Comparison of Density- and Pressure-Based Methods for Low Mach Number Computations

Venke Sankaran and Charles Merkle
Purdue University

Mathematical and Numerical Aspects of Low Mach Number Flows
Porquerolles, France
June 21-25, 2004

Outline

- Introduction
- Density-Based Approach
  - Role of time scales in convergence
- Pressure-Based Approach
  - Under-relaxation in Fluent
  - Discrete formulation
  - Segregated solution procedure
  - Pressure Poisson Solution
- Comparison of the Two Methods
- Conclusions

Introduction

- Two Families of CFD schemes
  - Density-based
  - Pressure-based
- Density-Based Methods
  - Time-marching (optionally with preconditioning)
  - Flux-difference and Flux-Vector schemes
  - Fully-coupled solution framework
- Pressure-Based Methods
  - Usually use under-relaxation for non-linear iterations
  - Staggered or co-located pressure-velocity coupling
  - Segregated solution procedure

Time-Marching Framework

- Steady Euler Equations:

$$\frac{\partial Q}{\partial \tau} + \frac{\partial E_i}{\partial x_i} = 0$$

- Convergence contrrolled by time-scales:
  - Eigenvalues of inviscid Jacobian
  - Expressed as non-dimensional CFL numbers
  - Preconditioned system employs pseudo-acoustic speeds

$$CFL_u = \frac{u\Delta t}{\Delta x_i} \qquad CFL_{u\pm c} = \frac{(u \pm c')\Delta t}{\Delta x_i}$$

Implicit Solution

- Delta Form in 2D:

$$\left[I + \Delta t \frac{\partial A \bullet}{\partial x_i} + \Delta t \frac{\partial B \bullet}{\partial y} \right] \Delta Q = \Delta t \left[\frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \right]^n$$

- ADI Approximate Factorization:

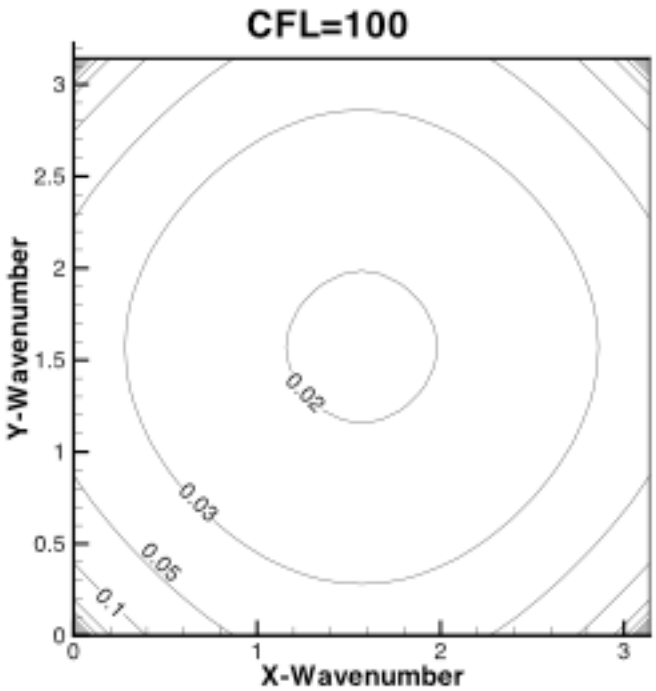
$$\left[I + \Delta t \frac{\partial A \bullet}{\partial x} \right] \left[I + \Delta t \frac{\partial B \bullet}{\partial y} \right] \Delta Q = \Delta t \left[\frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \right]^n$$

- Approximate Factorization Error:

