Moment realizability in Levermore's moment closure for kinetic equations: classical and relativistic cases.

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Contents

Introduction - Aim.

Moment Closure for Kinetic Theories

A generic kinetic equation : the Boltzmann equation

$$\partial_t f + \mathbf{v}.\nabla_x f = C(f)$$

where $f = f(t, \mathbf{x}, \mathbf{v})$ is the density function and $C(f)$ is an interaction term between particles acting only on the $\mathbf{v}$ dependance of f.

Levermore moments system.

1. Moments.
Let $\mathbb{M}$ be a linearly independent finite set of functions of $\mathbf{v}$ with vector basis $\mathbf{m}(\mathbf{v})$

2. Integration upon $\mathbf{v}$.

$$\partial_t \int \mathbf{m}(\mathbf{v})\, f\, d\mathbf{v} + \nabla_x . \int \mathbf{v}\mathbf{m}(\mathbf{v})\, f\, d\mathbf{v} = \int \mathbf{m}(\mathbf{v}) C(f) d\mathbf{v}$$

3. Closure relation : $f(.,.,\mathbf{v}) \longrightarrow G(.,.,\mathbf{v}) = \exp(\alpha(t,\mathbf{x}).\mathbf{m}(\mathbf{v}))$ with

$$\int \mathbf{m}(\mathbf{v}) \exp(\alpha.\mathbf{m}(\mathbf{v})) d\mathbf{v} = \int \mathbf{m}(\mathbf{v})\, f\, d\mathbf{v}$$

α are the Lagrange multipliers of the moments of f with respect to the entropy functional $H(f) = \int (f \ln f - f) d\mathbf{v}$

Setting of the problems.

1. What kind of spaces $\mathbb{M}$?

- (I) Conservation laws

$$\mathbb{M}_0 = span[1, v_1, v_2, v_3, |\mathbf{v}|^2] \subset \mathbb{M}$$

- (II) $\mathbb{M}$ is compatible with the Galilean invariance.
- (III) The cone

$$\mathbb{M}_c = \{m \in \mathbb{M} : \int \exp(\mathbf{m}(\mathbf{v}))\, d\mathbf{v} < +\infty\}$$

  has nonempty interior in $\mathbb{M}$.

$\Rightarrow$ Characterization of such spaces ?

2. Minimization Principle

G is formally the solution to

$$H(G) = \min_{g \in C_f} H(g),$$
$$C_f = \{g \;/\; \int \mathbf{m}(\mathbf{v}) g d\mathbf{v} = \int \mathbf{m}(\mathbf{v}) f d\mathbf{v}\}$$

$\Rightarrow$ Existence and uniqueness of the solution ?

3. Extension to the relativistic case

Main results in the classical case.

Admissible spaces : examples

[Bagland-Degond-Lemou, JSP, 125, 3, 2006]
The characterization is based on the representation theory of Lie algebra.

Classical moment spaces

$$\begin{aligned} \text{degree} = 2 \quad & \text{span}(1, v, |v|^2), & \text{(admissible)} \\ & \text{span}(1, v, v \otimes v), & \text{(admissible)} \\ \text{degree} = 3 \quad & \text{span}(1, v, v \otimes v, v|v|^2), & \text{(non admissible)} \end{aligned}$$

(Grad 13-moment space)

$$
\begin{aligned}
\text{degree} = 4 \quad & \mathrm{span}(1, v, v\otimes v, v|v|^2, |v|^4), && \text{(admissible)}\\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^2(v\otimes v - |v|^2/3 I_3)), && \text{(non admissible)}\\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v), && \text{(admissible)}
\end{aligned}
$$

Remark

Let $\mathbb{M}$ be an admissible space of maximal degree N then up to a linear combination of term of highest degree there exists a monomial $m_{sup}(\mathbf{v})$ which is equivalent to the norm $|\mathbf{v}|^N$

Preliminary result

Theorem

[J. S - 2004] The minimization problem has a unique solution in $\exp(\mathbb{M})$ *contained in the* L^1*-closure of* C_f.

(see also [Junk - 2000])

Example

Let $\mathbb{M} = \tilde{\mathbb{M}}_{N-1} \oplus \mathbb{R}|\mathbf{v}|^N$ where $\tilde{\mathbb{M}}_{N-1}$ is of maximal degree $N - 1$ then

$$\int m(\mathbf{v})\, G\, d\mathbf{v} = \int m(\mathbf{v})\, f d\mathbf{v}, \quad \forall m \in \tilde{\mathbb{M}}_{N-1}$$

but

$$\int m(\mathbf{v})\, G\, |\mathbf{v}|^N d\mathbf{v} \leq \int f\, |\mathbf{v}|^N d\mathbf{v}$$

The qualification of the moment(s) of highest degree may drop in !

