Gyrocenter Gauge Kinetic Theory and Algorithm for Radio-Frequency Waves in Plasmas

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http://www.pppl.gov/~hongqin/Gyrokinetics.php

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Classical gyrokinetics: average out gyrophase

Highly oscillatory

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+e_j\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial}{\partial \mathbf{p}}\right]f_j(\mathbf{x},\mathbf{p},t)=0\ ,$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\sum_j e_j\int d^3p\,\mathbf{v}f_j(\mathbf{x},\mathbf{p},t)+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\ ,$$

$$\nabla\cdot\mathbf{E}=4\pi\sum_j e_j\int d^3p f_j(\mathbf{x},\mathbf{p},t)\ ,$$

$$\nabla\cdot\mathbf{B}=0.$$

Does not work

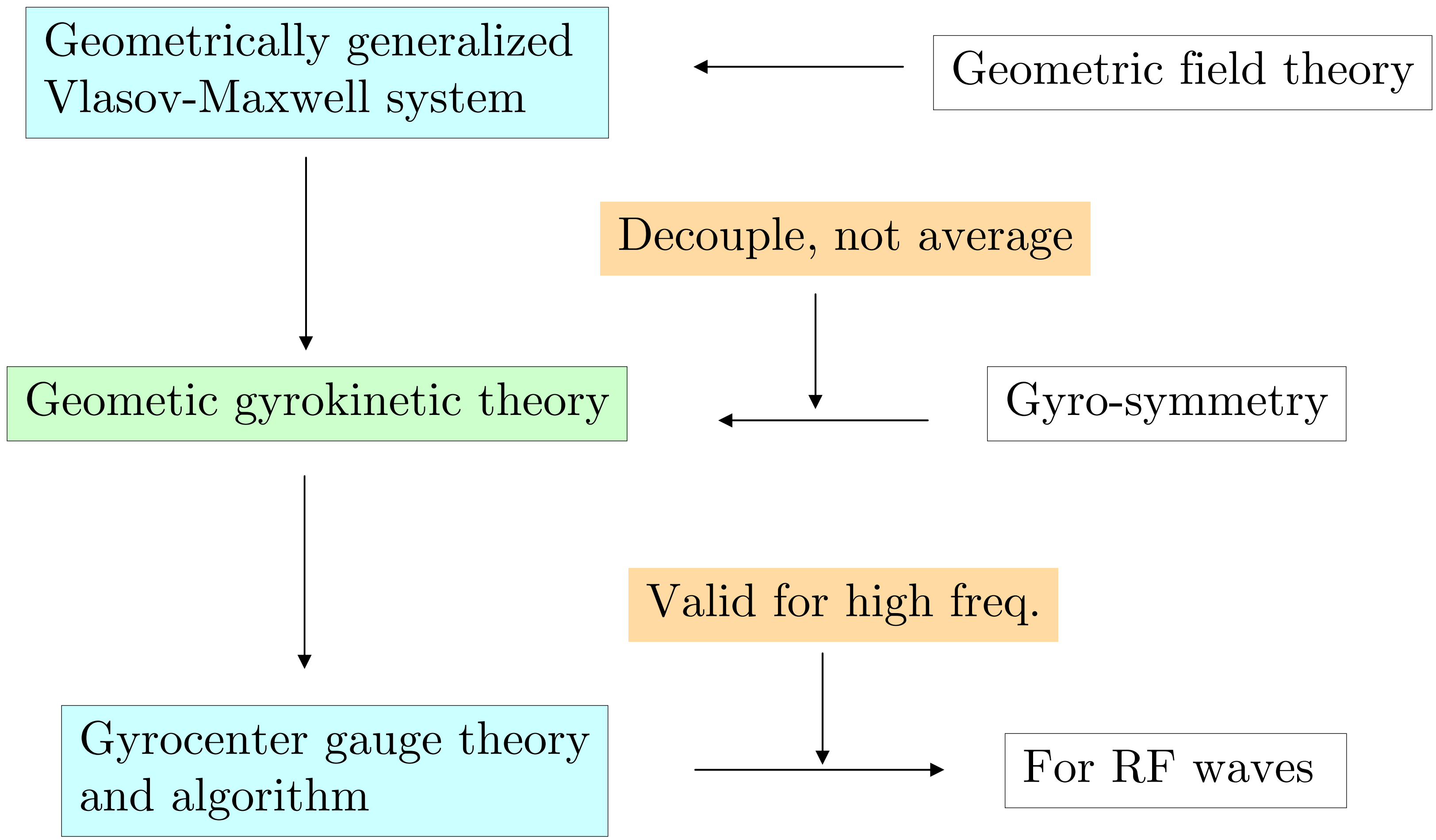
- Magnetized plasmas → fast gyromotion.
- "Average-out" the fast gyromotion
  - Theoretically appealing
  - Algorithmically efficient
- Only for low frequency waves.

