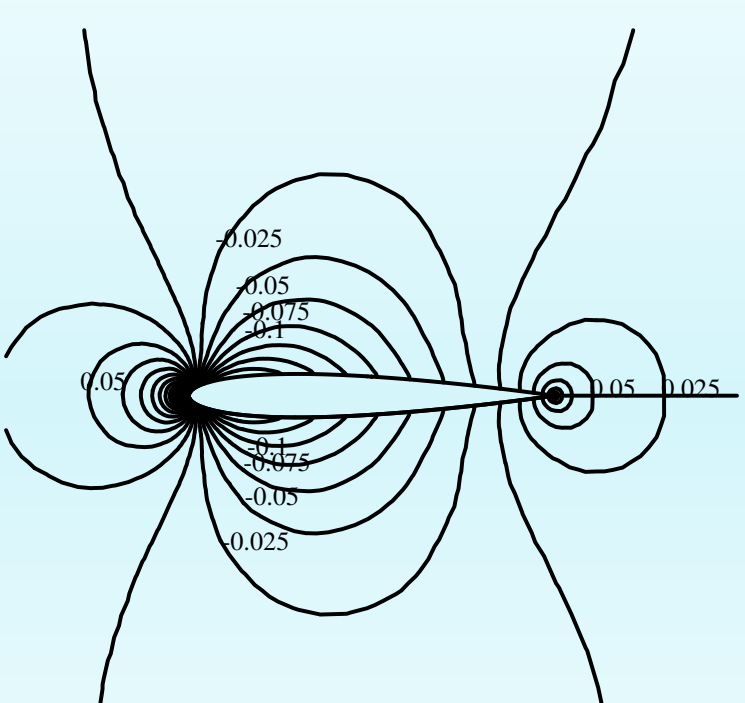
# *Unstructured staggered scheme*



# *Isobars for different Mach numbers*

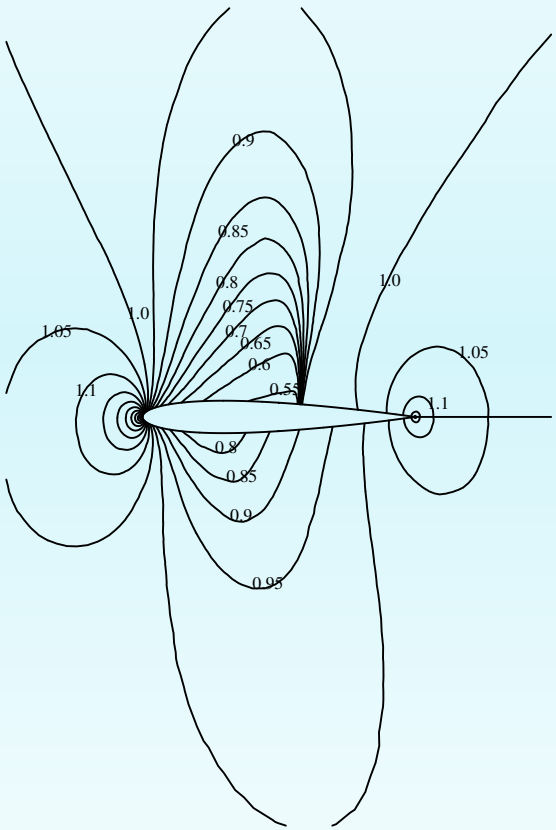



$\mathrm{M} = 0, \ \alpha = 0$

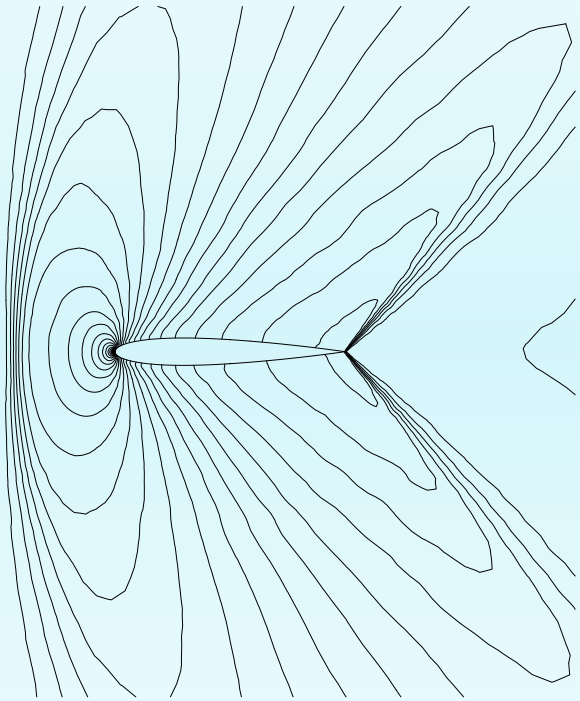



$\mathrm{M} = 0.1, \ \alpha = 0$

# *Isobars for different Mach numbers*




$\mathrm{M} = 0.8,\ \alpha = 1.25^o$



$\mathrm{isoMach} : \mathrm{M} = 1.2,\ \alpha = 0$

# *Mach-uniform formulation*

Dimensionless pressure:

$$p = \frac{\hat{p} - \hat{p}_0}{\hat{\rho}_r \hat{u}_r^2} \ .$$

Pressure-based governing equations:

$$\rho_t + \mathrm{div}\mathbf{m} = 0 \ ,$$

$$\mathbf{m}_t + \frac{\partial}{\partial x^\alpha}(u^\alpha \mathbf{m}) + \mathrm{grad}p = 0 \ ,$$

$$\mathrm{M}_0^2 \Big\{ \frac{\partial}{\partial t}\big[p + (\gamma - 1)\rho q\big] + \mathrm{div}\big[\mathbf{u}(\gamma p + (\gamma - 1)\rho q\big] \Big\} + \ \mathrm{div}\,\mathbf{u} = 0$$

# *Mach-uniform time stepping scheme*

$\lim_{\mathsf{M}\downarrow 0}$ scheme = incompressible staggered scheme.

