A Mach-uniform pressure correction algorithm

Krista Nerinckx, J. Vierendeels and E. Dick

Dept. of Flow, Heat and Combustion Mechanics
GHENT UNIVERSITY
Belgium

Mach-uniform algorithm

- Mach-uniform accuracy
- Mach-uniform efficiency
- For any level of Mach number
- Segregated algorithm: pressure-correction method

Discretization

- Finite Volume Method (conservative)
- Flux: AUSM+
- OK for high speed flow
- Low speed flow:
  scaling + pressure-velocity coupling

Convergence rate

- Low Mach: stiffness problem
- Highly disparate values of u and c
- Acoustic CFL-limit: $\dfrac{(u+c)\Delta t}{\Delta x} \leq CFL_{u+c}^{MAX}$

➡ breakdown of convergence

Remedy stiffness problem:

- Remove acoustic CFL-limit
- Treat acoustic information implicitly

?
Which terms contain acoustic information
?

Acoustic terms ?

- Consider Euler equations (1D, conservative) :

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \\ &\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial p}{\partial x} = 0 \\ &\frac{\partial \rho E}{\partial t} + \frac{\partial \rho H u}{\partial x} = 0 \end{aligned}}$$

- Expand to variables *p, u, T*
- Introduce fluid properties : $\rho=\rho(p,T)$
  $e=e(T)$ or $\rho e=\rho e(p,T)$

$$\rho_p\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)+\rho_T\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)+\rho\frac{\partial u}{\partial x}=0$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0$$

$$(\rho e)_p\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)+(\rho e)_T\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)+(\rho e+p)\frac{\partial u}{\partial x}=0$$

- Construct equation for pressure and for temperature from equations 1 and 3:

$$\boxed{\begin{aligned} &\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ &\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \frac{p}{\rho e_T}\frac{\partial u}{\partial x} = 0 \end{aligned}} \longrightarrow \text{quasi-linear system in p, u, T}$$

with $c^2 = \left(1 - \frac{\rho_T p}{\rho e_T}\right)\frac{1}{\rho_p}$ c: speed of sound (general fluid)

- Identify acoustic terms:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \frac{p}{\rho e_T}\frac{\partial u}{\partial x} = 0$$

- Go back to conservative equations to see where these terms come from:

- $$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0}$$

<u>momentum eq.</u>

$$\boxed{\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial p}{\partial x} = 0}$$

- $$\boxed{\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\rho c^2\frac{\partial u}{\partial x}=0}$$

write as

$$\boxed{\left[\rho_p(\rho e)_T-\rho_T(\rho e)_p\right]\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)+(\rho e)_T\underbrace{\rho\frac{\partial u}{\partial x}}-\rho_T\underbrace{(\rho e+p)\frac{\partial u}{\partial x}}=0}$$

<u>continuity eq.</u>

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}=0$$

<u>energy eq.</u>

$$\frac{\partial \rho E}{\partial t}+\frac{\partial \rho H u}{\partial x}=0$$

$$\rho H u=(\rho e+p)u+\frac{1}{2}\rho u^2 u$$

- Special cases:

  - constant density: ρ=cte

$$\boxed{(\rho e)_T \, \rho \frac{\partial u}{\partial x} - \rho_T (\rho e + p) \frac{\partial u}{\partial x}}$$

$\rightarrow$ 0 energy eq. contains no acoustic information

continuity eq. $(\nabla . \vec{v} = 0)$ determines pressure

- perfect gas: $\rho e = \dfrac{p}{\gamma - 1}$

$$\boxed{(\rho e)_T \, \rho \frac{\partial u}{\partial x} - \rho_T (\rho e + p) \frac{\partial u}{\partial x}}$$

0

continuity equation
contains no acoustic
information

energy eq.
determines pressure

- perfect gas, with heat conduction:

→ conductive terms in energy equation
→ energy and continuity eq. together determine pressure and temperature

→ treat conductive terms implicitly to avoid diffusive time step limit

Successive steps:

- Predictor (convective step):
  - continuity eq.: **ρ***
  - momentum eq.: **ρu***
- Corrector (acoustic / thermodynamic step):

$$p^{n+1} = p^{n} + p'$$

$$(\rho u)^{n+1} = (\rho u)^{*} + (\rho u)'$$

$$T^{n+1} = T^{*} + T'$$

- Momentum eq. :

$$(\rho u)' = f(p')$$

- Continuity eq. :

$$\rho_i^{n+1} = \rho^* + \rho_T T' + \rho_p p'$$

$$(\rho u)_{i+\frac{1}{2}}^{n+1} = (\rho u)^* + (\rho u)'$$

→ p'-T' equation

- energy eq. : $(\rho E)_i^{n+1} = (\rho E)^* + (\rho e)_p \, p' + (\rho e)_T \, T'$

$$(\rho H u)_{i+\frac{1}{2}}^{n+1} = (\rho e + p)^* (u^* + u') + \frac{1}{2} \rho^* u^{*2} u^*$$

$$q_{i+\frac{1}{2}}^{n+1} = \mathrm{K} \frac{\partial (T^* + T')}{\partial x}$$

→ p'-T' equation

- coupled solution of the 2 p'-T' equations

- special case: perfect gas AND adiabatic
  → energy eq. becomes p'-equation
  → no coupled solution needed

Test cases

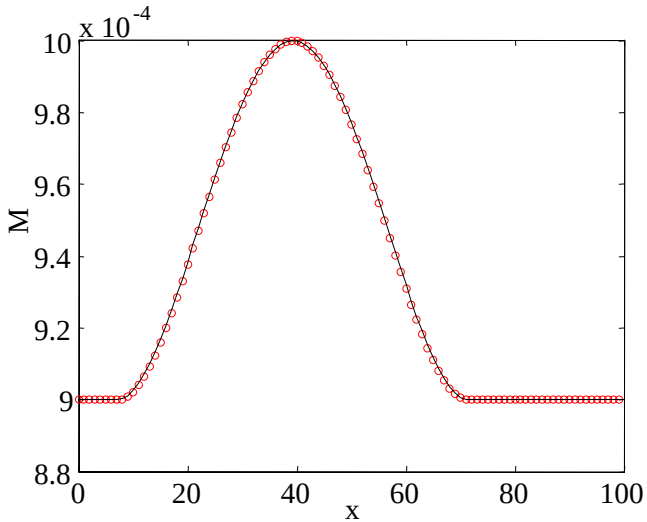
(perfect gas)

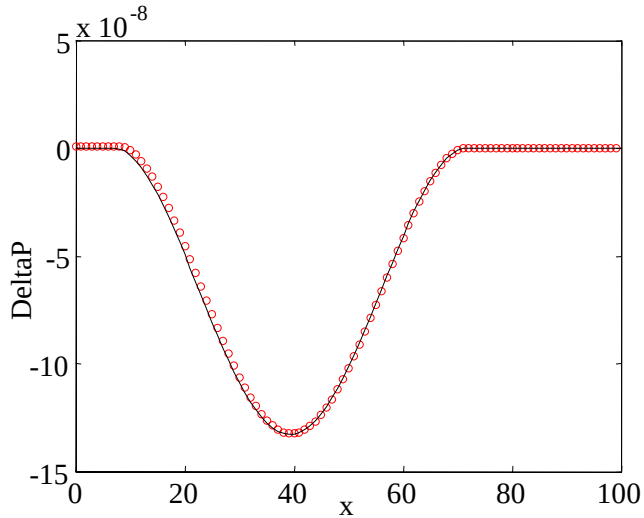
1. Adiabatic flow

→ p'-eq. based on the energy eq.
(NOT continuity eq.)