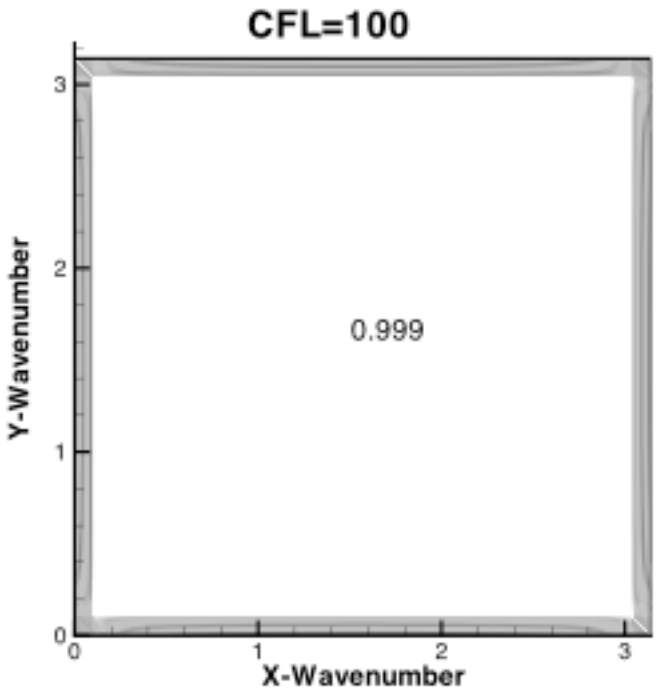
$$\left[I + \Delta t \frac{\partial A \bullet}{\partial x_i} + \Delta t \frac{\partial B \bullet}{\partial y} + \Delta t^2 \frac{\partial A \bullet}{\partial x_i} \frac{\partial B \bullet}{\partial y} \right] \Delta Q = \Delta t \left[\frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \right]^n$$

Von-Neumann Stability Results

Direct



ADI



- Direct Scheme is unconditionally stable
- Approximate factorization causes CFL Limit (5-10)
- Preconditioning insures all CFL numbers are optimum

Grid Aspect Ratio Effects

- Traditional Time-Step Definition:

$$\Delta t = \frac{CFL}{Max(\frac{u+c}{\Delta x}, \frac{v+c}{\Delta y})} \quad \text{or} \quad CFL = Max(CFL_x, CFL_y)$$

- Correct Time-Step Definition:

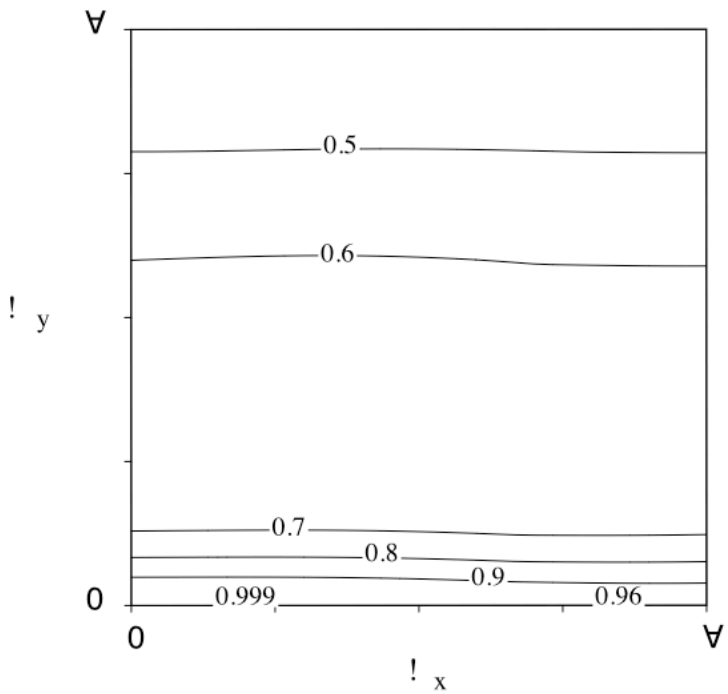
$$\Delta t = \frac{CFL}{Min(\frac{u+c}{\Delta x}, \frac{v+c}{\Delta y})} \quad \text{or} \quad CFL = Min(CFL_x, CFL_y)$$

