Comparison of Density- and Pressure-Based Methods for Low Mach Number Computations

Venke Sankaran and Charles Merkle

Purdue University
Outline

• Introduction
• Density-Based Approach
  – Role of time scales in convergence
• Pressure-Based Approach
  – Pressure Poisson Solution
  – Segregated solution procedure
  – Discrete formulation
  – Under-relaxation in Fluent
• Comparison of the Two Methods
• Conclusions
Introduction

Two Families of CFD schemes

- Density-based
- Pressure-based

Density-based Methods
- Time-marching (optionally with preconditioning)
- Fully-coupled solution framework

Pressure-based Methods
- Segregated or co-located pressure-velocity coupling
- Staggered or co-located pressure-velocity coupling
- Usually use under-relaxation for non-linear iterations
- Flux-difference and Flux-vector schemes
- Time-marching (optionally with preconditioning)

- Pressure-based
- Density-based

- Two Families of CFD schemes
Time-Marching Framework

Steady Euler Equations

Convergence controlled by time-scales:

\[ 0 = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \]

\[ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y} \]

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} - g \rho \]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y} \]

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} - g \rho \]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y} \]

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} - g \rho \]
Implicit Solution\n\textbf{Delta Form in 2D:} \\
\textbf{ADI Approximate Factorization:} \\
\textbf{Approximate Factorization Error:}

\[
\begin{align*}
\frac{\partial \psi}{\partial t} + \frac{x \partial \psi}{E \ell} &= \mathcal{O} \left( \frac{\partial \psi}{\partial \ell} \cdot \frac{x \partial \phi}{\partial \ell} \right) + \frac{x}{\partial \ell} + \frac{\partial \psi}{\partial \ell} + \mathcal{O}
\end{align*}
\]

\textbf{ADI Approximate Factorization:} \\
\textbf{Delta Form in 2D:} \\
Implict Solution
Von-Neumann Stability Results

- Approximate factorization causes CFL limit (5-10)
- Preconditioning insures all CFL numbers are optimum
- Direct Scheme is unconditionally stable

ADI

Direct
Some limitations remain in 3D.

Implicit schemes allow min-CFL choice in 2D.

Traditional Time-Step Definition:

\[
\text{CFL} = \frac{\lambda + \sqrt{\lambda^2 + 4\alpha (\nu + \lambda \sqrt{\nu^2 + 4\alpha})}}{2\alpha} = 1
\]

Correct Time-Step Definition:

\[
\text{CFL} = \frac{\lambda + \sqrt{\lambda^2 + 4\alpha (\nu + \lambda \sqrt{\nu^2 + 4\alpha})}}{2\alpha} = 1
\]
High Aspect Ratio Convergence

Max-CFL=5
Min-CFL=5
Line-Gauss-Seidel Scheme/AR=1000
Steady Navier-Stokes Equations

Preconditioned Equations:

- Time Scales

\[ H + \frac{\varphi}{\varepsilon} = \frac{\varphi}{\lambda} + \frac{\varphi}{\partial \varepsilon} \]

Preconditioned CFL:

\[ \text{CFL}^{\frac{n}{\varepsilon}} = \text{CFL} \]

Include the von Neumann Numbers (VNN)

- However, acoustic CFL's continue to be important
Optimal convergence performance

\[ \frac{\nu \nabla}{\nabla \nabla} \approx \frac{\nu \nabla}{\nabla \nabla} \approx \frac{\nu \nabla}{\nabla \nabla} \approx \nu^{\rho + n} \]

\[ \text{CFL} \]

\[ \frac{\nu}{\nu} \approx \rho \]

\[ \frac{\nu}{\nu} \approx \rho \]

\[ \frac{\nu}{\nu} \approx \rho \]

Preconditioning rescales acoustic-CFL and VNN

Time scales due to damping (VNN) and acoustic-CFL

Low Reynolds Number Limit

Optimization Navier-Stokes Solutions
Optimal convergence for all 2D and some 3D cases:

\[ \frac{M^2 \text{AR}^2}{d \partial} = d' \partial \]

**High Aspect Ratio Preconditioning Choice:**
- Optimize min-CFL and max-VNN in the cell
- Proper preconditioning and time-step necessary
- Viscous damping in the wall normal direction
- Relevant scales are acoustic in the flow direction and boundary layers

**High Reynolds Number Boundary Layers**
Pressure-Based Methods

- FLUENT Algorithm as Example:
  - Most commercial CFD codes are similar
  - Belongs to family of pressure-correction methods

- Non-Linear Under-Relaxation:
  - Relationship to time-marching
  - Belongs to family of pressure-correction methods
  - Most commercial CFD codes are similar