Therefore some terms must be implicit:

$$(\rho^{n+1} - \rho^n)/\Delta t + \mathsf{div}\,(\rho^{n+1}\mathbf{u}^n) = 0\,,$$

$$(m_\alpha^{n+1} - m_\alpha^n)/\Delta t + \mathsf{div}_h\,(m_\alpha^{n+1}\mathbf{u}^n) + \nabla_h p^{n+1} = 0\,,$$

$$\mathsf{M}_0^2\Big\{\frac{1}{\Delta t}\Big[p + (\gamma-1)\rho q\Big]_n^{n+1} + \mathsf{div}_h\,\Big[\mathbf{u}^{n+1}\big(\gamma p^{n+1} + (\gamma-1)\rho^{n+1}q^{n+1}\big)\Big]\Big\}$$
$$+\mathsf{div}_h\,\mathbf{u}^{n+1} = 0\,.$$

# *Pressure-correction method*

Mach $\downarrow 0 \;\Rightarrow$ Differential-algebraic system (stiffness $\rightarrow \infty$)
Computing work Mach-uniform $\Rightarrow$ Use pressure-correction method:

$$(\rho^{n+1} - \rho^n)/\Delta t + \mathsf{div}_h\,(\rho^{n+1}\mathbf{u}^n) = 0\,,$$

$$(m_\alpha^* - m_\alpha^n)/\Delta t + \mathsf{div}_h\,(m_\alpha^*\mathbf{u}^n) + p_{,\alpha}^{n} = 0\,,$$

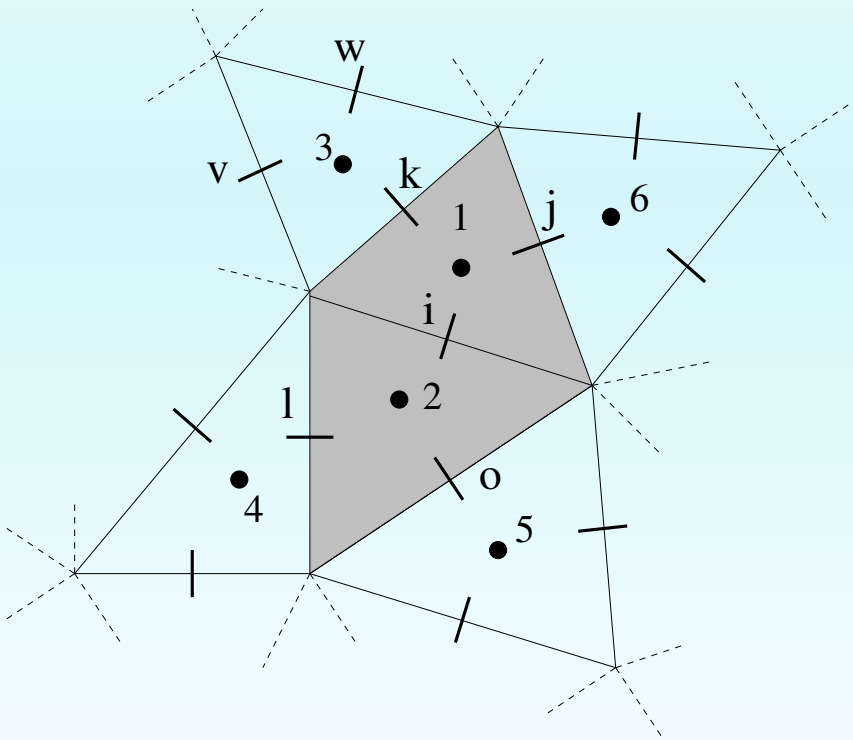
$$\mathbf{m}^{n+1} = \mathbf{m}^* - \Delta t \nabla_h (p^{n+1} - p^n)\,.$$

Substitute $\mathbf{u}^{n+1}$ in energy equation $\Rightarrow$ pressure-correction equation.

Sequential update ($\rho^{n+1},\ \mathbf{m}^*,\ p^{n+1},\ \mathbf{m}^{n+1}$) allows efficient iterative methods.

