Augmented projection methods for incompressible and dilatable flows

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- Goal: to design a numerical method for the simulation:
  - of low Mach number flows,
  - in the incompressible limit,
  - but with a space and time dependent density.

- Applications:
  - natural convection flows,
  - solutal convection flows (dissolution problems),
  - (a preliminary step toward) fire modelling,
  - ...

  beyond the domain of application of the Boussinesq formulation.

- Present work: a novel family of projection methods
  - for incompressible flows,
  - for dilatable flows.

Presentation for the evolutionnary Stokes Problem

$$\left|\begin{array}{ll} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_{\mathrm{D}} \\ \nabla u.n = 0 & \text{on } \partial\Omega_{\mathrm{N}} \end{array}\right.$$

. . . but:

- extrapolation of schemes to Navier-Stokes equations is straightforward, and numerical tests do include problems with non-vanishing Reynolds number,
- analyses may be extrapolated to Navier-Stokes equations with some care to technical problems associated to non-linear terms.

- The variational discrete scheme for the incremental projection method reads:

  (i) (velocity prediction) Find $\tilde{u}^{k+1} \in V_h$ such that, $\forall v \in V_h$:

  $$\frac{1}{\delta t}(\tilde{u}^{k+1} - u^k, v) + (\nabla \tilde{u}^{k+1}, \nabla v) + (\nabla p^k, v) = (f, v)$$

  (ii) (projection) Find $(\tilde{u}^{k+1}, \phi) \in X_h \times M_h$ such that:

  $$\begin{cases} \frac{1}{\delta t}(u^{k+1} - \tilde{u}^{k+1}, v) + (\nabla \phi, v) = 0 & \forall v \in X_h \\ (\nabla \cdot u^{k+1}, q) = 0 & \forall q \in M_h \end{cases}$$

  (iii) (Pressure Correction)

  $$p^{k+1} = p^k + \phi$$

  where:

  $V_h \subset \{v \in \mathbf{H}^1(\Omega)^d$ such that $v = 0$ on $\partial\Omega_\mathbf{D}\}$

  $M_h \subset \{q \in \mathbf{H}^1(\Omega)$ such that $q = 0$ on $\partial\Omega_\mathbf{N}\}$

  $X_h = V_h + \nabla M_h$

  and the pair $V_h \times M_h$ is inf-sup stable.

- Taking $v = \nabla \psi$, $\psi \in M_h$ in the first eq. of (ii) combined with the second one leads to the so-called Poisson problem for the projection step:

  $$(\nabla \phi, \nabla \psi) = \frac{1}{\delta t}(\tilde{u}^{k+1}, \nabla \psi)$$

  and taking $v \in V_h$ in the first eq. of (ii) allows to compute the end of step velocity.

Projection methods suffers from well-known limitations :

- splitting error:
  - Definition : difference between the velocity (resp. the pressure) obtained by the projection method and the velocity (resp. pressure) obtained by the coupled scheme.
  - Known to be of order two in time for the incremental scheme, at least for the velocity (Guermond, 99).
  - But suprinsingly large (in comparison with the coupled scheme errors) when a formally second order in time discretization is used.
- Spurious boundary conditions for the projector ϕ and then, by induction, for the pressure:
  - On $\partial\Omega_{\mathbf{D}}$, artificial homogeneous Neumann boundary condition, necessary to ensure that the normal to the boundary component of the end of step velocity obeys to Dirichlet boundary conditions.
  - On $\partial\Omega_{\mathbf{N}}$, artificial Dirichlet boundary condition necessary to ensure that the second step is a $\mathbf{L}^2(\Omega)$ orthogonal projection, *i.e.* $(u^{k+1} - \tilde{u}^{k+1}, v) = 0$ for any discretely divergence free velocity field in V_h

    . . . Space convergence is lost !

- The Augmented Rotational projection method reads in the semi-discrete form:

(i) (velocity prediction) Find $\tilde{u}^{k+1} \in V_h$ such that:

$$\frac{1}{\delta t}(\tilde{u}^{k+1} - u^k) - \Delta \tilde{u}^{k+1} - r\nabla(\nabla \cdot \tilde{u}^{k+1}) + \nabla p^k = f$$

(ii) (projection) Find $(\tilde{u}^{k+1}, \phi) \in X_h \times M_h$ such that

$$\begin{cases} \frac{1}{\delta t}(u^{k+1} - \tilde{u}^{k+1}) + \nabla \phi = 0 \\ \nabla \cdot u^{k+1} = 0 \end{cases}$$