  – Implicit schemes allow min-CFL choice in 2D
  – Some limitations remain in 3D

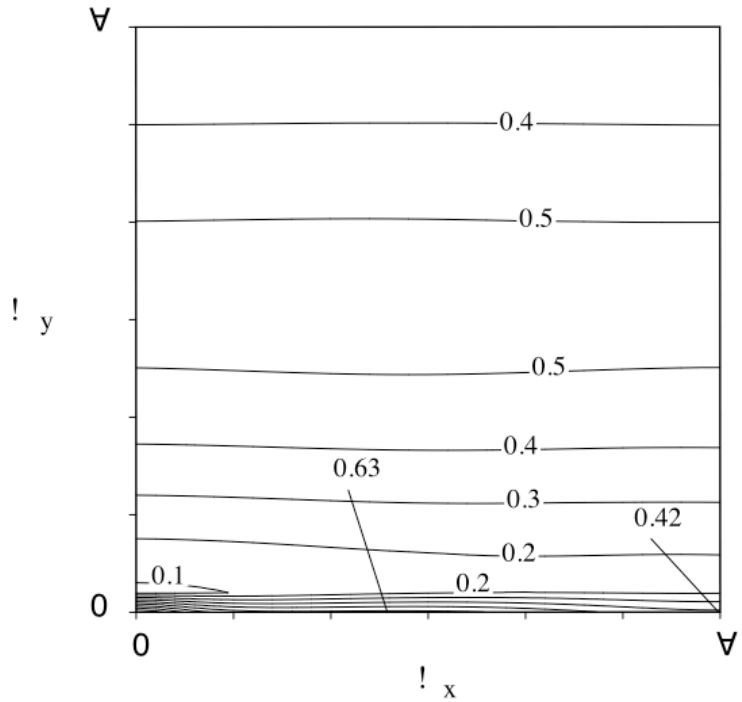
High Aspect Ratio Convergence

Line-Gauss-Seidel Scheme/AR=1000

Max-CFL=5



Min-CFL=5



Steady Navier-Stokes Equations

- Preconditioned Equations:

$$\Gamma \frac{\partial Q}{\partial t} + \frac{\partial E_i}{\partial x_i} = \frac{\partial V_i}{\partial x_i} + H$$

- Time Scales
  - Include the von Neumann Numbers (VNN)
  - However, acoustic-CFL's continue to be important

$$CFL_{u \pm c} = \frac{(u \pm c')\Delta t}{\Delta x_i} \qquad VNN = \frac{\Gamma \Delta t}{\Delta x_i^{\,2}}$$

Optimizing Navier-Stokes Solutions

- Low Reynolds Number Limit
  - Time scales due to damping (VNN) and acoustics (CFL)
  - Preconditioning rescales acoustic-CFL and VNN

$$\rho'_p = \frac{\rho_p}{M^2/Re^2} \quad \rightarrow \quad c' \approx \frac{u}{Re}$$

$$\rightarrow \quad CFL_{u+c'} \approx \frac{u\Delta t}{Re\Delta x} \approx \frac{\nu\Delta t}{\Delta x^2} \approx VNN$$

- Optimal convergence performance

Grid Aspect Ratio Effects

- High Reynolds Number Boundary Layers
  - Relevant scales are acoustics in the flow direction and viscous damping in the wall normal direction
  - Proper preconditioning and time-step necessary
  - Optimize min-CFL and max-VNN in the cell
- High Aspect Ratio Preconditioning Choice:

$$\rho'_p = \frac{\rho_p}{M^2\, AR^2/Re^2}$$

- Optimal convergence for all 2D and some 3D cases

Pressure-Based Methods

- FLUENT Algorithm as Example:
  - Most commercial CFD codes are similar
  - Belongs to family of pressure-correction methods
- Non-Linear Under-Relaxation
  - Relationship to time-marching
- Segregated Solution:
  - SIMPLE and SIMPLE-C algorithms
  - Von Neumann stability analysis
- Pressure-Poisson Solution:
  - Relationship to preconditioning
- Discretization:
  - Rhie-Chow co-located procedure

Under-Relaxation and Time-Step

- Discretized Equation with Under-Relaxation

$$(1-\alpha)a_p\left(u_p^{\,n+1}-u_p^{\,n}\right)=\alpha\left[a_p u_p-\sum_{nb} a_b u_b\right]^{n+1}+\alpha\frac{\partial p^{\,n}}{\partial x}$$

- Equivalent Time-Step Definition

$$\Delta t=\frac{\alpha}{1-\alpha}\frac{\Delta x\Delta y\rho}{a_P} \quad \text{where} \quad \frac{a_P}{\Delta x\Delta y}=\rho\left(\frac{u}{\Delta x}+\frac{v}{\Delta x}+\frac{2\Gamma}{\Delta x^2}+\frac{2\Gamma}{\Delta y^2}\right)$$

- Non-Dimensional Time-Step:

$$CFL_x+CFL_y+2VNN_x+2VNN_y=\frac{\alpha}{1-\alpha}$$

SIMPLE as Fractional Time-Stepping

- Predictor:

$$\rho\frac{u^{*}-u^{n}}{\Delta t}+\frac{\partial}{\partial x}\left(\rho u-\Gamma\frac{\partial}{\partial x}\right)u^{*}+\frac{\partial}{\partial y}\left(\rho v-\Gamma\frac{\partial}{\partial y}\right)u^{*}+\frac{\partial p^{n}}{\partial x}=0$$