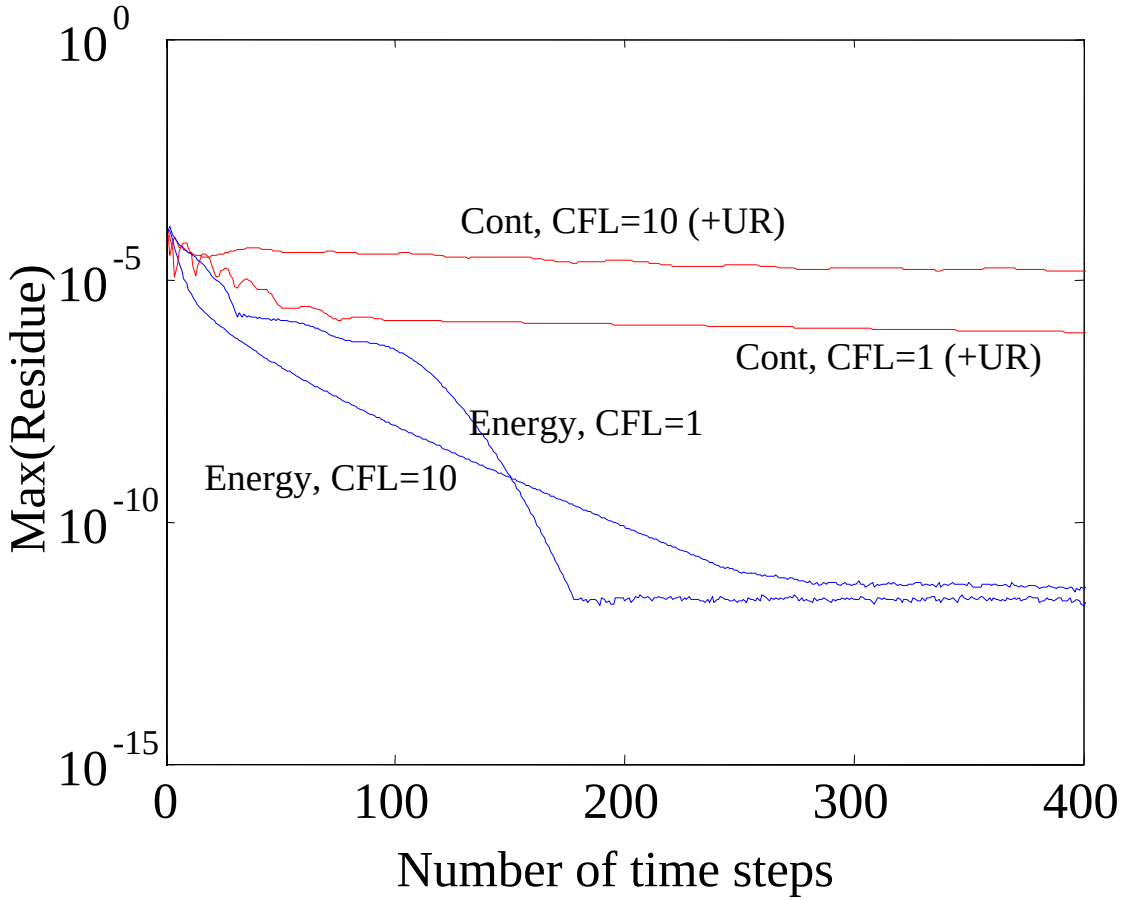
Low speed:

- Subsonic converging-diverging nozzle
- Throat Mach number: 0.001
- Mach number and relative pressure:





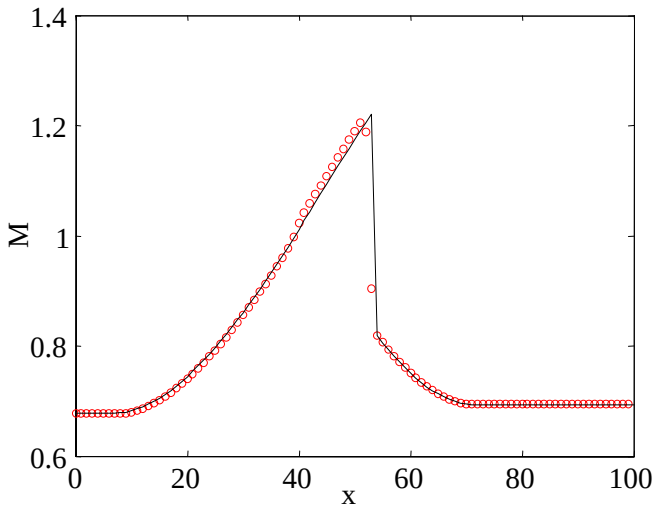
- No acoustic CFL-limit
- Convergence plot:

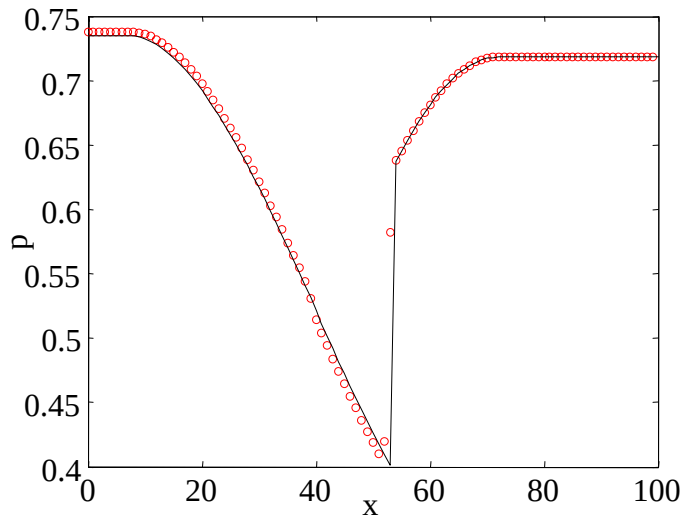


➡ Essential to use the energy equation!

High speed:

- Transonic converging-diverging nozzle
- Normal shock
- Mach number and pressure:





2. *non-adiabatic flow*

- if p'-eq. based on the energy eq.

  → explicit treatment of conductive terms
  → diffusive time step limit: $Ne = \frac{\kappa \Delta t}{\Delta x^2} \leq \frac{1}{2}$

- if coupled solution of p' and T' from continuity and energy eq.