(iii) (Pressure Correction)

$$p^{k+1} = p^k + \phi - r\nabla \cdot \tilde{u}^{k+1}$$

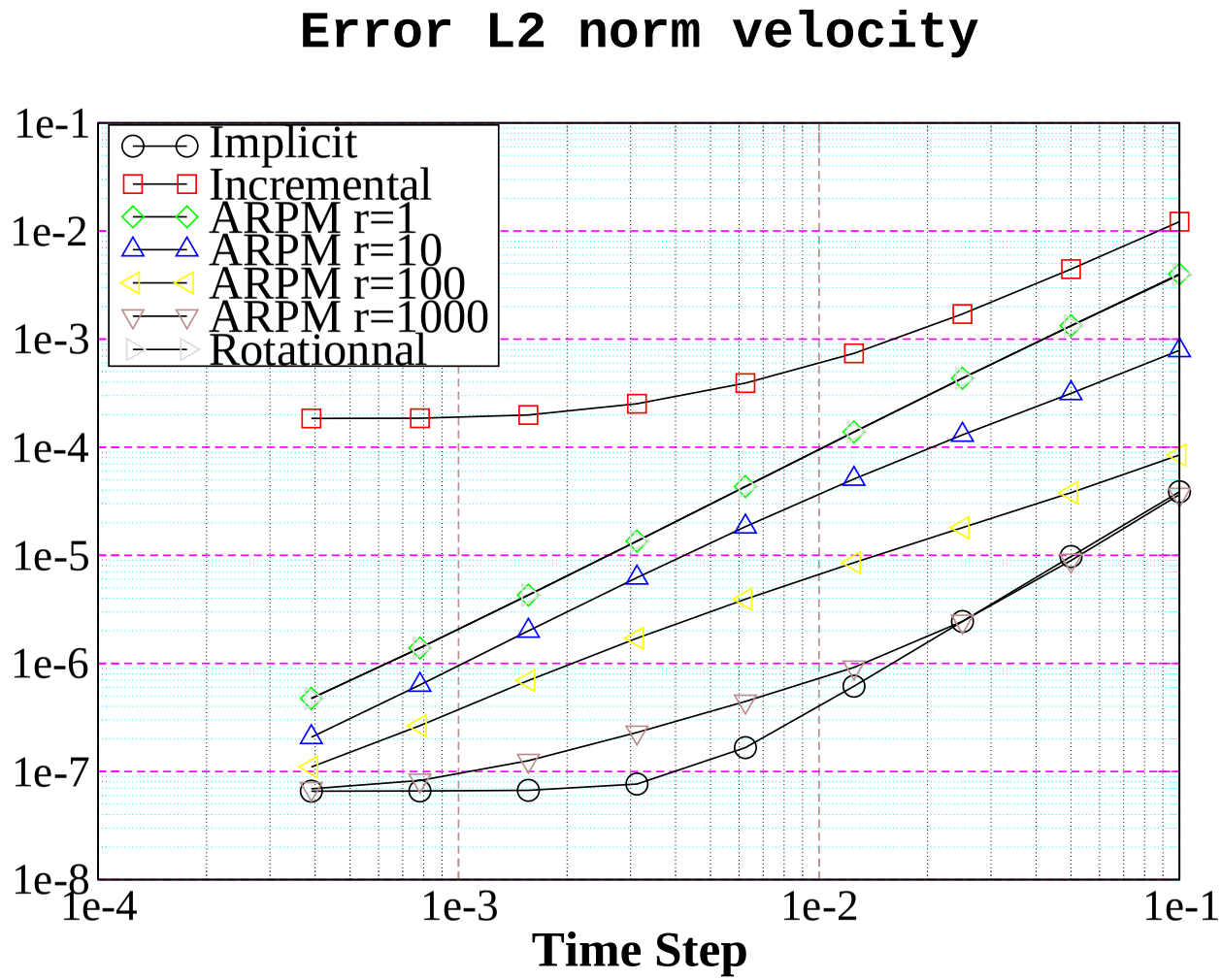
- Idea: the augmentation term in the velocity prediction step enforces, to some extent, the incompressibility constraint.
- To our knowledge, this augmentation has been introduced by Caltagirone & Breil, 1999, in the finite volume context and with a different projection step which gave the name of the method: the vectorial projection method. .
- This method presents also some similarities with the so-called rotational pressure correction projection method ($r = 0$ in the first step and r equal to the viscosity in the third one), introduced by Timmermans et al, 96, and analysed by Guermond & Shen, 2004.