- Corrector:

$$\rho\frac{u^{n+1}-u^{n}}{\Delta t}+\frac{\partial}{\partial x}\left(\rho u-\Gamma\frac{\partial}{\partial x}\right)u^{*}+\frac{\partial}{\partial y}\left(\rho v-\Gamma\frac{\partial}{\partial y}\right)u^{*}+\alpha\frac{\partial p^{n+1}}{\partial x}+(1-\alpha)\frac{\partial p^{n}}{\partial x}=0$$

$$\frac{\partial}{\partial x}\rho u^{n+1}+\frac{\partial}{\partial y}\rho v^{n+1}=0$$

- Current study limited to incompressible limit

SIMPLE-C in Delta Form

- Notation:

$$L_x = \frac{\partial}{\partial x}\left(\Gamma u \frac{\partial}{\partial x}\right) \qquad L_y = \frac{\partial}{\partial y}\left(\Gamma v \frac{\partial}{\partial y}\right)$$

- Predictor

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\Delta t(L_x+L_y) & 0 \\ 0 & 0 & 1+\Delta t(L_x+L_y) \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta u \\ \Delta v \end{bmatrix}^{*} = \Delta t \begin{bmatrix} R_c \\ R_x \\ R_y \end{bmatrix}^{n}$$

- Corrector

$$\begin{bmatrix} 0 & \Delta t \frac{\partial}{\partial x}\Gamma & \Delta t \frac{\partial}{\partial y}\Gamma \\ \Delta t \frac{\partial}{\partial x} & 1 & 0 \\ \Delta t \frac{\partial}{\partial y} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta u \\ \Delta v \end{bmatrix} = \Delta t \begin{bmatrix} \Delta p \\ \Delta u \\ \Delta v \end{bmatrix}^{*}$$

SIMPLE-C: Approximate Factorization

- Combining Predictor and Corrector:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\Delta t(L_x+L_y) & 0 \\ 0 & 0 & 1+\Delta t(L_x+L_y) \end{bmatrix} \begin{bmatrix} 0 & -\Delta t\frac{\partial}{\partial x} & -\Delta t\frac{\partial}{\partial y} \\ \Delta t\frac{\partial}{\partial x} & 1 & 0 \\ \Delta t\frac{\partial}{\partial y} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta u \\ \Delta v \end{bmatrix} = -\Delta t \begin{bmatrix} R_c \\ R_x \\ R_y \end{bmatrix}^n$$

- Continuity

$$\Delta t\left(\frac{\partial}{\partial x}\Delta u + \frac{\partial}{\partial y}\Delta v\right) = -\Delta t R_c^{\,n}$$

- X-Momentum

$$\Delta u + \Delta t(L_x+L_y)\Delta u + \Delta t\frac{\partial}{\partial x}\Delta p + \Delta t^2(L_x+L_y)\frac{\partial}{\partial x}\Delta p = -\Delta t R_x^{\,n}$$

Approximate Factorization Error

Pressure-Poisson Equation

- Corrector Continuity Equation

$$\frac{\partial}{\partial x}\rho u^{n+1} + \frac{\partial}{\partial y}\rho v^{n+1} = 0$$

- Substituting Corrector Momentum

$$\nabla^2 p = \frac{1}{\Delta t}\left[\frac{\partial}{\partial x}\rho u^* + \frac{\partial}{\partial y}\rho v^*\right]$$

- Pressure Under-Relaxation

$$p^{n+1} = p^n + \alpha_p p'$$

Discrete Form using Rhie-Chow

- Interfacial Velocity

$$u_{i+1/2}^{RC} = \bar{u}_{i+1/2} - \frac{\Delta t}{a_P}\left[\left(\frac{\partial p}{\partial x}\right)_{i+1/2} - \left(\frac{\partial \bar{p}}{\partial x}\right)_{i+1/2}\right]$$

- Equivalent Differential Form

$$u_{i+1/2}^{RC} = \bar{u}_{i+1/2} + \frac{\Delta t}{a_P}\frac{\Delta x^2}{4}\left[\frac{\partial^3 p}{\partial x^3}\right]_{i+1/2}$$

- Continuity Equation

$$\frac{\partial}{\partial x}\Delta u^{n+1} + \frac{\partial}{\partial y}\Delta v^{n+1} = \Delta t\left[\frac{1}{a_P}\left[\frac{\Delta x^2}{4}\frac{\partial^4 p}{\partial x^4} + \frac{\Delta y^2}{4}\frac{\partial^4 p}{\partial y^4}\right]\right]$$