# *Unstructured staggered scheme*

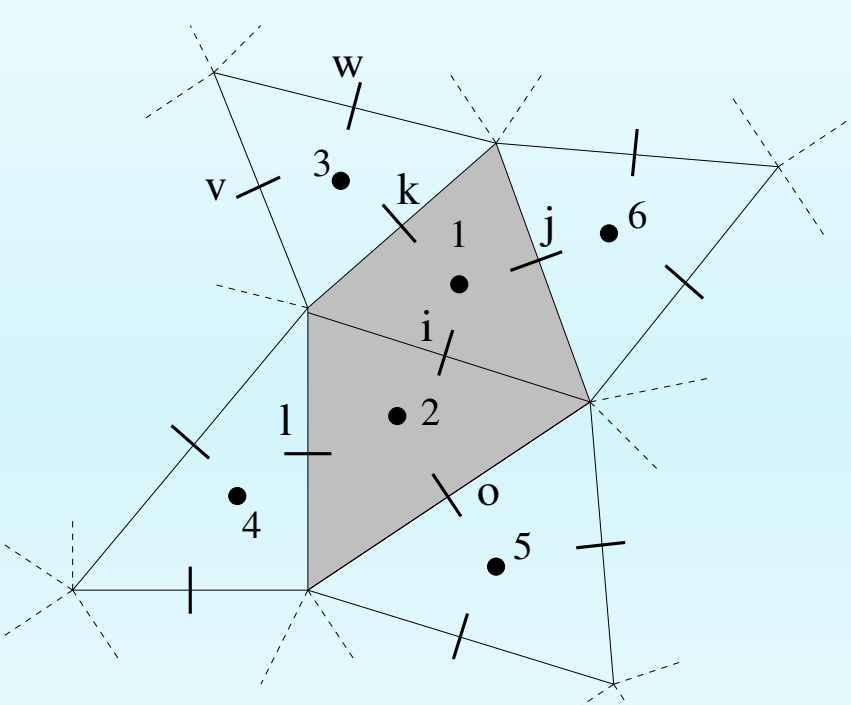
Cells: **arbitrary** triangles.
(triangles greater challenge than quads for staggered schemes).
Staggered: store only normal components.




$$\frac{\partial m_i}{\partial t} + \text{div}\,(\mathbf{u} m_i) = -\nabla p \cdot \mathbf{N}_i.$$

$$\int_{CV} \text{div}\,(\mathbf{u} m_i) \approx \sum_e u_e (\mathbf{m}_e \cdot \mathbf{N}_i) \bar{l}_e \ .$$

# *Unstructured staggered scheme*




$$(\mathbf{u}_k \cdot \mathbf{N}_k) = m_k/\rho_k$$

$$\rho_k = \frac{\Omega_3}{\Omega_1 + \Omega_3}\rho_1 + \frac{\Omega_1}{\Omega_1 + \Omega_3}\rho_3 \ .$$

Approximation of $(\mathbf{m}_k \cdot \mathbf{N}_i)$:

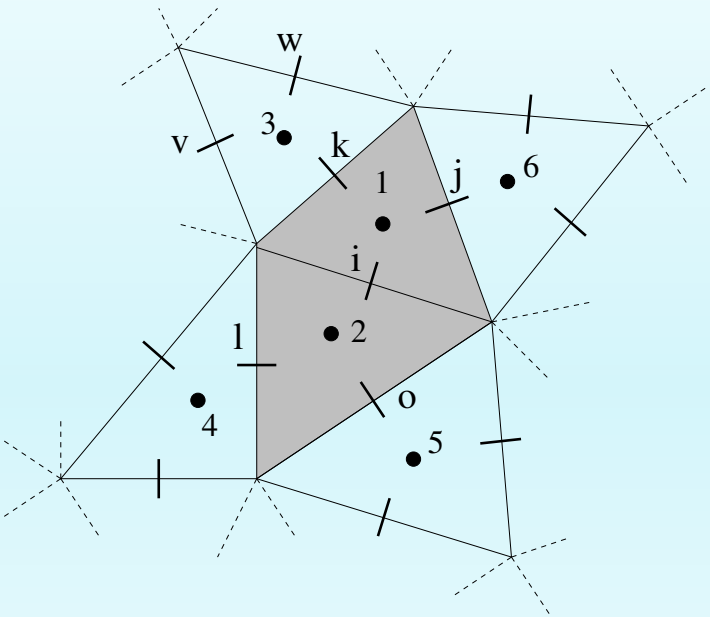
$$\mathbf{N}_i = \eta_v \mathbf{N}_v + \eta_w \mathbf{N}_w$$

$$\mathbf{m}_k \cdot \mathbf{N}_i \approx \eta_v m_v + \eta_w m_v \ .$$

Upwind:

$$\begin{aligned} \mathbf{m}_k \cdot \mathbf{N}_i &= \eta_v m_v + \eta_w m_v && \text{if } u_k \bar{l}_k < 0\,; \\ &= m_i && \text{if } u_k \bar{l}_k > 0\,. \end{aligned}$$

# *Unstructured staggered scheme*




Pressure gradient:
path integral method:

$$p_2 - p_1 \cong \nabla p_i \cdot (\mathbf{x}_2 - \mathbf{x}_1)\,,$$

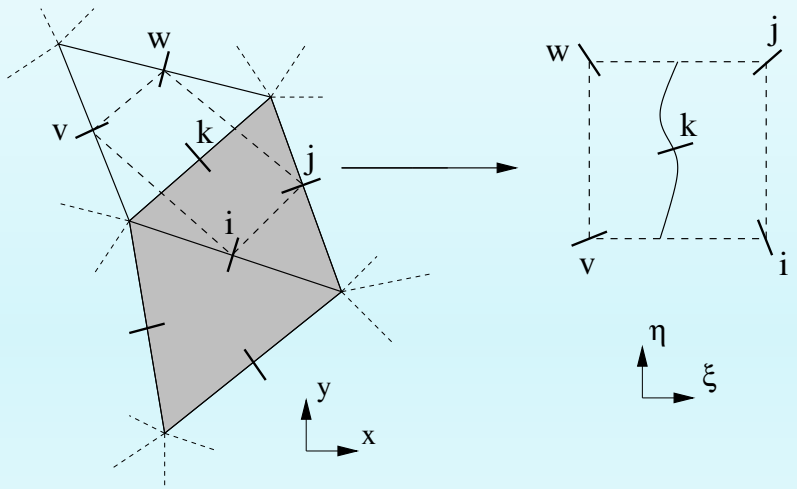
$$p_3 - p_6 + p_4 - p_5 \cong \nabla p_i \cdot (\mathbf{x}_3 - \mathbf{x}_6 + \mathbf{x}_4 - \mathbf{x}_5)\,.$$