General structure of the solution

Theorem

The solution G to the minimization problem (1) has one of the following structure :

- *either* $\int m_{sup}(\mathbf{v})\, G(\mathbf{v})\, d\mathbf{v} = \int m_{sup}(\mathbf{v})\, f(\mathbf{v})\, d\mathbf{v}$ *then* $G \in C_f$*, i.e* **all** *constraints are qualified,*
- *or* $\int m_{sup}(\mathbf{v})\, G(\mathbf{v})\, d\mathbf{v} < \int m_{sup}(\mathbf{v})\, f(\mathbf{v})\, d\mathbf{v}$ *then* **all** *constraints corresponding to a non signed homogeneous moment of degree N are not qualified and* **all** *Lagrange multipliers of nonnegative moments of G are either null or negative. Nevertheless* **all** *constraints of degree less or equal to* $N-1$ *are qualified.*

Relativistic case.

relativistic Boltzmann equation

Let

$$p = \gamma m v, \qquad \gamma = \left(1 - \frac{|v|^2}{c^2}\right)^{-\frac{1}{2}},$$

then the relativistic Boltzmann equation reads

$$\partial_t f + p \cdot \nabla_x f = C_R(f),$$

or with

$$\vec{x} = (ct, \mathbf{x}), \;\; \vec{p} = (\mathbf{e}/c,\, p) = (p^j)_{0\le j\le 3}, \;\; \mathbf{e} = c\sqrt{m^2c^2 + |p|^2} \tag{1}$$

it also reads

$$p^j \frac{\partial f}{\partial x^j} = m\,\gamma(p)\, C_R(f)$$

Levermore moments system.

1. Moments.
Let $\mathbb{M}$ be a linearly independent finite set of functions of $\vec{p}$ with vector basis $\boldsymbol{\varphi}(\vec{p})$
2. Integration upon p.

$$\frac{\partial}{\partial x^j}\int_{\mathbb{R}^3} p^j \mathbf{m}(\vec{p}) f(\vec{x},\vec{p}) \frac{dp}{\gamma(p)} = m \int_{\mathbb{R}^3} \mathbf{m}(\vec{p}) C_R(f,f)(\vec{x},\vec{p}) dp.$$

3. Closure relation : $f(.,p) \longrightarrow G(.,\vec{p}) = \exp(\alpha.\mathbf{m}(\vec{p}))$ with

$$\int \mathbf{m}(\vec{p}) \exp(\alpha.\mathbf{m}(\vec{p})) dp = \int \mathbf{m}(\vec{p}) f(\vec{p}) dp$$

1. What kind of spaces $\mathbb{M}$?

- (I) Conservation laws

$$\mathbb{M}_0 = span[1, p, \mathbf{e}] \subset \mathbb{M}$$

- (II) $\mathbb{M}$ is invariant under any proper Lorentz transformation and any rotation in the momentum space.
- (III) The cone

$$\mathbb{M}_c = \{m \in \mathbb{M} : \int \exp(\vec{p})\, dp < +\infty\}$$

has nonempty interior in $\mathbb{M}$.

Admissible spaces : examples

Result due to [Bagland-Degond-Lemou, 2006]

$$
\begin{array}{ll}
\text{maximal degree} = 1 & \mathbb{M} = \text{span}(1, \vec{p}), \\
\text{maximal degree} = 2 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p} \otimes \vec{p}), \\
\text{maximal degree} = 3 & \mathbb{M}_3 = \text{span}(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p}), \\
 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p}), \\
\text{maximal degree} = 4 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p}), \\
 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p}),
\end{array}
$$

Remark

Admissible spaces of odd maximal degree N exist due to $\mathbf{e}^N \subset \mathbb{M}$.

Example of moment system with $G \in \exp(\mathbb{M}_3)$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} G dp + \nabla_x \cdot \int_{\mathbb{R}^3} G\, v dp = 0,$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} G\, p dp + \nabla_x \cdot \int_{\mathbb{R}^3} G\, v \otimes p dp = 0,$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} G\, \mathbf{e} dp + c^2 \nabla_x \cdot \int_{\mathbb{R}^3} G\, p dp = 0,$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} G\, \mathbf{e} p \otimes p dp + c^2 \nabla_x \cdot \int_{\mathbb{R}^3} G\, p \otimes p \otimes p dp = \int_{\mathbb{R}^3} C_R(G) \mathbf{e} p \otimes p dp,$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} G\, p \otimes p \otimes p dp + \nabla_x \cdot \int_{\mathbb{R}^3} G\, v \otimes p \otimes p \otimes p dp = \int_{\mathbb{R}^3} C_R(G) p \otimes p \otimes p dp.$$

General structure of the solution

Theorem

The solution G to the minimization problem (1) has one of the following structure :

- *either* $\int \mathbf{e}^N G(\vec{p})\, dp = \int e^N G(\vec{p})\, dp$ *then* $G \in C_f$, *i.e* **all** *constraints are qualified,*
- *or* $\int \mathbf{e}^N G(\vec{p})\, dp < \int e^N G(\vec{p})\, dp$ *then* **all** *constraints corresponding to a non signed homogeneous moment of degree N are not qualified and* **all** *Lagrange multipliers of nonnegative moments of G are either null or negative. Nevertheless* **all** *constraints of degree less or equal to* $N-1$ *are qualified.*

Proof :

Lemma (Scheffé)

Let $(f_n)_n$ *be a sequence in* $L^1(\mathbb{R}^3)$ *for a given measure* μ *and assume that* $f_n \longrightarrow f$ *a.e. Then the two following assertion are equivalent*

i) $\int |f_n| \, d\mu \longrightarrow \int |f| \, d\mu$

ii) $\int |f_n - f| \, d\mu \longrightarrow 0$

Take $|f_n| = g_n \to G$ be a minimizing sequence in C_f and $d\mu = \mathbf{e}^N \, dp$