- Pressure-Poisson Solution:
  - Von Neumann stability analysis
  - SIMPLE and SIMPLE-C algorithms

- Segregated Solution:
  - Relationship to time-marching

Discretization:

- Rhie-Chow co-located procedure

Pressure-Based Methods
Under-Relaxation and Time-Step

Discretized Equation with Under-Relaxation

Equivalent Time-Step Definition

Non-Dimensional Time-Step:

\[
\frac{\partial}{\partial t} I = \text{CFL}_x + \text{CFL}_y + \text{UV}
\]

Equivalent Time-Step Definition

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) I = \frac{\partial \rho}{\partial \varphi}
\]

where

\[
\frac{\partial}{\partial \varphi} \left( \frac{\partial}{\partial \varphi} I \right) = 0
\]

Discretized Equation with Under-Relaxation

\[
\frac{x \varphi}{d \varphi} + \left[ q \varphi^q \varphi^{q_u} d n^d v \right] = \left( u^d n_{1+u} + d n \right) d \varphi (\text{CFL}_1)
\]
Current study limited to incompressible limit

\[ 0 = \frac{1}{1+u} \mathbf{r} \frac{\dot{\rho}}{\rho} + \frac{1}{1+u} n \frac{x_T}{\rho} \]

\[ 0 = \frac{x_T}{u} \frac{1}{d\rho} \left[ \mathbf{r} \mathbf{u} \right] + \frac{x_T}{d\rho} \mathbf{u} + n \left( \frac{\dot{\rho}}{\rho} \right) \frac{\dot{\rho}}{\rho} + n \left( \frac{x_T}{\rho} \right) \frac{x_T}{\rho} + \frac{1}{u} \frac{n}{n} \]

Corretor:

\[ 0 = \frac{x_T}{u} \frac{1}{d\rho} + n \left( \frac{\dot{\rho}}{\rho} \right) \frac{\dot{\rho}}{\rho} + n \left( \frac{x_T}{\rho} \right) \frac{x_T}{\rho} + \frac{1}{u} \frac{n}{n} \]

Predictor:

SIMPLE as Fractional Time-Stepping
SIMPLE-C in Delta Form

Notation:

- **Corretcor**

\[
\begin{bmatrix}
\lambda \\
\alpha \\
\mu \\
\delta
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{\partial \phi}{\partial x} \\
0 & 1 & \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0
\end{bmatrix}
\]

- **Predictor**

\[
\begin{bmatrix}
\lambda \\
\alpha \\
\mu \\
\delta
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{\partial \phi}{\partial x} \\
0 & 1 & \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0
\end{bmatrix}
\]

\[\frac{\partial^2 \phi}{\partial x^2} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}
\]

\[\begin{bmatrix}
\lambda \\
\alpha \\
\mu \\
\delta
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{\partial \phi}{\partial x} \\
0 & 1 & \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0
\end{bmatrix}
\]

\[\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}
\]

\[\begin{bmatrix}
\lambda \\
\alpha \\
\mu \\
\delta
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{\partial \phi}{\partial x} \\
0 & 1 & \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0
\end{bmatrix}
\]

\[\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}
\]

\[\begin{bmatrix}
\lambda \\
\alpha \\
\mu \\
\delta
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{\partial \phi}{\partial x} \\
0 & 1 & \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & 0
\end{bmatrix}
\]

\[\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}
\]
Approximate Factorization Error

\[ u \mathcal{R} = d \frac{x \phi}{\ell} (\hat{\mathcal{R}} + \mathcal{R}) + d \frac{x \phi}{\ell} \mathcal{I} + n (\hat{\mathcal{R}} + \mathcal{R}) + n \mathcal{I} \]

- **Continuity**
- **X-Momentum**
- **Combining Predictor and Corrector:** SIMPLE-C: Approximate Factorization
Pressure-Poisson Equation

Substituting Corrector Momentum

Corrector Continuity Equation

Pressure-Under-Relaxation
Discrete Form using Rhie-Chow • Interfacial Velocity • Equivalent Differential Form • Continuity Equation

Note viscosity coefficient has inviscid & viscous scales

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = \nabla \cdot (\rho \mathbf{v} - \rho \mathbf{w}) + \frac{\partial}{\partial x} \frac{\partial}{\partial x} p
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = \nabla \cdot (\rho \mathbf{w} - \rho \mathbf{v}) + \frac{\partial}{\partial y} \frac{\partial}{\partial y} p
\]
**Discrete PPE with Rhie-Chow**

\[
\frac{\partial \rho x}{\partial t} + \nabla \cdot (\rho \mathbf{u} x) = 0
\]

**Equivalence**

\[
\frac{\partial \rho x}{\partial t} + \nabla \cdot (\rho \mathbf{u} x) = 0
\]