A Stokes flow with Neumann BC

- Computational domain: $\Omega = [0, 1] \times [0, 1]$
- Solution : $u(x, y, t) = \begin{bmatrix} \sin(x)\sin(y+t) \\ \cos(x)\cos(y+t) \end{bmatrix}$

  $p(x, y, t) = \cos(x)\sin(y+t)$
- Boundary conditions: Neumann conditions at $x = 0$, Dirichlet conditions on the velocity on the rest of the boundary.
- Finite element: Taylor-Hood (P_2/P_1).
- Meshing: first build a 40x40 regular grid, then cut each quadrangle in four simplices.
- Second order time discretisation (BDF2).
- Result: difference between the analytical and computed velocity fields at $t = 1s$.

Contents:



- Advantages of augmented Rotational projection scheme :
  - Reduce error splitting

    From an intuitive point of view, the splitting error is due to the fact that the predicted velocity is not divergence free.

    This is confirmed by the analysis:

$$\left|\begin{array}{lcl} (\sum_{k=0}^{n} \delta t \, ||\tilde{e}^{k+1}||_o^2)^{1/2} & \leq & c \, \min(\delta t^2, \frac{\delta t^{3/2}}{r^{1/2}}) \\ (\sum_{k=0}^{n} \delta t \, ||\nabla \tilde{e}^{k+1}||_o^2)^{1/2} & \leq & c \, \frac{\delta t^{3/2}}{r^{1/2}} \\ (\sum_{k=0}^{n} \delta t \, ||\epsilon^{k+1}||_o^2)^{1/2} & \leq & c \, \frac{\delta t^{3/2}}{r^{1/2}} \end{array}\right.$$

    where $\tilde{e}^{k+1}$ and ϵ^{k+1} are respectively the velocity and pressure splitting errors.

  - Reduce pressure boundary layers
    We still have spurious boundary conditions for the projector ϕ ... but no more for the pressure (this point is shared with the rotational pressure correction method).
- Drawback: large values of the augmentation parameter r leads to poorly conditionned linear system in the prediction step.
- So: augmentation is interesting when time steps are large and must be used with caution otherwise (e.g. when a decoupling with the energy balance limits the time step to small values).

- The objective is to design a projection method for the following sub-problem of asympotic equations governing low Mach number flows:

$$\left|\begin{array}{ll} \dfrac{\partial \varrho u}{\partial t} + \nabla \cdot (\varrho u \otimes u) - \Delta u + \nabla p = f & \text{in } \Omega \\ \dfrac{\partial \varrho}{\partial t} + \nabla \cdot \varrho u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_{\mathrm{D}} \\ \nabla u . n = 0 & \text{on } \partial\Omega_{\mathrm{N}} \end{array}\right.$$

where $\varrho = \varrho(X, t)$ is a given fluid density assumed to be independent from the pressure.

- From a mathematical point of view, the pressure plays in the preceding system the same role as in incompressible flows: a Lagrange multiplier associated to a constraint which is the mass balance.

- An incremental projection method for dilatable flows reads:

(i) (velocity prediction) Find $\tilde{u}^{k+1} \in V_h$ such that:

$$\frac{1}{\delta t}\left(\varrho^{k+1}\tilde{u}^{k+1} - q^k\right) + \nabla \cdot (q^k \otimes \tilde{u}^{k+1}) - \Delta \tilde{u}^{k+1} + \nabla p^k = f$$

(ii) (projection) Find $(q^{k+1}, \phi) \in X_h \times M_h$ such that

$$\begin{cases} \frac{1}{\delta t}\left(q^{k+1} - \varrho^{k+1}\tilde{u}^{k+1}\right) + \nabla \phi = 0 \\ \nabla \cdot q^{k+1} = -\frac{1}{\delta t}(\varrho^{k+1} - \varrho^k) \end{cases}$$

(iii) (Pressure Correction)

$$p^{k+1} = p^k + \phi$$

where q^{k+1} stands for the mass flowrate at time $k+1$ (formally, $q^{k+1} = \varrho^{k+1}u^{k+1}$) and is used as advection field in the velocity prediction as well as in possible additional balance equations (e.g. energy balance).

- This method (and the following variants) enters the same variational framework as previously described for the incremental projection method.