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\,[\text{ Valsov} - \text{Maxwell eqs. }]$$

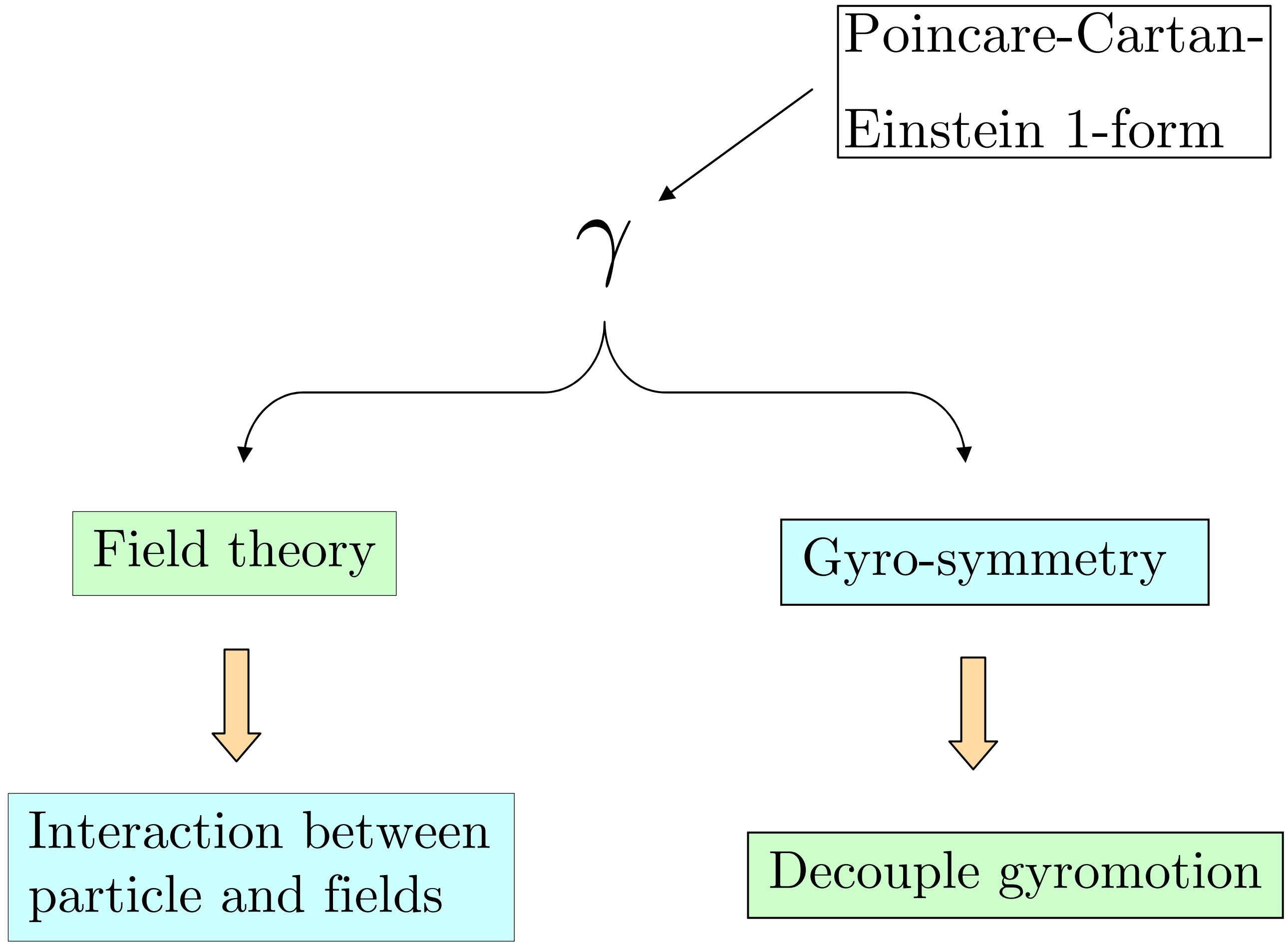
$$\left\langle\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}\right\rangle,\ \left\langle\frac{\mathbf{v}\times\mathbf{B}}{c}\cdot\frac{\partial f}{\partial \mathbf{p}}\right\rangle?$$

θ dependent

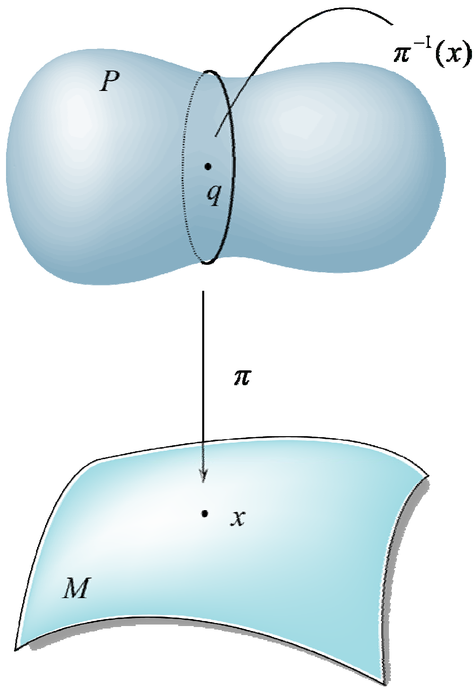
Motivations (presentation outline)



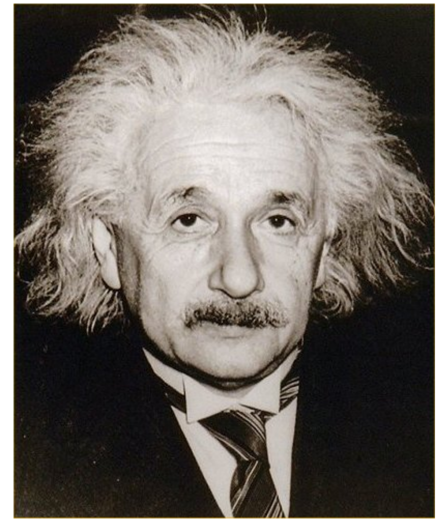
Modern gyrokinetics = field theory + gyro-symmetry



Where does γ live? -- phase space



Phase space is a fiber bundle $\pi : P \longrightarrow M$.



