Low Mach number limit of the full Navier-Stokes equations

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Governing equations

Full adimensioned Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \\ \rho \Big(\dfrac{\partial u}{\partial t} + u \cdot \nabla u \Big) + \dfrac{\nabla P}{\varepsilon^2} = \mu \Delta u, \\ \rho \Big(\dfrac{\partial e}{\partial t} + u \cdot \nabla e \Big) + P \operatorname{div} u = \kappa \Delta \mathcal{T}, \end{cases}$$

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with parameters

$\varepsilon \in (0,1]$, $\mu \in [0,1]$ and $\kappa \in [0,1]$,

$\varepsilon \approx Mach,$ $\mu \approx 1/Reynolds$ $\kappa \approx 1/Peclet$

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with parameters

$$\varepsilon \in (0,1], \quad \mu \in [0,1] \quad \text{and} \quad \kappa \in [0,1],$$

the variables are

$$t \in \mathbf{R} \quad \text{and} \quad x \in \mathbf{R}^d.$$

Structural Assumptions

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \\ \rho\Big(\dfrac{\partial u}{\partial t} + u \cdot \nabla u\Big) + \dfrac{\nabla P}{\varepsilon^2} = \mu \Delta u, \\ \rho\Big(\dfrac{\partial e}{\partial t} + u \cdot \nabla e\Big) + P \operatorname{div} u = \kappa \Delta \mathcal{T}, \end{cases}$$

Structural Assumptions

We choose to work with the pressure P and temperature $\mathcal{T}$.
$\rightarrow$ The system is supplemented with two state laws

$$\rho = \rho(P, \mathcal{T}) \quad \text{and} \quad e = e(P, \mathcal{T}),$$

such that

1. existence of an entropy S : $\mathrm{d}S = \frac{1}{\mathcal{T}} \left\{ \mathrm{d}e + P\, \mathrm{d}\left(\frac{1}{\rho}\right) \right\}$;
2. positivity of the coefficients of isothermal compressibility and thermal expansion and of the specific heats:

$$\rho > 0, \quad \frac{\partial \rho}{\partial P} > 0, \quad \frac{\partial \rho}{\partial \mathcal{T}} < 0 \quad \text{and} \quad \frac{\partial e}{\partial \mathcal{T}} \frac{\partial \rho}{\partial P} > \frac{\partial e}{\partial P} \frac{\partial \rho}{\partial \mathcal{T}}.$$

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Example : $\rho(P, \mathcal{T}) := \dfrac{P}{R\mathcal{T}}$ and $e(P, \mathcal{T}) := C_V \mathcal{T}$ $(R > 0\ ,\ C_V > 0)$.

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Motivation : Low Mach number combustion.

Setting

→ We consider: regular solutions.
→ We are interested in the limit $\varepsilon \to 0$.
→ We take into account: general initial data, thermal conduction and general state laws.
→ The analysis contains (at least) two parts:

1. an existence and uniform boundedness result for a time interval independent of the small parameters ε, μ and κ.
2. a convergence result to the solution of a limit equation.

→ We (essentially) restrict ourselves to: the whole space case.

Many results available

Five key assumptions:

$\rightarrow$ isentropic / non-isentropic;

$\rightarrow$ perfect gases / general state laws;

$\rightarrow$ inviscid / viscous fluid;

$\rightarrow$ unbounded / bounded domains;

$\rightarrow$ prepared / general initial data;

Many results available

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Existence of the solutions for a time independent of ε:

S. Klainerman and A. Majda (isentropic, inviscid or vicous fluids, general data, $\mathbf{R}^d$ or $\mathbf{T}^d$).

S. Schochet (non-isentropic, Euler, bounded domains, prepared data).

G. Métivier and S. Schochet (non-isentropic, Euler, general data, $\mathbf{R}^d$ or $\mathbf{T}^d$).

R. Danchin (isentropic N.S. $\mathbf{R}^d$ or $\mathbf{T}^d$, global solutions with critical regularity, small data)

T. A. (non-isentropic, Euler, general data, bounded or exterior domains).

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Incompressible limit

S. Klainerman and A. Majda, S. Schochet for prepared data.

S. Ukai (isentropic, Euler, general initial data, whole space).

H. Isozaki (isentropic, Euler, general data, exterior domain).

G. Métivier and S. Schochet (results in $\mathbf{R}^d$, advances in $\mathbf{T}^d$).

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Many other results. Among others: D. Bresch, R. Danchin, B. Desjardins, I. Gallagher, E. Grenier, D. Hoff, C.-K. Lin, P.-L. Lions, N. Masmoudi, S. Schochet ... .