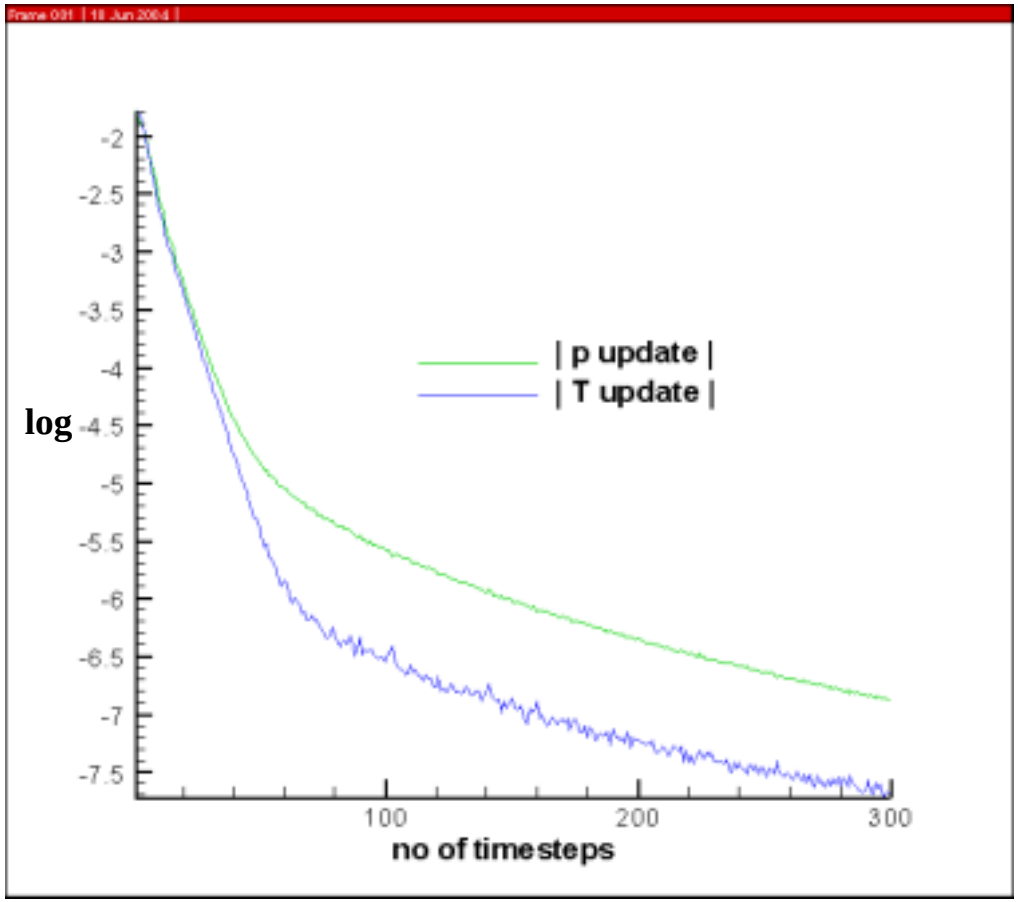
  → no diffusive time step limit

1D nozzle flow

NOZZLE	κ	Ne	*stable?*		*timesteps*		*calc time*	
			exp	*coup*	*exp*	*coup*	*exp*	*coup*
Subsonic	10^{-5}	10^{-3}	yes	yes	220	208	19.6	23.5
$M_t = 0.01$	0.01	1	yes	yes	259	224	22.3	24.5
$Ne \approx \kappa \frac{CFL_u}{0.01}$	0.1	10	no	yes	-	337	-	36.8
	1	100	no	yes	-	646	-	70.4
	10	1000	no	yes	-	222	-	24.5
Subsonic	10^{-5}	10^{-4}	yes	yes	1186	1180	101.2	128.8
$M_t = 0.1$	0.1	1	yes	yes	1182	1176	101.1	128.4
$Ne \approx \kappa \frac{CFL_u}{0.1}$	1	10	no	yes	-	1211	-	132.2
	10	100	no	yes	-	2019	-	228.7
Transsonic	10^{-5}	10^{-5}	yes	yes	3280	3309	290.0	372.9
$Ne \approx \kappa \frac{CFL_u}{1}$	1	1	yes	yes	3169	3200	279.9	359.0
	10	10	no	yes	-	4850	-	549.7
	100	100	no	yes	-	3696	-	413.3

Thermal driven cavity

- Ra = 10^3
- ε = 0.6
- very low speed: $M_{max}=O(10^{-7})$
  → no acoustic CFL-limit allowed
- diffusion >> convection
  in wall vicinity: very high ΔT
  → no diffusive Ne-limit allowed