Cotagent bundle

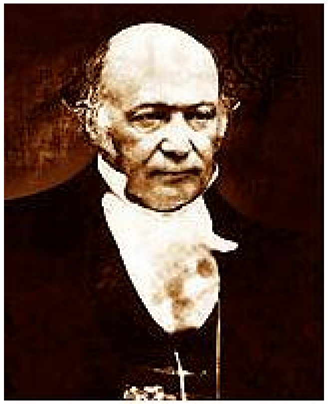
$$P = \{(x,p)\,|\,x \in M,\, p \in T_x^* M,\, g^{-1}(p,p) = -m^2c^2\}$$

7D spacetime

Inverse of metric

Poincare-Cartan-Einstein 1-form $\gamma \rightarrow$ particle dynamics

Hamilton's Eq.



Inner product

Exterior derivative

$$i_\tau d\gamma = 0$$

Worldline

Different for different particles: charged particle, neutron, polymer, virus, ...

Geometrically generalized Vlasov-Maxwell system --- A field theory

Liouville 6-form

$$\Omega \equiv -\frac{1}{3!m^3} d\gamma \wedge d\gamma \wedge d\gamma ,$$

$$L_\tau \Omega = i_\tau d\Omega + d(i_\tau \Omega) = 0.$$

Liouville Theorem

Vlasov Eq.

$$L_\tau f = i_\tau df = 0.$$

$$L_\tau (f\Omega) = (L_\tau f)\Omega + (L_\tau \Omega) f = 0.$$

Conservative form

Geometrically generalized Vlasov-Maxwell system --- A field theory

$$S = \int_x L,$$

$$L(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

Fiber integral

Variational derivative

$$\frac{\delta S}{\delta A} = E(L) = 0$$

Euler derivative

$$d * dA = 4\pi * j,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\delta \gamma(x')}{\delta A_{\alpha}(x)}, \ (\alpha = 0,1,2,3).$$

Second order field theory → 4-diamagnetic current

$$j^{\alpha}(x) = \int_{x'} \delta\left(x - x'\right)\int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha}(x)}$$

$$-\frac{\partial}{\partial x^{\beta}}\left[\int_{x'} \delta\left(x - x'\right)\int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha,\beta}(x)}\right], (\alpha = 0,1,2,3).$$

4-diamagnetic current

$$\frac{\partial A_{\alpha}}{\partial x^{\beta}}$$

Valid for any γ. Exact conservation properties.

Allow physics models, approximations build into γ.

Example: 1-form for charge particles with Lorentz force

$$\gamma = A + p = (\mathbf{A} + \mathbf{v}) \cdot d\mathbf{x} - \left[\frac{v^2}{2} + \phi\right] dt \ ,$$

$A \equiv (-\phi, \mathbf{A})$, four vector potential, 1-form

$p \equiv (-v^2/2, \mathbf{p})$, 1-form momentum

$\times dt$

$$L = \mathbf{A} \cdot \mathbf{v} + \frac{1}{2} v^2 - \phi$$
$$= (\mathbf{A} + \mathbf{v}) \cdot \mathbf{v} - \left(\frac{1}{2} v^2 + \phi\right)$$

Action

$$L(x) = -\frac{1}{2} dA \wedge * dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

Vlasov

$$df(\tau) = 0, \ i_\tau d\gamma = 0$$

Maxwell

$$d * dA = 4\pi \int_{\pi^{-1}(x)} f\Omega$$

What is symmetry?

- Coordinate dependent version:

$$\frac{\partial L}{\partial \theta} = 0\,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0\,.$$

Problem, what is θ?

- Geometric version:

S. Lie (1890s)

Symmetry is group

$$L_\eta \gamma = dS$$

Lie derivative → L_η

Symmetry vector field → η

Advantage: general, stronger, enables techniques to find θ.

Symmetry is invaraint

Noether's Theorem (1918)

Cartan's formula

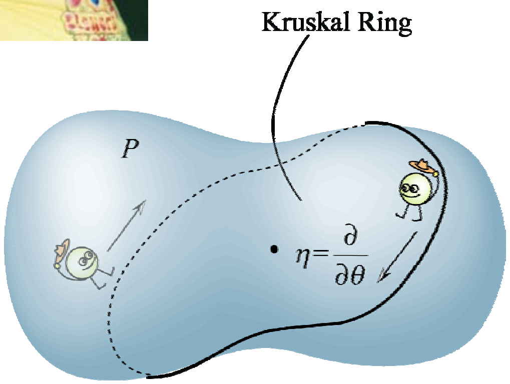
$$L_\eta \gamma = d\left(i_\eta \gamma\right) + i_\eta d\gamma = dS$$

⇓

$$d(\gamma \cdot \eta) \cdot \tau = dS \cdot \tau$$

$\gamma \cdot \eta - S$ is conversed.

What is gyrosymmetry?

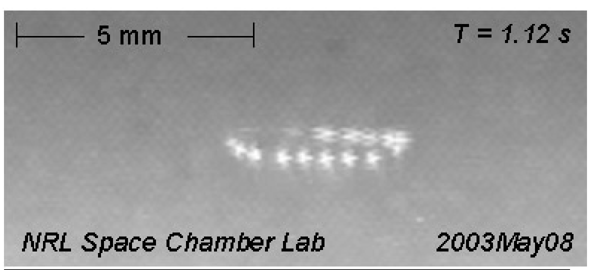


$$\eta = v_x\left(\frac{1}{B}\frac{\partial}{\partial x} + \frac{\partial}{\partial v_y}\right) + v_y\left(\frac{1}{B}\frac{\partial}{\partial y} - \frac{\partial}{\partial v_x}\right) \Rightarrow \eta = \frac{\partial}{\partial \theta}$$