Solve for $\nabla p_i$:

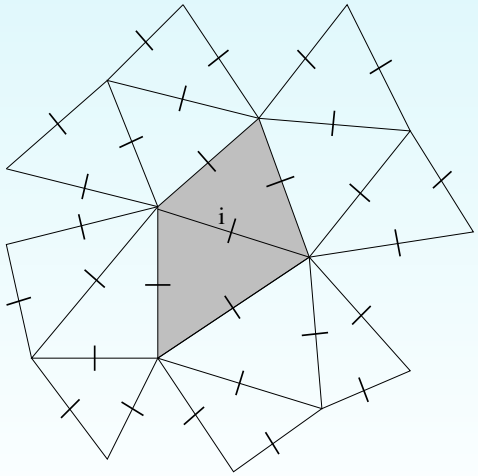
$$(\nabla p \cdot \mathbf{N})_i = \sum_{j=1}^{6} \gamma_j p_j .$$

# *Unstructured staggered scheme*

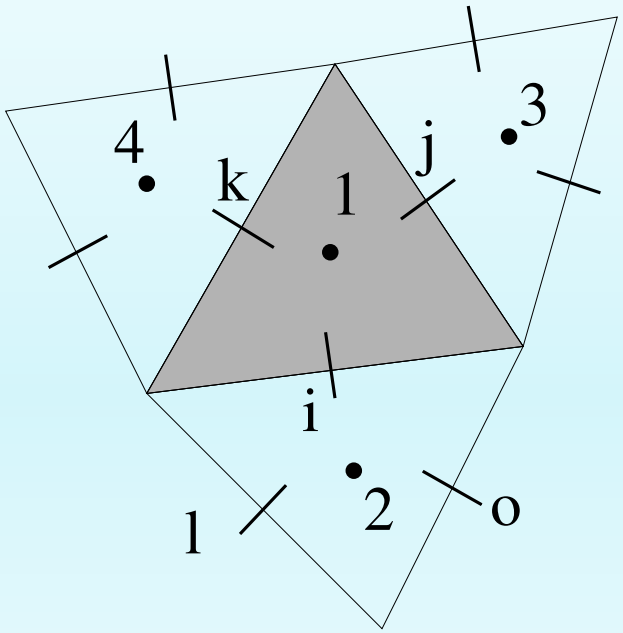
**Viscous term**: velocity gradients by bilinear interpolation:




Tangential components by interpolation from neighbors. Resulting viscous stencil:



# *Mass conservation*




$$\frac{\Omega_1}{\Delta t}(\rho_1^{n+1} - \rho_1^n) + \sum_{e(1)} \rho_e^{n+1} u_e^n l_e = 0\,.$$

Central: $\rho_i = (\rho_1 + \rho_2)/2\,,$

Upwind: $\rho_i = \rho_1$ or $\rho_2$.

# *Energy conservation*

Energy equation: $\delta p \equiv p^{n+1} - p^n$ .

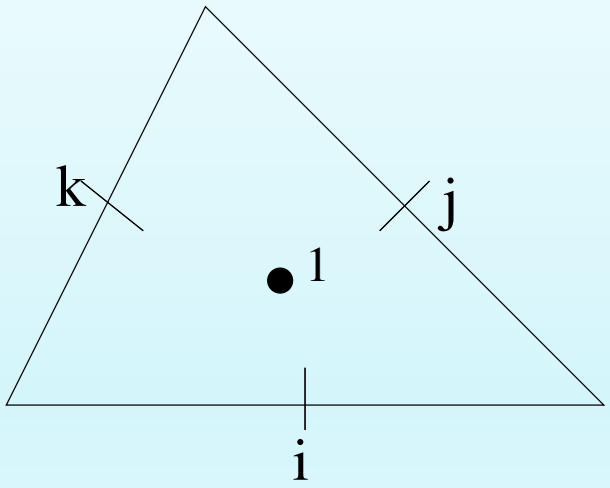
$$\mathsf{M}_0^2 \Big\{ \frac{\delta p}{\Delta t} + \frac{\gamma - 1}{2\Delta t} \Big( \rho^{n+1} |\mathbf{u}^* - \frac{\Delta t}{\rho^{n+1}} \nabla \delta p|^2 - \rho^n |\mathbf{u}^n|^2 \Big)$$

$$+\mathsf{div}_h \Big[ (\mathbf{u}^* - \frac{\Delta t}{\rho^n} \nabla \delta p)(\gamma(p^n + \delta p) + \frac{\gamma - 1}{2} \rho^{n+1} |\mathbf{u}^*|^2) \Big] \Big\}$$

$$+\mathsf{div}_h \, (\mathbf{u}^* - \frac{\delta t}{\rho^n} \nabla \delta p) = 0 \, .$$

Convection-diffusion equation for $\delta p$. Upwind scheme.
Downstream influence by convection term,
upstream influence by diffusion term.
Linearized in $\delta p$ and solved for $\delta p$ with ILU-preconditioned GMRES.