Uniform stability

Theorem (**T. A. 04**) Uniform existence of solutions having general initial data in appropriate Sobolev spaces $H^s(\mathbf{R}^3)$, together with uniform estimates.

Uniform stability

Theorem (**T. A. 04**) Let s be an integer large enough. For all positive $\underline{P}$, $\underline{\mathcal{T}}$ and M_0, there is a positive T such that for all $a \in A$ and all initial data $(P_0^a, u_0^a, \mathcal{T}_0^a)$ in the Sobolev space $H^{s+1}(\mathbf{R}^3)$ such that P_0^a and $\mathcal{T}_0^a$ take positive values, and

$$\varepsilon^{-1} \|P_0^a - \underline{P}\|_{H^{s+1}} + \|u_0^a\|_{H^{s+1}} + \|\mathcal{T}_0^a - \underline{\mathcal{T}}\|_{H^{s+1}} \leqslant M_0,$$

the Cauchy problem has a unique solution such that

$$(P^a - \underline{P}, u^a, \mathcal{T}^a - \underline{\mathcal{T}}) \in C^0([0,T]; H^s(\mathbf{R}^3))$$

and P^a and $\mathcal{T}^a$ take positive values. In addition there is a positive M depending only on M_0, $\underline{P}$, $\underline{\mathcal{T}}$, and s such that

$$\sup_{a\in A} \sup_{t\in[0,T]} \left\{ \varepsilon^{-1} \|\nabla(P^a(t) - \underline{P})\|_{H^{s-1}} + \|\nabla u^a(t)\|_{H^{s-1}} + \|\mathcal{T}^a(t) - \underline{\mathcal{T}}\|_{H^s} \right\} \leqslant M.$$

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Remark : one can replace $\mathbf{R}^3$ by $\mathbf{R}^d$ with $d \geqslant 3$. In addition, for perfect gases, one can replace $\mathbf{R}^3$ by $\mathbf{R}^d$ or $\mathbf{T}^d$ with $d \geqslant 1$.

Singular system : Step 0 of the proof

The triple (p, u, θ) defined by

$$p := \frac{1}{\varepsilon} \log\Big(\frac{P}{\underline{P}}\Big) \quad \text{and} \quad \theta := \log\Big(\frac{T}{\underline{T}}\Big),$$

solves

$$\begin{cases} \dfrac{\partial p}{\partial t} + u \cdot \nabla p + \dfrac{1}{\varepsilon} g_1(\theta, \varepsilon p) \operatorname{div} u - \dfrac{\kappa}{\varepsilon} \chi_1(\theta, \varepsilon p) \operatorname{div} q = 0, \\ \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \dfrac{1}{\varepsilon} g_2(\theta, \varepsilon p) \nabla p - \mu \chi_2(\theta, \varepsilon p) \Delta u = 0, \\ \dfrac{\partial \theta}{\partial t} + u \cdot \nabla \theta + g_3(\theta, \varepsilon p) \operatorname{div} u - \kappa \chi_3(\theta, \varepsilon p) \operatorname{div} q = 0, \end{cases} \tag{ANS}$$

where $q = e^\theta \nabla \theta$.
The coefficients satisfy

$$\boxed{g_i > 0, \quad \chi_i > 0 \quad \text{and} \quad g_3 \chi_1 < g_1 \chi_3.}$$

Important Features of the System

1) Instability: The linearized equations are not uniformly well-posed. The reason is

$$\nabla g_1(\theta, \varepsilon p) = \mathcal{O}(1) \quad \text{or} \quad \nabla g_2(\theta, \varepsilon p) = \mathcal{O}(1).$$