Noether's Theorem

$$\mu = \frac{v_x^2 + v_y^2}{2B}$$

Gyrophase coordinate



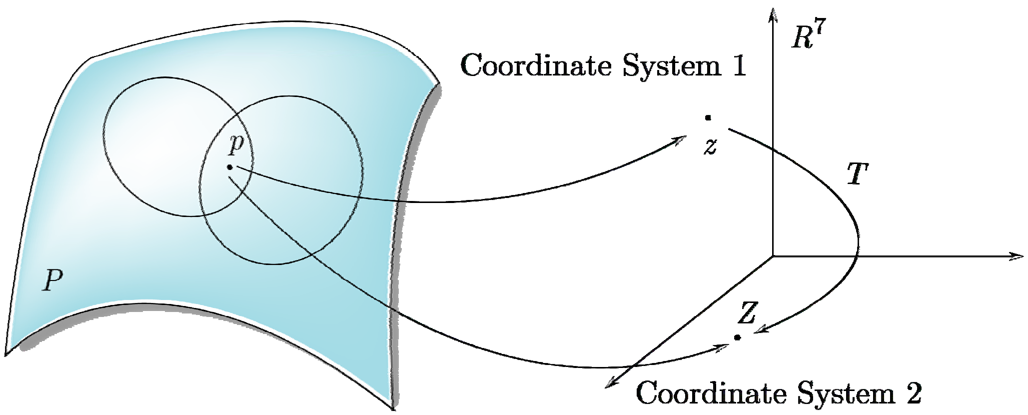
Amatucci, Pop 11, 2097 (2004).

Coordinate perturbation to find η.

Q: How does the dynamics transform?

A: It transforms as an 1-form!

Dynamics under Lie group of coordinate transformation



Continuous Lie group,

$$g : z \mapsto Z = g(z,\varepsilon)$$

Vector field (Lie algbra),

$$G = dg/d\varepsilon\,|_{\varepsilon=0} \quad ,$$

$$-G = dg^{-1}/d\varepsilon\,|_{\varepsilon=0} \; .$$

γ in Z

Pullback

Taylor expansion

$$\Gamma(Z) = g^{-1*}\gamma(z) = \gamma\left[g^{-1}(Z)\right] = \gamma(Z) - L_{G(Z)}\gamma(Z) + O(\varepsilon^2)$$

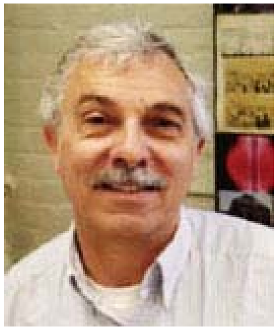
$$= \gamma(Z) - i_{G(Z)}d\gamma(Z) - d\left[\gamma \cdot G(Z)\right] + O(\varepsilon^2)\,.$$

Cartan’s formula

Insignificant

No need for
Poisson bracket

0^{th} order gyrocenter coordinates



Catto, 1979

$$g_0 : z = (\mathbf{x}, \mathbf{v}, t) \mapsto \bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta}, t)$$

$$\mathbf{x} \equiv \bar{\mathbf{X}} + \rho(\bar{\mathbf{X}}, \mathbf{v}) \,,\; \bar{u} \equiv u(\bar{\mathbf{X}}, \mathbf{v}) \,,\; \bar{w} \equiv w(\bar{\mathbf{X}}, \mathbf{v}),$$

$$\sin\bar{\theta} \equiv -\mathbf{c}(\bar{\mathbf{X}}) \cdot \mathbf{e}_1(\bar{\mathbf{X}}) \,,\mathbf{v} = \mathbf{D}(\bar{\mathbf{X}}) + \bar{u}\mathbf{b}(\bar{\mathbf{X}}) + \bar{w}\mathbf{c}(\bar{\mathbf{X}}) \,.$$

↑

$$u(y, \mathbf{v}_x)\mathbf{b}(y) \equiv [\,\mathbf{v}_x - \mathbf{D}(y)\,] \cdot \mathbf{b}(y)\; \mathbf{b}(y) \,,$$

$$w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \equiv [\,\mathbf{v}_x - \mathbf{D}(y)\,] \times \mathbf{b}(y) \times \mathbf{b}(y) \,,$$

$$\mathbf{c}(y, \mathbf{v}_x) \cdot \mathbf{c}(y, \mathbf{v}_x) = 1,$$

$$\mathbf{a}(y, \mathbf{v}_x) \equiv \mathbf{b}(y) \times \mathbf{c}(y, \mathbf{v}_x) \,,$$

$$\rho(y, \mathbf{v}_x) \equiv \frac{\mathbf{b}(y) \times [\,\mathbf{v}_x(y) - \mathbf{D}(y)\,]}{B_0(y)}.$$

$$\mathbf{D}(y) \equiv \frac{\mathbf{E}_0(y) \times \mathbf{B}_0(y)}{[\,B_0(y)\,]^2} \,,\quad \mathbf{b}(y) \equiv \frac{\mathbf{B}_0(y)}{B_0(y)} \,,$$

$$\mathbf{v}_x(y) \equiv \mathbf{D}(y) + u(y, \mathbf{v}_x)\mathbf{b}(y) + w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \,.$$