Note viscosity coefficient has inviscid & viscous scales

Rhie-Chow and PPE

- Discrete PPE without Rhie-Chow

$$\nabla^2 p = \frac{p_{i+2} - 2p_i + p_{i-2}}{4\Delta x^2}$$

- Equivalence

$$\left(\frac{\partial^2}{\partial x^2}\right)_{5pt} = \left(\frac{\partial^2}{\partial x^2}\right)_{3pt} + \frac{\Delta x^2}{4}\left(\frac{\partial^4}{\partial x^4}\right)_{5pt}$$

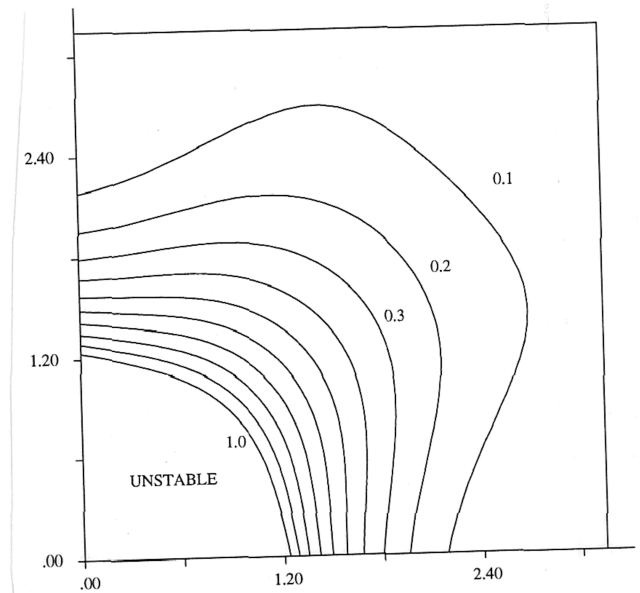
- Discrete PPE with Rhie-Chow

$$\nabla^2 p = \frac{p_{i+1} - 2p_i + p_{i-1}}{2\Delta x^2}$$

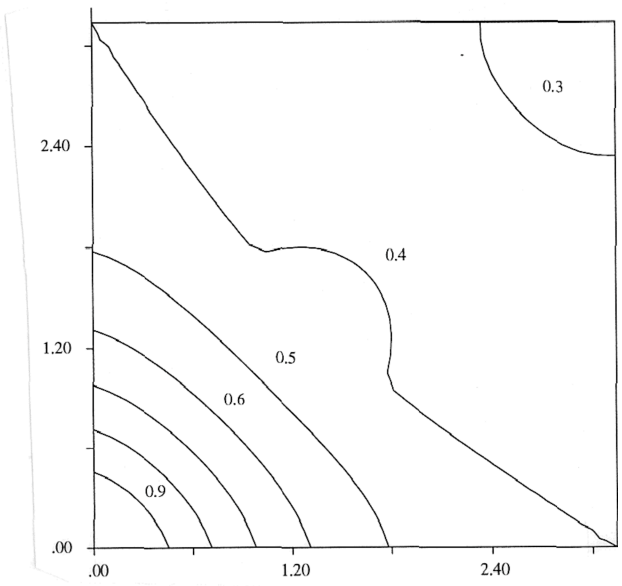
SIMPLE Stability Results

Direct PPE Solution

$\alpha_u = 1, \alpha_p = 1$



$\alpha_u = 0.6, \alpha_p = 0.8$

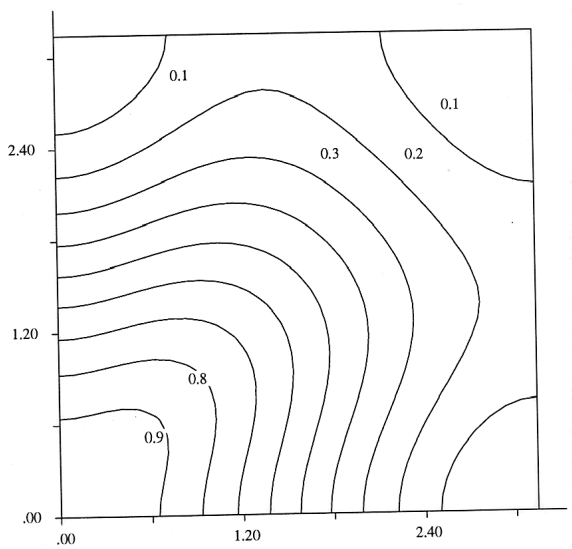


- Unconditionally unstable without pressure under-relaxation
- Conditionally stable when pressure is under-relaxed