2) No $L_t^\infty L_x^2$ estimates available when

$$\kappa >> \varepsilon \quad \text{and} \quad \nabla \left(\frac{g_1(\theta, \varepsilon p)}{\chi_1(\theta, \varepsilon p)} \right) = \mathcal{O}(1).$$

The reason is that we cannot factorize

$$\frac{1}{\varepsilon} g_1(\theta, \varepsilon p) \operatorname{div} u - \frac{\kappa}{\varepsilon} \chi_1(\theta, \varepsilon p) \operatorname{div} q.$$

Step 1 of the proof : the norm

DEFINITION Let $T \in [0,+\infty)$, $s \in \mathbf{R}$, $a \in A$.

$$\mathcal{H}^a_{s,T} := \Big\{ (p,u,\theta) \,|\, \mathcal{X}^a_{s,T}(p,u) + \mathcal{P}^a_{s,T}(\varepsilon p, \varepsilon u, \theta) < +\infty \Big\},$$

with

$$\mathcal{X}^a_{s,T}(p,u) := \|\nabla p\|_{L^\infty_T H^{s-1}_x} + \|\nabla u\|_{L^\infty_T H^{s-1}_x} + \mu^{1/2}\, \|\nabla u\|_{L^2_T H^s_x}$$
$$+ \, e\, \|\nabla p\|_{L^2_T H^s_x} + e\, \|\mathrm{div}\, u\|_{L^2_T H^s_x} \,,$$

$$\mathcal{P}^a_{s,T}(\varepsilon p, \varepsilon u, \theta) := \big\|\Lambda_e\big(\varepsilon p, \varepsilon u, \theta\big)\big\|_{L^\infty_T H^s_x}$$
$$+ \, \kappa^{1/2}\, \|\Lambda_e \theta\|_{L^2_T H^{s+1}_x} + \mu^{1/2}\, \|\Lambda_e \varepsilon u\|_{L^2_T H^{s+1}_x} \,.$$

where

$$e := \sqrt{\mu + \kappa} \quad \text{and} \quad \Lambda_e := (I - e^2 \Delta)^{1/2}.$$

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$$\mathcal{P}^a_{s,T}(\varepsilon p, \varepsilon u, \theta) := \big\|\Lambda_e\big(\varepsilon p, \varepsilon u, \theta\big)\big\|_{L^\infty_T H^s_x} + \kappa^{1/2}\,\|\Lambda_e \theta\|_{L^2_T H^{s+1}_x} + \mu^{1/2}\,\|\Lambda_e \varepsilon u\|_{L^2_T H^{s+1}_x}\,.$$

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$$\mathcal{H}^a_{s,0} := \Big\{(p,u,\theta) \,|\, \|\nabla p\|_{H^{s-1}} + \|\nabla u\|_{H^{s-1}} + \|\Lambda_e(\varepsilon p, \varepsilon u, \theta)\|_{H^s} < +\infty \Big\}.$$

THEOREM Let $d \geqslant 3$ be an integer. For all integer $s > 1 + d/2$ and for all real $M_0 > 0$, there is a positive T and a positive M such that for all $a \in A$ and all initial data in $B\big(\mathcal{H}^a_{s,0}; M_0\big)$ the Cauchy problem has a unique classical solution in $B\big(\mathcal{H}^a_{s,T}; M\big)$.

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$\rightarrow$ Estimates for the “linearized” system (additional smoothing effect);
$\rightarrow$ Estimate from the slow components: $(\varepsilon p^a, \varepsilon u^a, \theta^a)$;
$\rightarrow$ Estimate for the incompressible component ($\mathrm{curl}\big(g_2^{-1}(\theta^a, \varepsilon p^a) u^a\big)$);
$\rightarrow$ High frequencies estimates : the commutators between $I - J_{\varepsilon e}$ and the singular operators can be seen as source terms);

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The main part: Low frequencies estimates

$\rightarrow$ Firstly, estimates for $J_{\varepsilon e}(\varepsilon \partial_t)^s (p^a, u^a)$. Secondly, induction using the equations.