Lie perturbations

Cary, Littlejohn 1980s

Noncanonical coordinates: $\bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta})$

$$\gamma = \bar{\gamma}_0 + \bar{\gamma}_1 + \bar{\gamma}_2 + \dots,$$

$$\frac{\partial \bar{\gamma}_0}{\partial \bar{\theta}} = 0, \quad \frac{\partial \bar{\gamma}_1}{\partial \bar{\theta}} \neq 0,$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1, \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1,$$

$$\mathbf{E}_0 \sim \frac{\mathbf{v}\times\mathbf{B}_0}{c}, \ \mathbf{E}_1 \sim \varepsilon_1 \frac{\mathbf{v}\times\mathbf{B}_0}{c}, \ \mathbf{B}_1 \sim \varepsilon_1 \mathbf{B}_0,$$

$$\left(|\rho|\frac{\nabla E_0}{E_0}, \frac{1}{\Omega E_0}\frac{\partial E_0}{\partial t}\right) \sim \left(|\rho|\frac{\nabla B_0}{B_0}, \frac{1}{\Omega B_0}\frac{\partial B_0}{\partial t}\right) \sim \varepsilon_0,$$

$$\left(|\rho|\frac{\nabla E_1}{E_1}, \frac{1}{\Omega E_1}\frac{\partial E_1}{\partial t}\right) \sim \left(|\rho|\frac{\nabla B_1}{B_1}, \frac{1}{\Omega B_1}\frac{\partial B_1}{\partial t}\right) \sim 1,$$

$$\gamma(Z) = \gamma_0(Z) + \gamma_1(Z) + O(\varepsilon^2),$$

$$\gamma_0(Z) = \bar{\gamma}_0(Z),$$

$$\gamma_1(Z) = \bar{\gamma}_1(Z) - i_{G_1(Z)} d\gamma_0(Z) - dS_1,$$

$$\gamma_2(Z) = \bar{\gamma}_2(Z) - L_{G_1(Z)}\bar{\gamma}_1(Z) + \left(\frac{1}{2}L^2_{G_1(Z)} - L_{G_2(Z)}\right)\gamma_0(Z) + dS_2.$$

$g: \bar{Z} \to Z = g(\bar{Z})$ such that $\partial\gamma/\partial\theta = 0$.

$G_1,\ G_2,\ S_1,\ S_2$

Gyrocenter gauges

γ in the 0th order gyrocenter coordinates

$$\gamma = (\mathbf{A}+\mathbf{v})\cdot d\mathbf{x} - \left[\frac{v^2}{2}+\phi\right]dt = \bar{\gamma}_0 + \bar{\gamma}_1 + O(\varepsilon^2),$$

$$\bar{\gamma}_0 = \left(A_0 + \bar{u}\mathbf{b} + \mathbf{D}\right)\cdot d\bar{\mathbf{X}} + \frac{\bar{w}^2}{2B_0}d\bar{\theta} - \left(\frac{\bar{u}^2+\bar{w}^2+D^2}{2}+\phi_0\right)dt\,,$$

$$\bar{\gamma}_1 = \left[\frac{\bar{w}}{B_0}\nabla\mathbf{a}\cdot\left(\bar{u}\mathbf{b}+\frac{\bar{w}\mathbf{c}}{2}\right)+\frac{1}{2}\rho\cdot\nabla\mathbf{B}_0\times\rho-\frac{\bar{w}}{B_0}\nabla\mathbf{D}\cdot\mathbf{a}+\mathbf{A}_1(\bar{\mathbf{X}}+\rho)\right]\cdot d\bar{\mathbf{X}}$$
$$+\left[-\frac{\bar{w}^3}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b}+\frac{\bar{w}}{B_0}\mathbf{A}_1(\bar{\mathbf{X}}+\rho)\cdot\mathbf{c}\right]d\bar{\theta}+\left[\frac{1}{\bar{w}}\mathbf{A}_1(\bar{\mathbf{X}}+\rho)\cdot\mathbf{a}\right]d\bar{\mu}$$
$$-\left[\phi_1(\bar{\mathbf{X}}+\rho)+\rho\cdot\frac{\partial\mathbf{D}}{\partial t}-\frac{1}{2}\rho\cdot\nabla\mathbf{E}_0\cdot\rho-\left(\bar{u}\mathbf{b}+\frac{\bar{w}\mathbf{c}}{2}\right)\cdot\frac{\bar{w}}{B_0}\frac{\partial\mathbf{a}}{\partial t}\right]dt\,.$$

⇓

$$Z = g_1(\bar{Z},\varepsilon),\ \ \frac{dg_1}{d\varepsilon}\Big|_{\varepsilon=0} = G_1(\bar{Z}),$$

⇒

$$\gamma_1(Z) = \bar{\gamma}_1(Z) - i_{G_1(Z)}d\gamma_0(Z) + dS_1(Z),$$

γ under coordinate perturbation

$$
\begin{aligned}
\gamma_1(Z) = \Big[& \mathbf{G}_{1\mathrm{X}} \times \mathbf{B}^{\dagger} - G_{1u}\mathbf{b} + \nabla S_1 + \frac{w}{B_0}\nabla\mathbf{a}\cdot\left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right) + \frac{1}{2}\rho\cdot\nabla\mathbf{B}_0\times\rho \\
& - \frac{w}{B_0}\nabla\mathbf{D}\cdot\mathbf{a} + \mathbf{A}_1(\mathbf{X}+\rho)\Big]\cdot d\mathbf{X} + \left[\mathbf{G}_{1\mathrm{X}}\cdot\mathbf{b} + \frac{\partial S_1}{\partial u}\right]du + \Big[G_{1\theta} + \frac{\partial S_1}{\partial \mu} + \\
& + \frac{1}{w}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{a}\Big]dw + \Big[-G_{1\mu} + \frac{\partial S_1}{\partial \theta} - \frac{w^3}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b} \\
& + \frac{w}{B_0}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{c}\Big]d\theta + \Big[-\mathbf{E}_0^{\dagger}\cdot\mathbf{G}_{1\mathrm{X}} + uG_{1u} + B_0 G_{1\mu} + \frac{\partial S_1}{\partial t} - \phi_1(\mathbf{X}+\rho) \\
& - \rho\cdot\frac{\partial\mathbf{D}}{\partial t} + \frac{1}{2}\rho\cdot\nabla\mathbf{E}_0\cdot\rho + \left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right)\cdot\frac{w}{B_0}\frac{\partial\mathbf{a}}{\partial t}\Big]dt\,.
\end{aligned}
$$