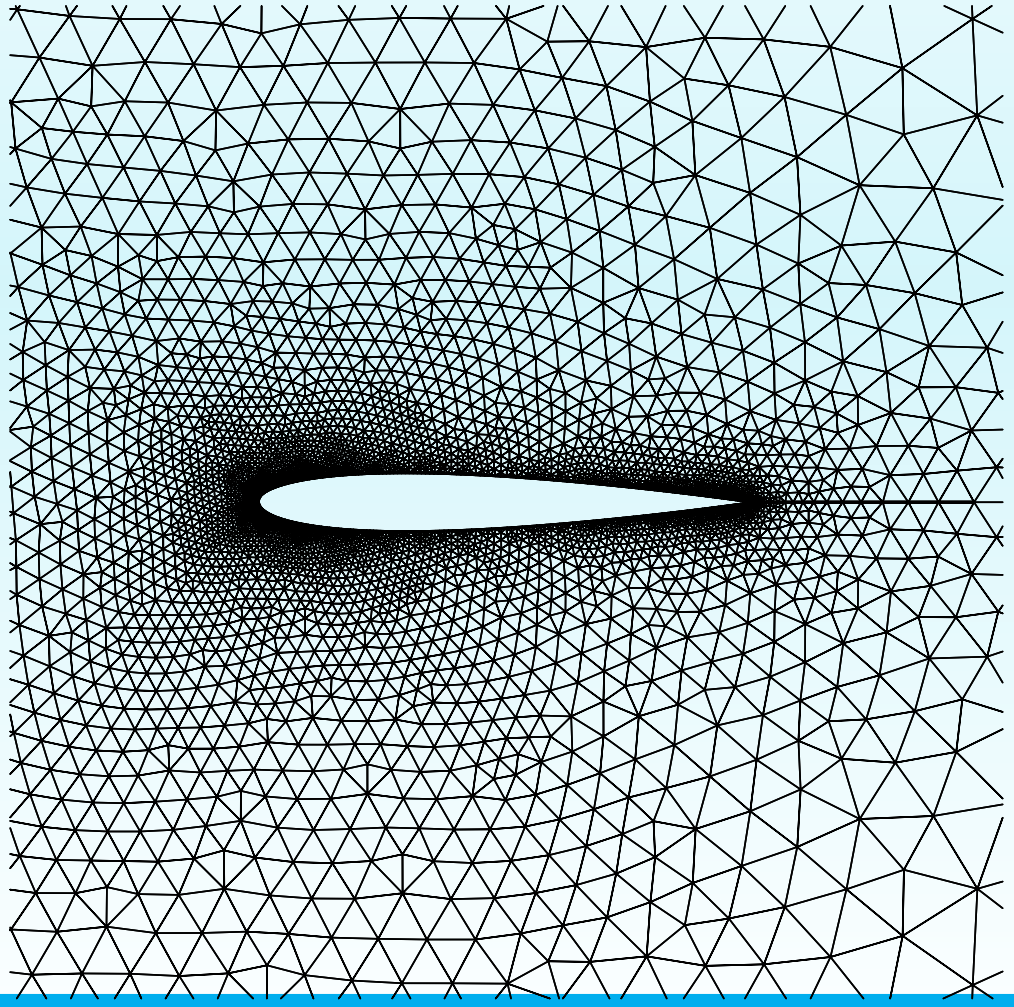
# *Kinetic energy*




Evaluation of $\mathbf{m}_1$: Least squares.

