A Mach-uniform pressure correction algorithm

Krista Nerinckx, J. Vierendeels and E. Dick

Dept. of Flow, Heat and Combustion Mechanics
GHENT UNIVERSITY
Belgium
Mach-uniform algorithm

- Mach-uniform accuracy
- Mach-uniform efficiency
- For any level of Mach number
- Segregated algorithm: pressure-correction method
Discretization

- Finite Volume Method (conservative)
- Flux: AUSM+
- OK for high speed flow
- Low speed flow: scaling + pressure-velocity coupling
Mach-uniform efficiency

Convergence rate

- Low Mach: stiffness problem
- Highly disparate values of \( u \) and \( c \)
- Acoustic CFL-limit: \( \frac{(u + c)\Delta t}{\Delta x} \leq CFL_{u+c}^{MAX} \)

\[\text{breakdown of convergence}\]
Mach-uniform efficiency

Remedy stiffness problem:

- Remove acoustic CFL-limit
- Treat acoustic information implicitly

Which terms contain acoustic information?
Mach-uniform efficiency

**Acoustic terms?**

- **Consider Euler equations (1D, conservative):**

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0 \\
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu + \frac{\partial \rho}{\partial x}}{\partial x} &= 0 \\
\frac{\partial \rho E}{\partial t} + \frac{\partial \rho Hu}{\partial x} &= 0
\end{align*}
\]
Mach-uniform efficiency

- Expand to variables $p, u, T$
- Introduce fluid properties: $\rho = \rho(p, T)$
  $e = e(T)$ or $\rho e = \rho e(p, T)$

\[
\rho_p \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \right) + \rho_T \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) + \rho \frac{\partial u}{\partial x} = 0
\]

\[
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0
\]

\[
(\rho e)_p \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \right) + (\rho e)_T \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) + (\rho e + p) \frac{\partial u}{\partial x} = 0
\]
Mach-uniform efficiency

- Construct equation for pressure and for temperature from equations 1 and 3:

\[
\begin{align*}
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + \frac{p}{\rho e_T} \frac{\partial u}{\partial x} &= 0
\end{align*}
\]

quasi-linear system in \( p, u, T \)

with \( c^2 = \left(1 - \frac{\rho_T p}{\rho e_T}\right) \frac{1}{\rho_p} \)
\( c \): speed of sound (general fluid)
Mach-uniform efficiency

- **Identify acoustic terms:**

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0 \]
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \]
\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + \frac{p}{\rho e_r} \frac{\partial u}{\partial x} = 0 \]
• Mach-uniform efficiency

• Go back to conservative equations to see where these terms come from:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0
\]

momentum eq.

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} + \frac{\partial p}{\partial x} = 0
\]
Mach-uniform efficiency

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0 \]

write as

\[
\left[ \rho_p (\rho e)_T - \rho_T (\rho e)_p \right] \left[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \right] + (\rho e)_T \rho \frac{\partial u}{\partial x} - \rho_T (\rho e + p) \frac{\partial u}{\partial x} = 0
\]

continuity eq.

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \]

energy eq.

\[ \frac{\partial \rho E}{\partial t} + \frac{\partial \rho H u}{\partial x} = 0 \]

\[ \rho H u = (\rho e + p) u + \frac{1}{2} \rho u^2 \]
• Special cases:

- **constant density**: \( \rho = \text{cte} \)

\[
(\rho e)_T \, \rho \frac{du}{dx} - \rho_T (\rho e + p) \frac{du}{dx} = 0
\]

energy eq. contains no acoustic information

continuity eq. \((\nabla \cdot \vec{v} = 0)\)
determines pressure
Mach-uniform efficiency

- **perfect gas:**

\[
\rho e = \frac{p}{\gamma - 1}
\]

\[
(\rho e)_T \frac{\partial u}{\partial x} - \rho_T (\rho e + p) \frac{\partial u}{\partial x} = 0
\]

continuity equation contains no acoustic information

energy eq. determines pressure

Department of Flow, Heat and Combustion Mechanics
Mach-uniform efficiency

- perfect gas, with heat conduction:

  → conductive terms in energy equation
  → energy and continuity eq. together
determine pressure and temperature

  → treat conductive terms implicitly
to avoid diffusive time step limit
Successive steps:

- **Predictor (convective step):**
  - continuity eq.: $\rho^*$
  - momentum eq.: $\rho u^*$

- **Corrector (acoustic / thermodynamic step):**

\[
\begin{align*}
  p^{n+1} &= p^n + p' \\
  (\rho u)^{n+1} &= (\rho u)^* + (\rho u)' \\
  T^{n+1} &= T^* + T'
\end{align*}
\]
Implementation

- **Momentum eq.**:
  \[(\rho u)' = f(p')\]