- A tentative augmented projection method for dilatable flows reads:

(i) (velocity prediction) Find $\tilde{u}^{k+1} \in V_h$ such that:

$$\frac{1}{\delta t}(\varrho^{k+1}\tilde{u}^{k+1} - q^k) + \nabla \cdot (q^k \otimes \tilde{u}^{k+1}) - \Delta \tilde{u}^{k+1} - r\nabla(\nabla \cdot \varrho^{n+1}\tilde{u}^{k+1}) + \nabla p^k = f + \frac{r}{\delta t}\nabla(\varrho^{k+1} - \varrho^k)$$

(ii) (projection) Find $(q^{k+1}, \phi) \in X_h \times M_h$ such that

$$\begin{cases} \frac{1}{\delta t}(q^{k+1} - \varrho^{k+1}\tilde{u}^{k+1}) + \nabla \phi = 0 \\ \nabla \cdot q^{k+1} = \frac{-1}{\delta t}(\varrho^{k+1} - \varrho^k) \end{cases}$$

(iii) (Pressure Correction)

$$p^{k+1} = p^k + \phi - r(\nabla \cdot \varrho^{n+1}\tilde{u}^{k+1} + \frac{1}{\delta t}(\varrho^{k+1} - \varrho^k))$$

A test against an analytical solution

- Computational domain: $\Omega = [0, 1] \times [0, 1]$
- Solution :

$$u(x, y, t) = \begin{bmatrix} 0.5y(1-y)(2+cos(2\pi t)) \\ 0 \end{bmatrix}$$

$$p(x, y, t) = -(2+cos(2\pi t))(x-0.5)$$

$$\rho(x, y, t) = \rho_0(X(x, y, t)) = 1 + \frac{(X-x_0)^2(-2X+3x_1-x_0)}{(x_1-x_0)^3}$$

with: $X(x, y, t) = x - 0.5y(1-y)(2t + \frac{sin(2\pi t)}{2\pi})$

$$(x_0, x_1) = (0.2, 0.8)$$

- Boundary conditions:

$$\nabla u.\vec{n} = \begin{pmatrix} -p_N \\ 0 \end{pmatrix} \quad \text{at } x = 1$$

Dirichlet conditions on the velocity on the rest of the boundary.

- Finite element: Taylor-Hood (P_2/P_1).
- Meshing: first build a regular 40x40 grid, then cut each quadrangle in four simplices.
- Second order time discretisation (BDF2).
- Result: difference between the analytical and the computed velocity field at $t = 1s$.

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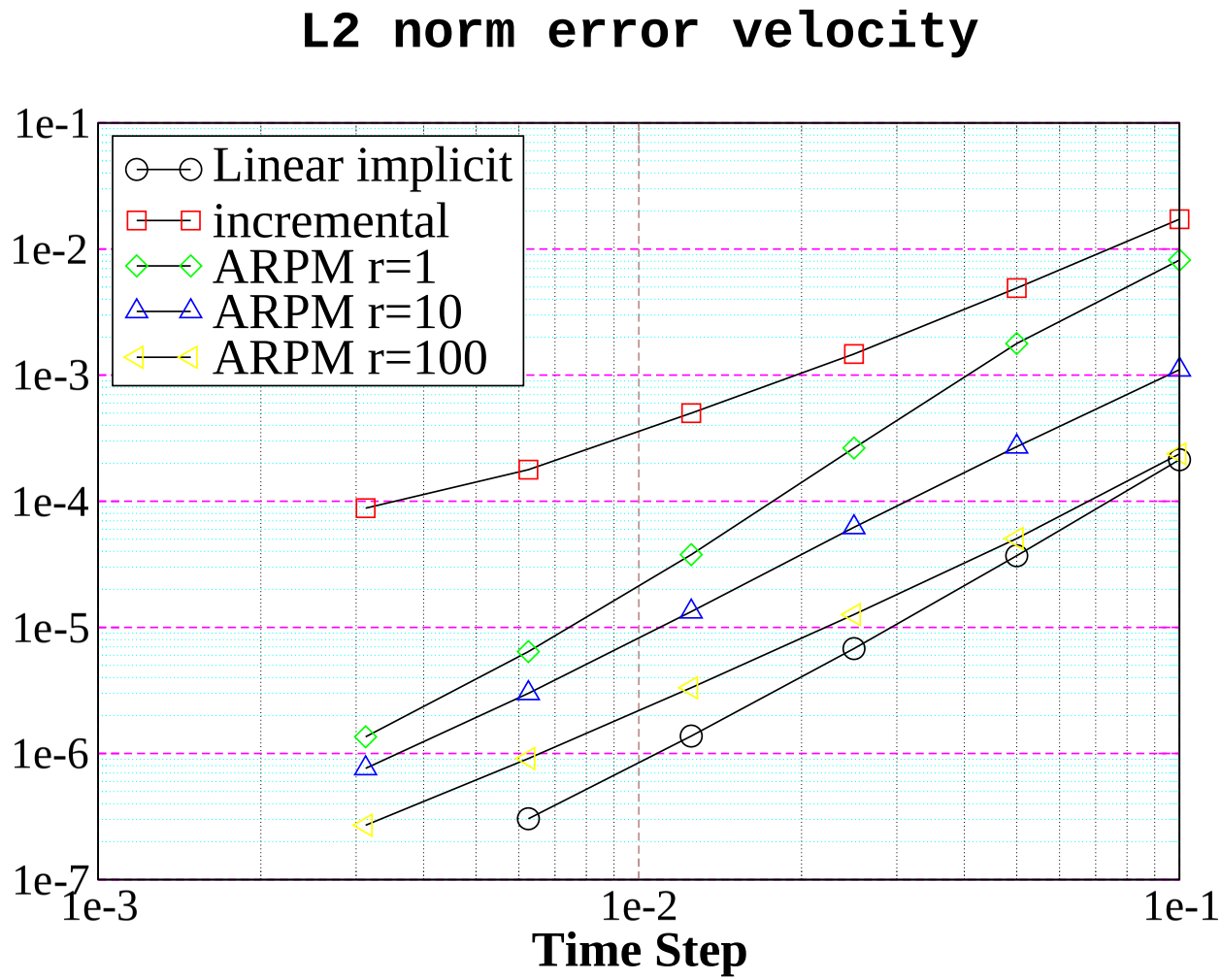