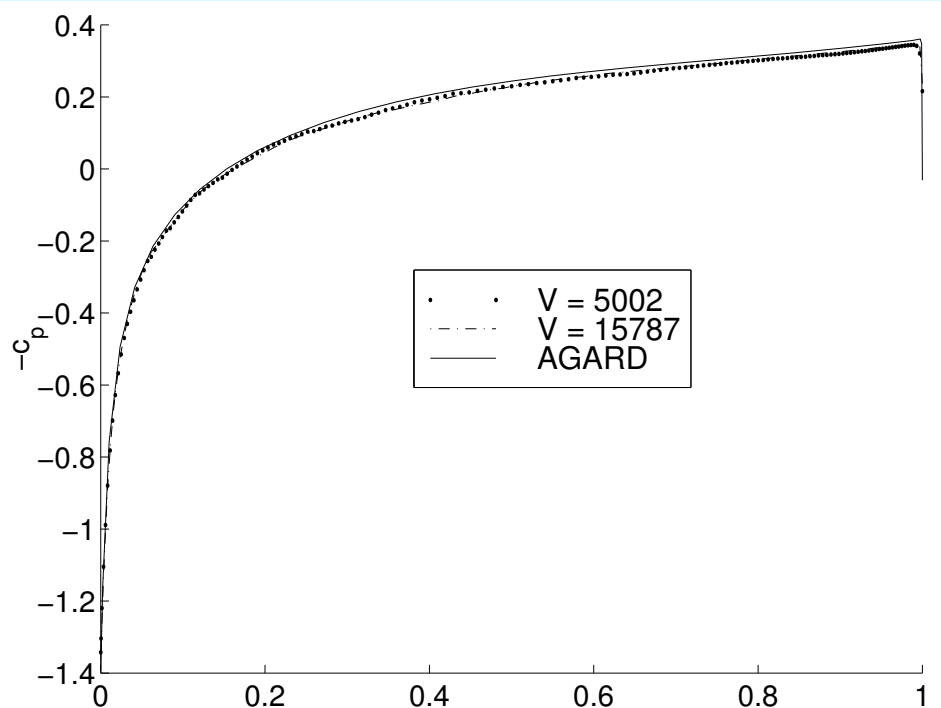
# *Euler solutions*

NACA 0012, 9610 cells, 5002 vertices

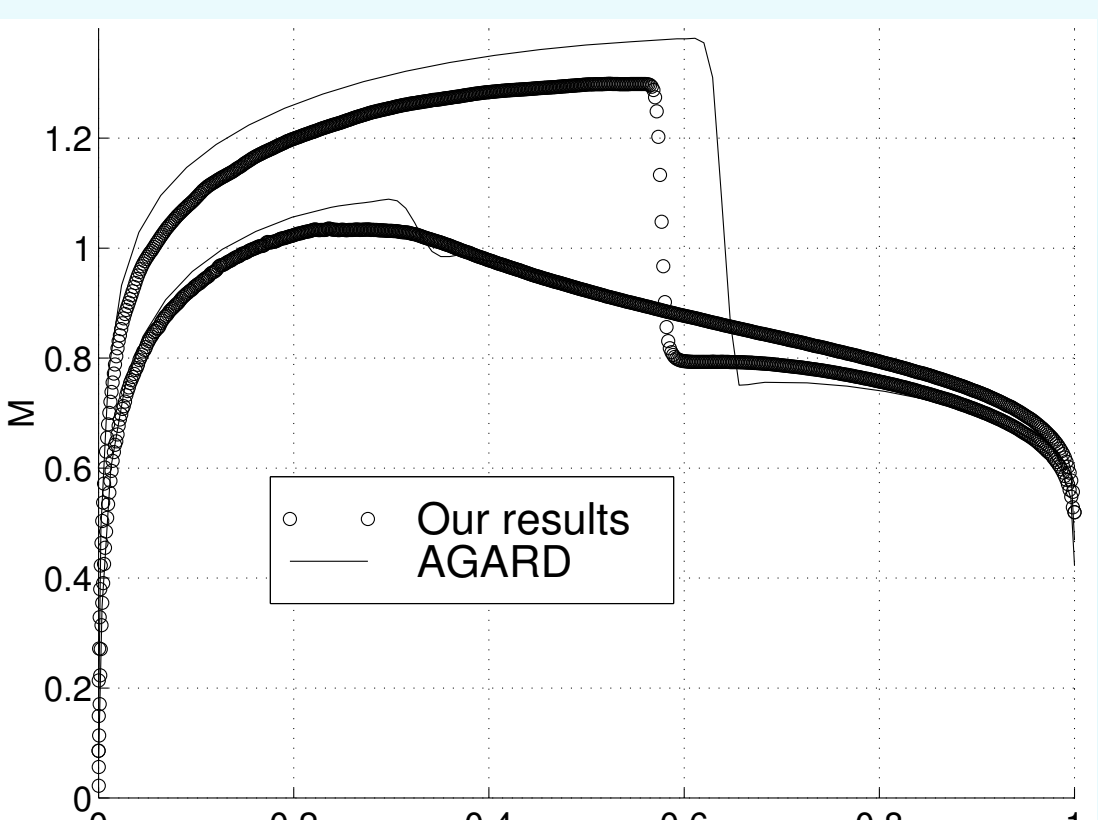


# *Mach-uniform efficiency*

| $M_\infty$ | 0 | 0.1 | 0.63 | 0.8 | 1.2 |
|---|---|---|---|---|---|
| $\Delta t$ | 0.08 | 0.08 | 0.09 | 0.08 | 0.08 |
| CFL | | 800 | 233 | 180 | 150 |
| Tend | 4.7 | 3.0 | 17 | 50 | 53 |
| CPU/step | 4.7 | 4.5 | 3.5 | 3.3 | 2.6 |
| CPU/tot | 279 | 165 | 668 | 2021 | 1645 |