+

$\partial\gamma_1/\partial\theta = 0.$ ⇒ $G_1,\ S_1$

Freedoom
Gyro-center gauges

γ in the 1st order gyrocenter coordinate, $\partial\gamma/\partial\theta=0$,

$$\gamma(Z) = \gamma_0(Z) + \gamma_1(Z),$$

$$\gamma_0 = (A_0 + u\boldsymbol{b} + \boldsymbol{D})\cdot d\boldsymbol{X} + \frac{w^2}{2B_0}d\theta - \left(\frac{u^2+w^2+D^2}{2} + \phi_0\right)dt,$$

$$\gamma_1(Z) = -\frac{w^2}{2B_0}\boldsymbol{R}\cdot d\boldsymbol{X} - H_1 dt,$$

Gyrocenter dynamics

$$H_1 = \left(\boldsymbol{E}_{0\perp}^{\dagger} - \boldsymbol{B}_{\perp}^{\dagger}\times u\boldsymbol{b}\right)\cdot\frac{w^2}{4B_0^2 B_{\parallel}^{\dagger}}\nabla\boldsymbol{B}_0 + \frac{w^2 u}{4B_0}\boldsymbol{b}\cdot\nabla\times\boldsymbol{b}$$
$$-\frac{w^2}{4B_0^2}(\nabla\cdot\boldsymbol{E}_0 - \boldsymbol{bb}:\nabla\boldsymbol{E}_0) - \frac{w^2}{2B_0}R_0 + \langle\psi_1\rangle$$

$$\boldsymbol{R} \equiv \nabla\boldsymbol{c}\cdot\boldsymbol{a}, \quad R_0 \equiv -\frac{\partial\boldsymbol{c}}{\partial t}\cdot\boldsymbol{a},$$

Gyro-gauge invariant

$$R \longrightarrow R' + \nabla\delta(X),$$

$$\theta \longrightarrow \theta' + \delta(X).$$

$$R = (R_0, \mathbf{R}), \quad X = (t, \mathbf{X}),$$

$$\nabla = (-\partial/\partial t, \nabla).$$

$$\psi_1 \equiv \phi_1(\boldsymbol{X}+\rho) - \left(\frac{\boldsymbol{E}_0^{\dagger}\times\boldsymbol{b} + \boldsymbol{B}_{\perp}^{\dagger}u}{B_{\parallel}^{\dagger}} + u\boldsymbol{b} + w\boldsymbol{c}\right)\cdot\boldsymbol{A}_1(\boldsymbol{X}+\rho),$$

$$\langle\alpha\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi}\alpha d\theta, \quad \tilde{\alpha} \equiv \alpha - \langle\alpha\rangle.$$

$$\boldsymbol{B}^{\dagger} \equiv \nabla\times(\boldsymbol{A}_0 + u\boldsymbol{b} + \boldsymbol{D}),$$

$$\boldsymbol{E}_0^{\dagger} \equiv \boldsymbol{E}_0 - \nabla\left[\mu B_0 + \frac{D^2}{2}\right] - u\frac{\partial\boldsymbol{b}}{\partial t} - \frac{\partial\boldsymbol{D}}{\partial t}.$$

Transformation g_1 and gyrocenter gauge S_1

$$G_{1\boldsymbol{X}} = -\frac{\partial S_1}{\partial u}\left(\boldsymbol{b} + \frac{\boldsymbol{B}_\perp^\dagger}{B_\parallel^\dagger}\right) + \frac{w^2}{2B_0^2 B_\parallel^\dagger}\boldsymbol{aa}\cdot\nabla\boldsymbol{B}_0 + \frac{wu}{B_0 B_\parallel^\dagger}(\nabla\boldsymbol{a}\cdot\boldsymbol{b})\times\boldsymbol{b}$$

$$-\frac{w}{B_0 B_\parallel^\dagger}(\nabla\boldsymbol{D}\cdot\boldsymbol{a})\times\boldsymbol{b} + \frac{\nabla S_1 + \boldsymbol{A}_1(\boldsymbol{X}+\rho)}{B_\parallel^\dagger}\times\boldsymbol{b}$$

$$G_{1u} = \left(\boldsymbol{B}_\perp^\dagger\times\boldsymbol{b}\right)\cdot\boldsymbol{G}_{1\boldsymbol{X}}\,\frac{w^2}{2B_0^2}\boldsymbol{a}\cdot\nabla\boldsymbol{B}_0\cdot\boldsymbol{c} + \frac{wu}{B_0}\boldsymbol{b}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{b}$$

$$-\frac{w}{B_0}\boldsymbol{b}\cdot\nabla\boldsymbol{D}\cdot\boldsymbol{a} - \boldsymbol{b}\cdot[\nabla S_1 + \boldsymbol{A}_1(\boldsymbol{X}+\rho)],$$

$$G_{1\mu} = \frac{\partial S_1}{\partial\theta} - \frac{w^2}{2B_0^3}\boldsymbol{a}\cdot\nabla\boldsymbol{B}_0\cdot\boldsymbol{b} + \frac{w}{B_0}\boldsymbol{c}\cdot\boldsymbol{A}_1(\boldsymbol{X}+\rho),$$

$$G_{1\theta} = -\frac{\partial S_1}{\partial\mu} - \frac{1}{w}\boldsymbol{a}\cdot\boldsymbol{A}_1(\boldsymbol{X}+\rho).$$