Under adimentionnalized form, the problem under consideration (balance equation for low Mach number flows) reads:

$$\left|\begin{array}{ll}
\varrho\left(\frac{\partial T}{\partial t} + u \cdot \nabla T\right) - \frac{1}{RePr}\nabla \cdot (\kappa \nabla T) = \frac{\gamma - 1}{\gamma}\frac{dP}{dt} & \text{in } \Omega \\
\varrho = \frac{P}{T} & \\
\frac{\partial \varrho u}{\partial t} + \nabla \cdot (\varrho u \otimes u) - \frac{1}{Re}\nabla \cdot \tau + \nabla p = -\frac{\varrho - 1}{Fr^2} e_z & \text{in } \Omega \\
\frac{\partial \varrho}{\partial t} + \nabla \cdot \varrho u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega_{\mathrm{D}} \\
(\tau - p\mathrm{I})(n) = 0 & \text{on } \partial\Omega_{\mathrm{N}}
\end{array}\right.$$

with:

$$\tau = \mu\left(\nabla u + \nabla^t u - \frac{2}{3}(\nabla \cdot u) I\right)$$

IRSN

Contents:

(i) (energy balance) Find $T^{k+1} \in W_h$ such that:

$$\frac{\varrho^k (T^{k+1} - T^k)}{\delta t} + q^k \cdot \nabla T^{k+1} - \nabla \cdot \left(\frac{\kappa^k}{RePr} \nabla T^{k+1}\right) = \frac{\gamma - 1}{\gamma} \frac{P^{k+1} - P^k}{\delta t}$$

(ii) (thermodynamic pressure computation) Find P^{k+1} such that:

$$P^{k+1} = \frac{P^k \int_\Omega \frac{1}{T^k}}{\int_\Omega \frac{1}{T^{k+1}}}$$

(iii) (density update, at each integration point) :

$$\varrho^{k+1} = \frac{P^{k+1}}{T^{k+1}}$$

(iv) (velocity prediction) Find $\tilde{u}^{k+1} \in V_h$ such that:

$$\frac{\varrho^{k+1} \tilde{u}^{k+1} - q^{k+1}}{\delta t} + \nabla \cdot (q^k \otimes \tilde{u}^{k+1}) - \frac{1}{Re} \nabla \cdot \tau(\tilde{u}^{k+1}) + \nabla p^k = \frac{\varrho^{k+1} - 1}{Fr^2} e_z$$

(v) (projection) Find $(q^{k+1}, \phi) \in X_h \times M_h$ such that

$$\begin{cases} \frac{1}{\delta t} (q^{k+1} - \varrho^{k+1} \tilde{u}^{k+1}) + \nabla \phi = 0 \\ \nabla \cdot q^{k+1} = \frac{-1}{\delta t} (\varrho^{k+1} - \varrho^k) \end{cases}$$

(vi) (Pressure Correction)

$$p^{k+1} = p^k + \phi$$