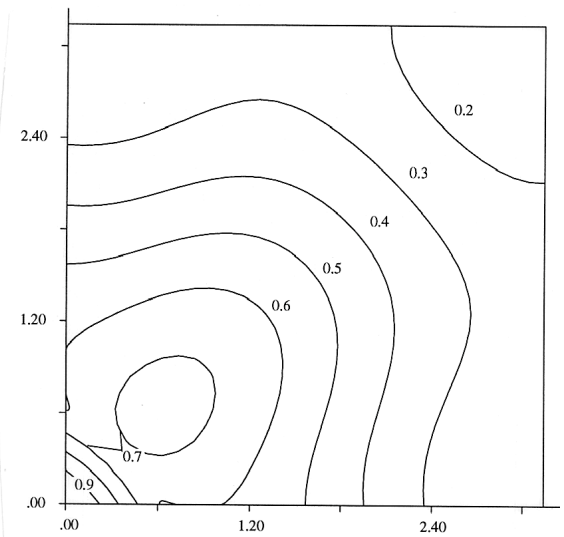
SIMPLE-C Stability Results

Direct PPE Solution

$\alpha_u = 1, \alpha_p = 1$



$\alpha_u = 0.8, \alpha_p = 1$



- Unconditionally stable but stiff without under-relaxation
- Optimal convergence when momentum is under-relaxed

PPE Linear Solver

- PPE Direct Solution
  - Not necessary to introduce time-step in continuity/PPE
- Iterative PPE Solution
  - Introduce pseudo-time step similar to preconditioning
- Continuity Equation

$$\rho_p \frac{p^{n+1} - p^n}{\Delta t} + \frac{\partial}{\partial x}(\rho u^{n+1}) + \frac{\partial}{\partial y}(\rho v^{n+1}) = 0$$

- PPE

$$\rho_p \frac{p^{n+1} - p^n}{\Delta t} + \nabla^2 p = \frac{1}{\Delta t}\left[\frac{\partial}{\partial x}(\rho u^*) + \frac{\partial}{\partial y}(\rho v^*)\right]$$

PPE Time-Step Choice

- Finite-Time Step
  - Required for explicit schemes
  - Required for non-diagonally dominant schemes (ADI)
  - Not required for line relaxation or point-Gauss-Seidel
- Time-Step Definition

$$CFL^c = Min(CFL^c{}_x, CFL^c{}_y)$$

- Definition of Preconditioning Parameter
  - High-Reynolds number, $1/M^2$
  - Low-Reynolds number, Re^2/M^2
  - High aspect ratio grids, Re^2/M^2AR^2

Comparative Study

- Choice of Primary Dependent Variable
  - Both methods use pressure for all speed performance
- Non-Linear Iterative Framework
  - Under-Relaxation equivalent to time-step
  - Time-step definition similar to explicit time-step
  - Potential limitations for high aspect ratio problems
- Discrete Formulation
  - Rhie-Chow adds artificial viscosity to continuity
  - Similar to scalar artificial dissipation schemes
  - Similar to AUSM- and CUSP-like schemes

Comparative Study (Contd)

- Solution Procedure
  - Both methods use a form of approximate factorization
  - Segregated scheme has an AF error that requires an explicit-like time-step restriction on the transport equations
  - Under-relaxation of pressure for optimal convergence
- Pressure-Poisson Solution
  - Finite time-step for explicit & some implicit (ADI) schemes
  - Scaling of the time-step identical to preconditioning
  - Other implicit methods do not require finite time-step

Conclusions

- Not really two distinct families
  - Non-linear iterative framework
  - Discrete formulation
  - Linear system solution
- Iterative Framework
  - Both under-relaxation and time-marching do the job
  - But, time-stepping is crucial for insight
- Discrete Formulation
  - Many approaches to obtaining correct discretizations
  - Asymptotic analysis provides crucial insight

Conclusions

- Coupled or Segregated Solutions
  - Implicit non-linear framework is attractive
  - Linear Solver: coupled and segregated are competetive
- Potential Limitations of Segregated
  - Limit on the transport equation time-step
  - High aspect ratio problems
  - Strong compressible effects
  - Combustion introduces additional transport eqns
- General Conclusion
  - Not really distinct methods
  - Each approach enhances understanding

Is Preconditioning Essential?

Is Preconditioning Essential?

- NO

Is Preconditioning Essential?

- NO
- But, the view is great!