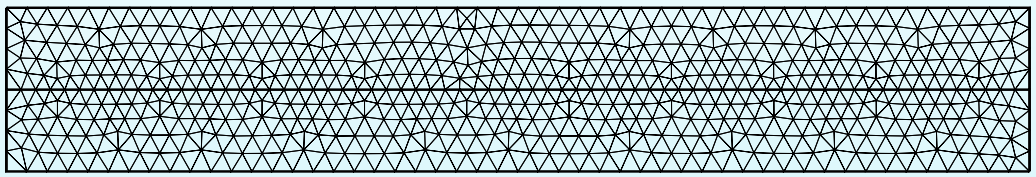
# *Transonic accuracy*

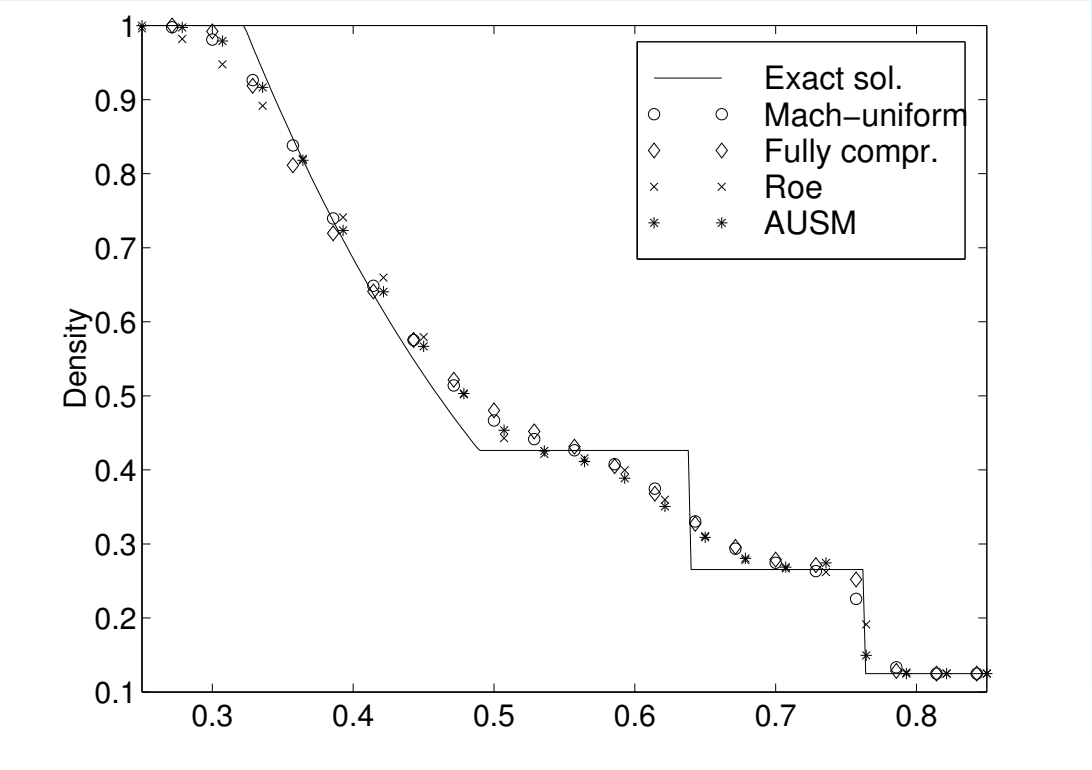



Mach number distribution. Second order scheme required.