$$\frac{\partial S_1}{\partial t} + \left(\frac{\boldsymbol{E}_0^\dagger\times\boldsymbol{b} + \boldsymbol{B}_\perp^\dagger u}{B_\parallel^\dagger} + u\boldsymbol{b}\right)\cdot\nabla S_1 + \left(E_{0\parallel}^\dagger + \frac{E_{0\parallel}^\dagger\cdot\boldsymbol{B}_\perp^\dagger}{B_\parallel^\dagger}\right)\frac{\partial S_1}{\partial u} + B_0\frac{\partial S_1}{\partial\theta} =$$

$$\left(\boldsymbol{E}_{0\perp}^\dagger - \boldsymbol{B}_\perp^\dagger\times u\boldsymbol{b}\right)\cdot\left[\frac{w^2}{2B_0^3}\widetilde{\boldsymbol{aa}}\cdot\nabla B_0 + \frac{wu}{B_0^2}(\nabla\boldsymbol{a}\cdot\boldsymbol{b})\times\boldsymbol{b} - \frac{w}{B_0^2}(\nabla\boldsymbol{D}\cdot\boldsymbol{a})\times\boldsymbol{b}\right]$$

$$-\frac{w^2u}{2B_0^2}\nabla\boldsymbol{B}_0:\widetilde{\boldsymbol{ca}} - \frac{wu^2}{B_0}\boldsymbol{b}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{b} + \frac{wu}{B_0}\boldsymbol{b}\cdot\nabla\boldsymbol{D}\cdot\boldsymbol{a} + \frac{w^3}{2B_0^2}\boldsymbol{a}\cdot\nabla\boldsymbol{B}_0\cdot\boldsymbol{b}$$

$$+\frac{w}{B_0}\boldsymbol{a}\cdot\frac{\partial\boldsymbol{D}}{\partial t} - \frac{w^2}{2B_0^2}\nabla\boldsymbol{E}_0:\widetilde{\boldsymbol{aa}} + \frac{uw}{B_0}\boldsymbol{a}\cdot\frac{\partial\boldsymbol{b}}{\partial t} + \widetilde{\psi}_1.$$

2^{nd} order

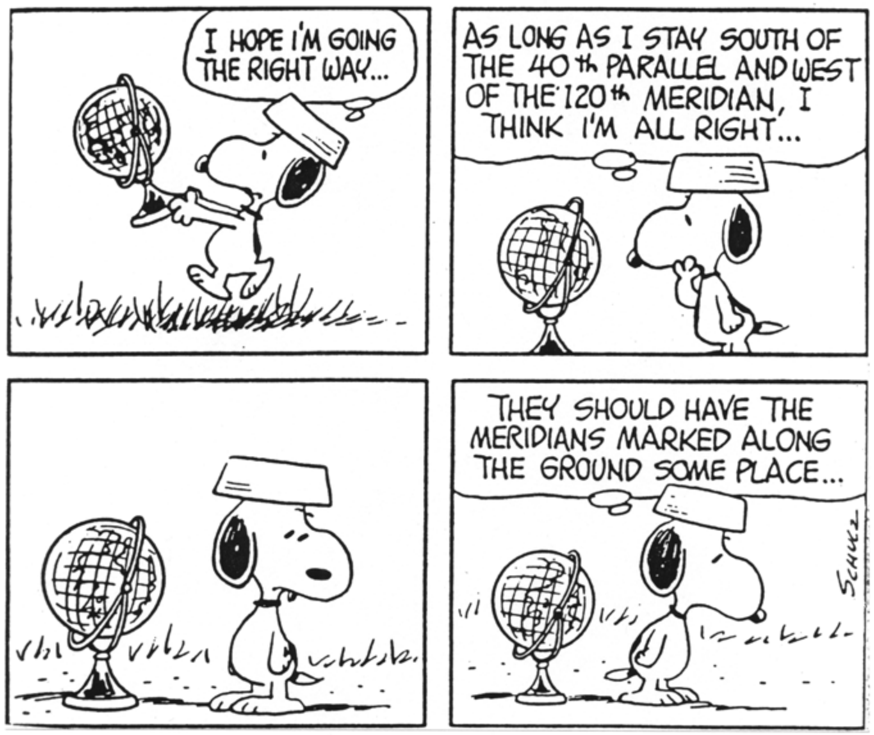
$$\gamma_2 = -\langle \psi_2 \rangle dt,$$

$$\psi_2 \equiv \frac{1}{2}\boldsymbol{E}_{0\perp}\cdot\left[\left(\boldsymbol{G}_1^{\dagger}\times\boldsymbol{B}_1\right)\times\boldsymbol{b}\right] - \frac{1}{2}(u\boldsymbol{b}+w\boldsymbol{c})\cdot\left(\boldsymbol{G}_1^{\dagger}\times\boldsymbol{B}_1\right)$$
$$+\frac{1}{2}\left[\boldsymbol{G}_{1\boldsymbol{x}}\cdot\boldsymbol{E}_1 + \left(\boldsymbol{E}_1\cdot\frac{\boldsymbol{a}}{\sqrt{2\mu B_0}} - \frac{\partial\langle\psi_1\rangle}{\partial\mu}\right)G_{1\mu} + \boldsymbol{E}_1\cdot\boldsymbol{c}\sqrt{\frac{2\mu}{B_0}}G_{1\theta}\right],$$

$$\boldsymbol{G}_1^{\dagger} \equiv \boldsymbol{G}_{1\boldsymbol{x}} + G_{1\mu}\frac{\boldsymbol{a}}{\sqrt{2\mu B_0}} + G_{1\theta}\sqrt{\frac{2\mu}{B_0}}\boldsymbol{c},$$

$$\boldsymbol{E}_1^{*} \equiv -\boldsymbol{\nabla}\phi_1 - \frac{\partial \boldsymbol{A}_1}{\partial t}.$$

Perturbation techniques — quest of good coordinates



Peanuts by Charles Schulz. Reprint permitted by UFS, Inc.