The limit system

Fix μ and κ.
The triple $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ solves

$$\begin{cases} \dfrac{\partial p^\varepsilon}{\partial t} + u^\varepsilon \cdot \nabla p^\varepsilon + \dfrac{1}{\varepsilon} g_1(\theta^\varepsilon, \varepsilon p^\varepsilon) \operatorname{div} u^\varepsilon - \dfrac{\kappa}{\varepsilon} \chi_1(\theta^\varepsilon, \varepsilon p^\varepsilon) \operatorname{div} q^\varepsilon = 0, \\ \dfrac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \cdot \nabla u^\varepsilon + \dfrac{1}{\varepsilon} g_2(\theta^\varepsilon, \varepsilon p^\varepsilon) \nabla p^\varepsilon - \mu \chi_2(\theta^\varepsilon, \varepsilon p^\varepsilon) \Delta u^\varepsilon = 0, \\ \dfrac{\partial \theta^\varepsilon}{\partial t} + u^\varepsilon \cdot \nabla \theta^\varepsilon + g_3(\theta^\varepsilon, \varepsilon p^\varepsilon) \operatorname{div} u^\varepsilon - \kappa \chi_3(\theta^\varepsilon, \varepsilon p^\varepsilon) \operatorname{div} q^\varepsilon = 0, \end{cases}$$

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with appropriate initial data.

References : P. Embid, R. Danchin, S. Dellacherie, P.-L.Lions, ... ?

Decay to 0 of the local energy

Only partial result!

PROPOSITION (T. A.) Fix $\mu \in [0,1]$ and $\kappa \in [0,1]$. Assume that $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ satisfy (ANS) and

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in [0,T]} \|(\nabla p^\varepsilon, \nabla u^\varepsilon, \theta^\varepsilon)(t)\|_{H^s} < +\infty,$$

for some fixed $T > 0$ and s large enough. Suppose that the initial data $\theta^\varepsilon(0)$ converge in $H^s(\mathbf{R}^3)$ to a function θ_0 decaying sufficiently rapidly at infinity in the sense that

$$|\theta_0(x)| \leqslant C\,|x|^{-1-\delta}\,, \quad |\nabla\theta_0(x)| \leqslant C\,|x|^{-2-\delta}\,,$$

for some positive constants C and δ. Then $(\nabla p^\varepsilon, \varepsilon\partial_t p^\varepsilon)$ converge strongly to 0 in $L^2_{loc}([0,T] \times \mathbf{R}^3)$.

Proof

The proof relies upon a Theorem of G. Métivier and S. Schochet :
If

$$\varepsilon^2 \partial_t (a^\varepsilon(t,x)\partial_t \phi^\varepsilon) - \mathrm{div}(b^\varepsilon(t,x)\nabla\phi^\varepsilon) = \varepsilon f^\varepsilon(t,x),$$

where

ϕ^ε is bounded in $C^0([0,T];H^2(\mathbf{R}^d))$,

f^ε is bounded in $L^2([0,T];L^2(\mathbf{R}^d))$,

a^ε and b^ε decay to zero at spatial infinity :

$$a^\varepsilon(t,x) \geqslant c, \quad |a^\varepsilon(t,x) - \underline{a}| = \mathcal{O}(|x|^{-1-\delta}), \quad |\nabla a^\varepsilon(t,x)| = \mathcal{O}(|x|^{-2-\delta}),$$

$$b^\varepsilon(t,x) \geqslant c, \quad |b^\varepsilon(t,x) - \underline{b}| = \mathcal{O}(|x|^{-1-\delta}), \quad |\nabla b^\varepsilon(t,x)| = \mathcal{O}(|x|^{-2-\delta}),$$

Then ϕ^ε converges strongly to 0 in $L^2_{loc}([0,T]\times\mathbf{R}^d)$ to $(0,0)$.
We apply this rersult with $(\varepsilon\partial_t)p^\varepsilon$. Then we multiply the momentum equation by ∇p^ε to deduce convergence for ∇p^ε.

Conclusion

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$$p^\varepsilon \to 0,$$

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$$g_1(\theta^\varepsilon, \varepsilon p^\varepsilon) \operatorname{div} u^\varepsilon - \kappa\chi_1(\theta^\varepsilon, \varepsilon p^\varepsilon) \operatorname{div} q^\varepsilon \to 0.$$

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$\rightarrow$ Additional results for perfect gases.