Low Mach number limit for viscous compressible flows

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It is common sense that *slightly compressible* flows and *incompressible* flows do not differ much from one another. As a matter of fact, the so-called incompressible Navier-Stokes equations

$$(NS) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = g, \\ \operatorname{div} u = 0, \end{cases}$$

are generally considered as relevant for describing slightly compressible liquids as water for instance. In the talk, we shall discuss to what extent the above considerations may be made rigorous from a mathematical viewpoint.

For the sake of simplicity, we restrict ourselves to the case of *barotropic* fluids with periodic boundary conditions or in the whole space. After suitable rescaling, the equations for a slightly compressible fluid read

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div} \rho^\varepsilon u^\varepsilon = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu \Delta u^\varepsilon - (\lambda + \mu)\nabla \operatorname{div} u^\varepsilon + \frac{\nabla P^\varepsilon}{\varepsilon^2} = \rho^\varepsilon f^\varepsilon, \end{cases}$$

where the Mach number ε is bound to tend to 0.

We focus on the case of *ill-prepared* data : the initial density is assumed to be of the form $\rho_0^\varepsilon = 1 + \varepsilon b_0^\varepsilon$ with $b_0^\varepsilon \to b_0$, the initial velocity u_0^ε tends to some u_0 and the external force f^ε tends to some vector field f. As regards regularity, we assume that the data belong to some functional space with *critical* exponent with respect to the *scaling* of the equations.

In both cases – the periodic one and the whole space one – we show that u^ε tends to the solution u of (NS) with data $\mathcal{P}u_0$ and $g = \mathcal{P}f$ where $\mathcal{P}$ stands for the projector on solenoidal vector fields.

In the whole space case, the proof of convergence relies on dispersive inequalities. As a result, it holds in a strong sense in a suitable functional space. In contrast, in the periodic case, only weak convergence may be proved and the gradient part of the velocity is likely to oscillate forever. In both cases however, global existence may be shown for small Mach number provided that the corresponding limit system (NS) has a global strong solution. This is of particular interest in dimension $N = 2$ where the limit system has a global solution for possibly large data.