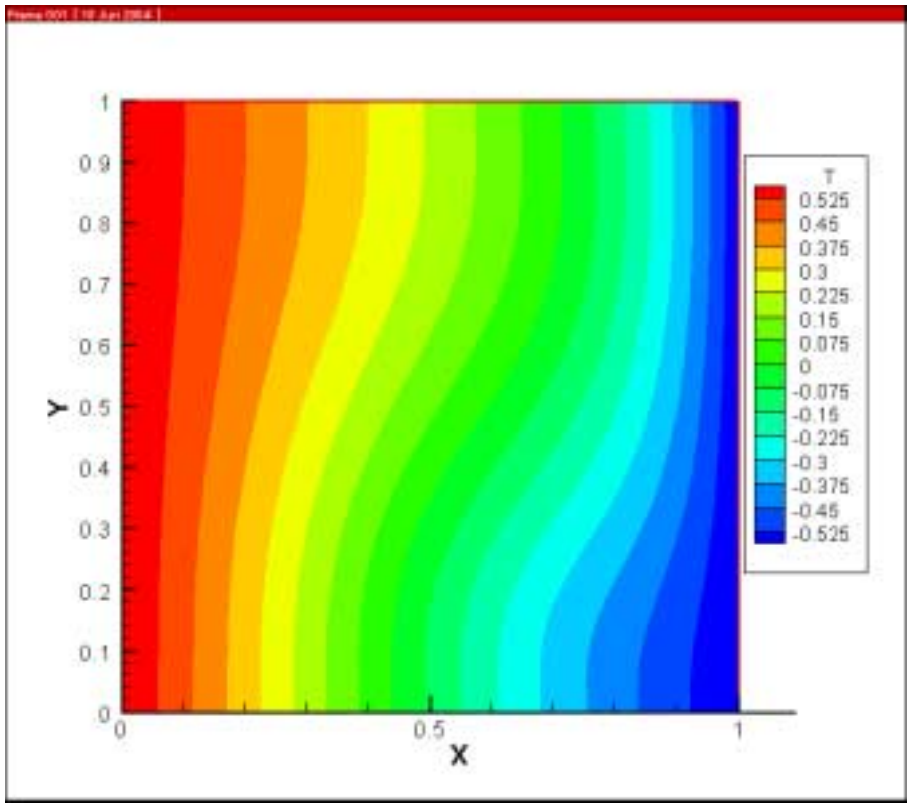


$$CFL_u \approx O(1)$$

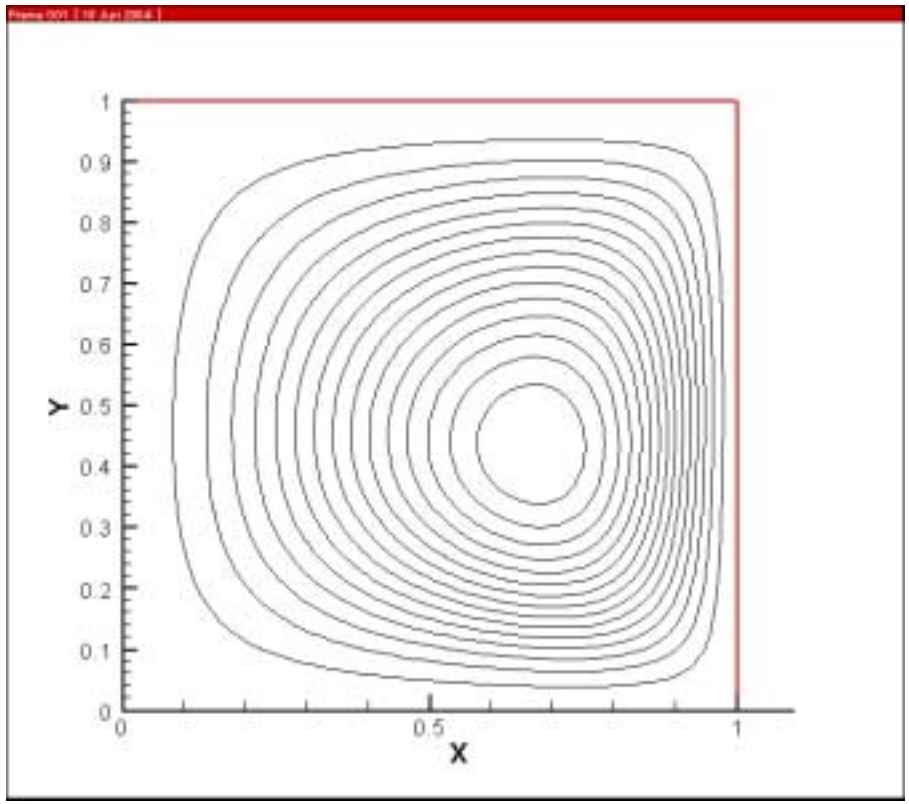
$$CFL_{u+c} \approx O(10^6)$$

$$Ne \approx 70$$

Temperature distribution:



Streamlines:



Conclusions:

- Pressure-correction algorithm with
  - **Mach-uniform accuracy**
  - **Mach-uniform efficiency**
- Essential:
  convection (predictor) separated from acoustics/thermodynamics (corrector)
  - **General case: coupled solution of energy and continuity eq. for p'-T'**
  - **Perfect gas, adiabatic: p' from energy eq.**
- Results confirm the developed theory