TU Delft

# *Shocktube problem*

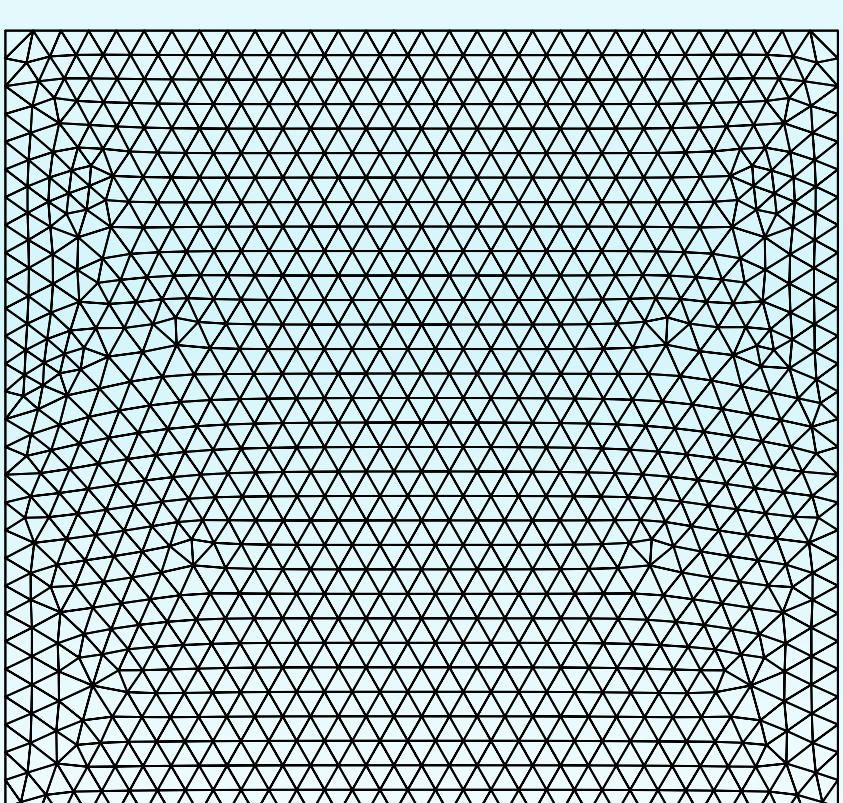




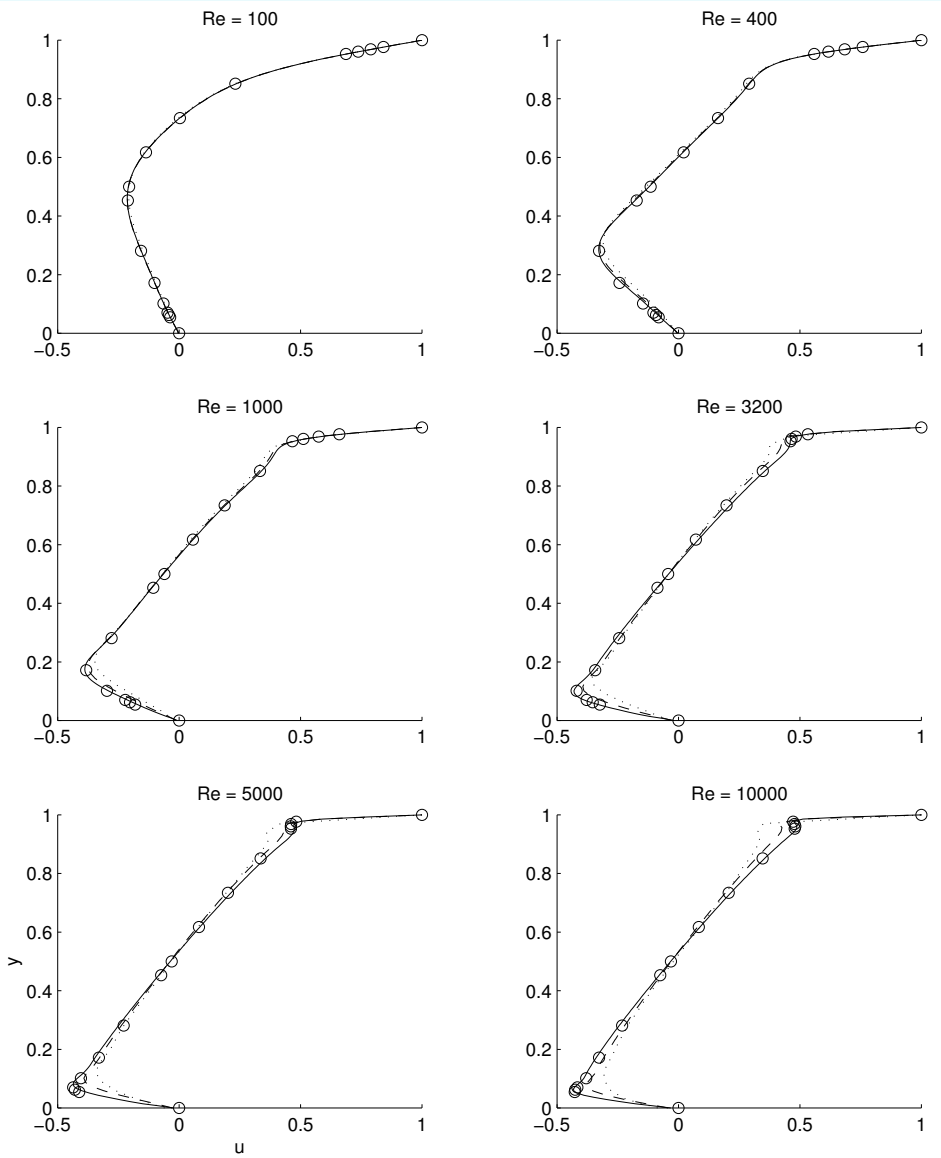

Comparison with classical schemes on 1D uniform grid.

# *Incompressible Navier-Stokes*

Driven cavity.

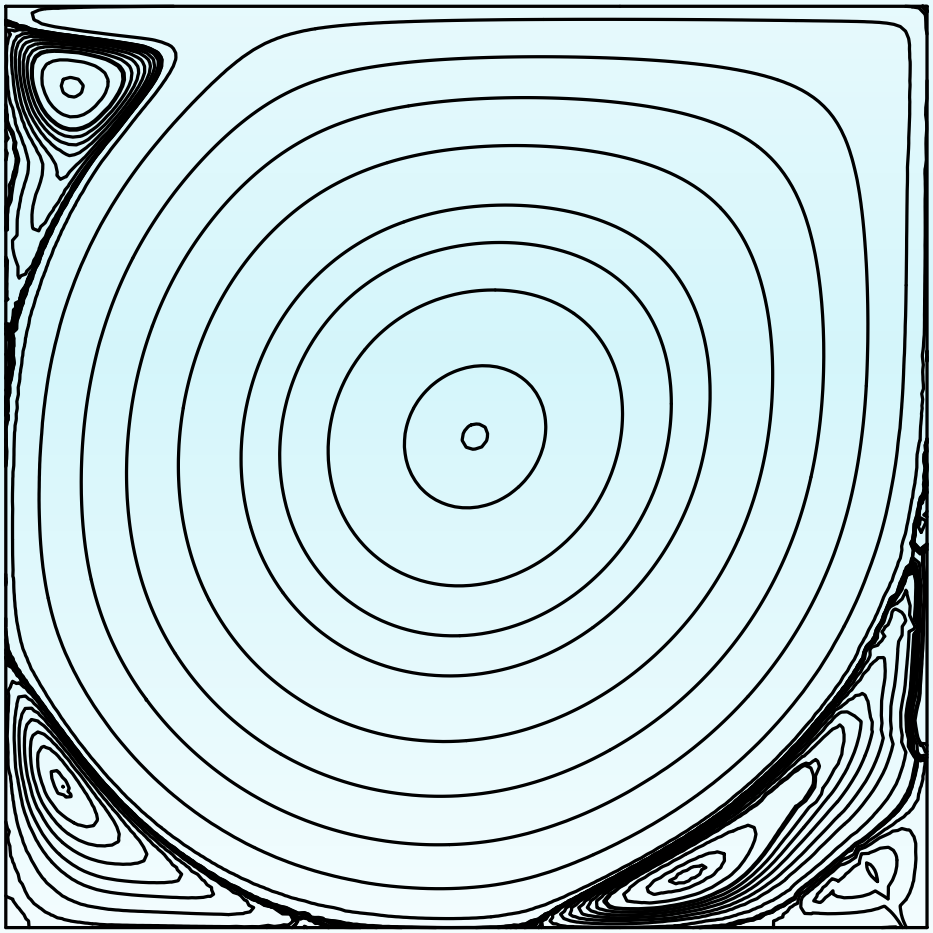


# *Driven cavity*




Comparison of horizontal velocities along the vertical centerline for several Reynolds numbers.
$n = 30$ ($\cdots$); $n = 60$ (- -); $n = 120$ (—); Ghia et al. ($\circ$).
$n$ is no. of cells per wall.

# *Driven cavity*



Streamlines for $\mathrm{Re} = 10,000$ obtained on grid with $n = 120$.

# *Conclusions*

Generalization of classical staggered scheme to unstructured triangles and all speeds.
Mach-uniform method.
Accuracy similar to structured first order colocated for compressible flows.
For transonic flow, second order upwind required.