**Discrete PPE without Rhie-Chow**

\[
\frac{\partial \rho x}{\partial t} + \nabla \cdot (\rho \mathbf{u} x) = 0
\]

**Rhie-Chow and PPE**
SIMPLE Stability Results

Conditionally stable when pressure is under-relaxed

Unconditionally unstable without pressure under-relaxation

\[ w_u = 0.6, \quad w_p = 0.8 \]

Unconditionally unstable without pressure under-relaxation

Direct PPE Solution

SIMPLE Stability Results
UNCONDITIONALLY STABLE BUT STIFF WITHOUT UNDER-RELAXATION

\[ w_u = 0.8, \quad w_p = 1 \]

\[ \beta \text{ Unconditionally stable when momentum is under-relaxed} \]

SIMPLE-C Stability Results

Direct PPE Solution

\[ I = d \Box n = 1 \]

\[ I = d \Box I = 1 \]
**PPE Linear Solver**

- Not necessary to introduce time-step in continuity/PPE

**PPE Iterative Solution**

- Introduce pseudo-time step similar to preconditioning
- Not necessary to introduce time-step in continuity/PPE

**PPE Direct Solution**

- **Continuity Equation**

\[
0 = \frac{\kappa \rho}{\epsilon} + \frac{\chi \rho}{\epsilon} \frac{d}{d} + \frac{1}{ud} \frac{d}{d}^{d}
\]
Definition of Preconditioning Parameter

\[ \text{CFL} = \min(\text{CFL}_x, \text{CFL}_y) \]

Time-Step Definition

- Not required for line relaxation or point-Gauss-Seidel
- Required for non-diagonally dominant schemes (ADI)
- Required for explicit schemes

Finite-Time Step Choice

- Finite-Time Step
- Required for explicit schemes
- Not required for line relaxation or point-Gauss-Seidel
- Required for non-diagonally dominant schemes (ADI)
- Required for explicit schemes

- High-Reynolds number, \( Re^2 \)
- Low-Reynolds number, \( Re \)
- High-aspect ratio grids, \( AR^2 \)
- Required for explicit schemes
- Required for non-diagonally dominant schemes (ADI)
- Not required for line relaxation or point-Gauss-Seidel

\[ CFLc = \min(CFL_x, CFL_y) \]

- Required for explicit schemes
- Not required for line relaxation or point-Gauss-Seidel
- Required for non-diagonally dominant schemes (ADI)
Comparative Study

Choice of Primary Dependent Variable

- Both methods use pressure for all speed performance
- Non-Linear Iterative Framework
  - Both methods use pressure for all speed performance
  - Under-Relaxation equivalent to time-step
  - Time-step definition similar to explicit time-step
  - Potential limitations for high aspect ratio problems

Discrete Formulation

- Similar to AUSM- and CUSP-like schemes
- Similar to scalar artificial dissipation schemes
- Rhie-Chow adds artificial viscosity to continuity

- Similar to scalar artificial dissipation schemes
- Similar to AUSM- and CUSP-like schemes
Comparative Study (Contd) • Solution Procedure
– Both methods use a form of approximate factorization
– Segregated scheme has an AP error that requires an explicit
  – Like time-step restriction on the transport equations
  – Scaling of the time-step identical to preconditioning
  – Finite time-step for explicit & some implicit (ADI) schemes
  – Other implicit methods do not require finite time-step

Pressure-Poisson Solution
– Under-relaxation of pressure for optimal convergence

Solution Procedure
Conclusions

Not really two distinct families

- Asymptotic analysis provides crucial insight
- Many approaches to obtaining correct discretizations

Discrete Formulation

- But, time-stepping is crucial for insight
- Both under-relaxation and time-marching do the job

Iterative Framework

- Linear system solution
- Discrete formulation
- Non-linear iterative framework

Iterative Framework

- Both under-relaxation and time-marching do the job

Discrete Formulation

- Many approaches to obtaining correct discretizations

Iterative Framework

- But, time-stepping is crucial for insight

Discrete Formulation

- Asymptotic analysis provides crucial insight
Conclusions

- Coupled or segregated solutions
  - Implicit non-linear framework is attractive
  - Linear solver: coupled and segregated are competitive

Potential Limitations of Segregated
  - Limit on the transport equation time-step
  - High aspect ratio problems
  - Strong compressible effects
  - Combustion introduces additional transport equations

General Conclusion
  - Not really distinct methods
  - Each approach enhances understanding
  - Coupled or segregated solutions
    - Linear solver: coupled and segregated are competitive
    - Implicit non-linear framework is attractive
Is Preconditioning Essential?
Is Preconditioning Essential? • NO
Is Preconditioning Essential?

NO

But, the view is great!