- **Continuity eq.**:
  \[
  \rho_{i}^{n+1} = \rho^* + \rho_T T' + \rho_p p' \\
  (\rho u)_{i+\frac{1}{2}}^{n+1} = (\rho u)^* + (\rho u)'
  \]
  \[\rightarrow p'-T'\text{ equation}\]
Implementation

- **energy eq.**:
  \[
  (\rho E)^{n+1}_i = (\rho E)^* + (\rho e)_p p' + (\rho e)_T T'
  \]
  \[
  (\rho Hu)^{n+1}_{i+\frac{1}{2}} = (\rho e + p)^*(u^* + u') + \frac{1}{2} \rho^* u^* u^*
  \]
  \[
  q^{n+1}_{i+\frac{1}{2}} = K \frac{\partial (T^* + T')}{\partial x}
  \]
  \[\rightarrow \text{p'}-T' \text{ equation}\]

- **coupled solution of the 2 p'-T' equations**

- **special case: perfect gas AND adiabatic**
  \[\rightarrow \text{energy eq. becomes p'}-\text{equation}\]
  \[\rightarrow \text{no coupled solution needed}\]
Test cases

(perfect gas)

1. Adiabatic flow

→ $p'$-eq. based on the energy eq. (NOT continuity eq.)
Results

**Low speed:**

- **Subsonic converging-diverging nozzle**
- **Throat Mach number: 0.001**
- **Mach number and relative pressure:**

![Graph of Mach number and relative pressure over distance.]
• No acoustic CFL-limit

• Convergence plot:

\[ \text{Max(Residue)} \]

\[ \begin{align*}
0 \quad & \text{Cont, CFL=10 (+UR)} \\
10^{-5} \quad & \text{Cont, CFL=1 (+UR)} \\
10^{-10} \quad & \text{Energy, CFL=1} \\
10^{-15} \quad & \text{Energy, CFL=10}
\end{align*} \]

Number of time steps

\[ \begin{align*}
0 \quad & 100 \quad 200 \quad 300 \quad 400
\end{align*} \]

Essential to use the energy equation!

Department of Flow, Heat and Combustion Mechanics
High speed:

- Transonic converging-diverging nozzle
- Normal shock
- Mach number and pressure:
2. **non-adiabatic flow**

- if $p'$-eq. based on the energy eq.
  
  → explicit treatment of conductive terms
  
  → diffusive time step limit: $Ne = \frac{k\Delta t}{\Delta x^2} \leq \frac{1}{2}$

- if coupled solution of $p'$ and $T'$ from continuity and energy eq.
  
  → no diffusive time step limit
## 1D nozzle flow

<table>
<thead>
<tr>
<th>NOZZLE</th>
<th>( \kappa )</th>
<th>( Ne )</th>
<th>stable?</th>
<th>timesteps</th>
<th>calc time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>exp</td>
<td>coup</td>
<td>exp</td>
</tr>
<tr>
<td>Subsonic, ( M_t = 0.01 )</td>
<td>10^{-5}</td>
<td>10^{-3}</td>
<td>yes</td>
<td>yes</td>
<td>220</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>1</td>
<td>yes</td>
<td>yes</td>
<td>259</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>10</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>100</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1000</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
<tr>
<td>Subsonic, ( M_t = 0.1 )</td>
<td>10^{-5}</td>
<td>10^{-4}</td>
<td>yes</td>
<td>yes</td>
<td>1186</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>1</td>
<td>yes</td>
<td>yes</td>
<td>1182</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>10</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>100</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
<tr>
<td>Transsonic</td>
<td>10^{-5}</td>
<td>10^{-5}</td>
<td>yes</td>
<td>yes</td>
<td>3280</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>yes</td>
<td>yes</td>
<td>3169</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>no</td>
<td>yes</td>
<td>-</td>
</tr>
</tbody>
</table>
Thermal driven cavity

- $Ra = 10^3$
- $\varepsilon = 0.6$

- very low speed: $M_{max} = O(10^{-7})$
  \(\Rightarrow\) no acoustic CFL-limit allowed

- diffusion $>>$ convection
  in wall vicinity: very high $\Delta T$
  \(\Rightarrow\) no diffusive Ne-limit allowed
Results

\[ \text{CFL}_u \approx O(1) \]
\[ \text{CFL}_{u+c} \approx O(10^6) \]
\[ Ne \approx 70 \]
Results

Temperature distribution:

Streamlines:
Conclusions:

- **Pressure-correction algorithm with**
  - Mach-uniform accuracy
  - Mach-uniform efficiency

- **Essential:**
  convection (predictor) separated from acoustics/thermodynamics (corrector)
  - General case: coupled solution of energy and continuity eq. for $p'-T'$
  - Perfect gas, adiabatic: $p'$ from energy eq.

- **Results confirm the developed theory**