Gyrocenter dynamics

$i_\tau d\gamma = 0\,.$

Curvature drift

$\boldsymbol{E}\times\boldsymbol{B},\ \nabla B$ drift

∇D & curvature $\boldsymbol{D}$ drift

drift by spacetime inhomogeneties of $\boldsymbol{E_0}$

Banos drift

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\Big(u + \frac{\mu}{2}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}\Big) - \frac{\boldsymbol{b}\times\boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{du}{dt} = \frac{\boldsymbol{B}^\dagger\cdot\boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{d\theta}{dt} = B_0 + \boldsymbol{R}\cdot\frac{d\boldsymbol{X}}{dt} - R_0 + \frac{\boldsymbol{E}_0\cdot\boldsymbol{\nabla}B_0}{B_0^2} + \frac{u}{2}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}$$
$$- \frac{1}{2B_0}\left[\boldsymbol{\nabla}\cdot\boldsymbol{E}_0 - \boldsymbol{bb}:\boldsymbol{\nabla}\boldsymbol{E}_0\right] + \frac{\partial}{\partial\mu}\langle\psi_1 + \psi_2\rangle$$

$$\frac{d\mu}{dt} = 0$$

$$\boldsymbol{B}^\dagger \equiv \boldsymbol{\nabla}\times(\boldsymbol{A}_0 + u\boldsymbol{b} + \boldsymbol{D}),\ \ B_\parallel^\dagger = \boldsymbol{B}^\dagger\cdot\boldsymbol{b}$$

$$\boldsymbol{E}^\dagger \equiv \boldsymbol{E}_0 - \boldsymbol{\nabla}\left[\mu B_0 + \frac{D^2}{2} + \langle\psi_1 + \psi_2\rangle\right] - u\frac{\partial\boldsymbol{b}}{\partial t} - \frac{\partial\boldsymbol{D}}{\partial t}$$

Gyrokinetic equations

$$L_{\tau} F = i_{\tau} dF = 0.$$

$$\frac{dZ_j}{dt}\frac{\partial F}{\partial Z_j} = 0\,,(0 \le j \le 6).$$

$$F = \langle F \rangle.$$

$$\frac{\partial}{\partial \theta}\left(\frac{dZ}{dt}\right) = 0\,,$$

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\boldsymbol{X}}{dt} \cdot \boldsymbol{\nabla}_{\boldsymbol{X}} \langle F \rangle + \frac{du}{dt}\frac{\partial \langle F \rangle}{\partial u} = 0\,,$$

Gyrokinetic field theory

$$\Omega \equiv -\frac{1}{3!m^3} d\gamma \wedge d\gamma \wedge d\gamma ,$$

↓

$$\Omega = B_{\|}^{\dagger} dx^1 \wedge dx^2 \wedge dx^3 \wedge du \wedge d\mu \wedge d\theta$$
$$+\left[A_{j,t}^{\dagger} b_i - A_{i,j}^{\dagger} u - b_j H_{,i} \right] dt \wedge dx^j \wedge dx^i \wedge du \wedge d\mu \wedge d\theta$$
$$+\left[A_{i,j}^{\dagger} H_{,l} + A_{i,j}^{\dagger} A_{l,t}^{\dagger} \right] dx^j \wedge dx^i \wedge dt \wedge d\mu \wedge d\theta$$
$$-A_{i,j}^{\dagger} b_l H_{,\mu} dx^j \wedge dx^i \wedge dt \wedge du \wedge d\mu .$$

γ

↓

$$d * dA = 4\pi * j ,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\delta\gamma(x')}{\delta A_{\alpha}(x)} , (\alpha = 0,1,2,3).$$

Pullback of distribution function

Particle distribution

Don't know $f(z)$.

$\Downarrow$

Gyrocenter distribution

pullback from $F(Z)$:

$$f(z) = g^* \left[F(Z) \right] = F\left(g(z) \right).$$

Gyrocenter gauge theory for waves in magnetized plasmas

$$F = F_0 + F_1$$
$$\boldsymbol{B}_0 = const.$$

Gyrocenter gauge function

$$\frac{\partial F_1}{\partial t} + u\boldsymbol{b}\cdot\boldsymbol{\nabla}F_1 = \frac{1}{m}\boldsymbol{b}\cdot\boldsymbol{\nabla}\langle\psi_1\rangle\frac{\partial F_0}{\partial u}$$

$$\frac{\partial S}{\partial t} + \dot{\boldsymbol{X}}\cdot\frac{\partial S}{\partial \boldsymbol{X}} + u\frac{\partial S}{\partial u} + \dot{\theta}\frac{\partial S}{\partial \theta} = \tilde{\phi}(\boldsymbol{X}+\boldsymbol{\rho}_0,t) - \widetilde{\boldsymbol{v}\cdot\boldsymbol{A}}(\boldsymbol{X}+\boldsymbol{\rho}_0,t)$$

$$F_1 = \frac{-k_z}{(\omega - k_z u)}[J_0(\phi - uA_\parallel) + V_\perp J_1 A_y]\frac{\partial F_0}{\partial u}$$

$$S^* = S - \frac{H_1}{i(\omega - k_z u)}$$

$$S^* = \sum_{n=-\infty}^{n=\infty}\{\frac{I_n(\lambda)\,\mathrm{e}^{\,\mathrm{i}n\theta}}{i(n-\overline{\omega}+\overline{k_z u})}(\phi - uA_z)$$
$$+\frac{nI_n(i\lambda)\,\mathrm{e}^{\,\mathrm{i}n\theta}}{-i\lambda(n-\overline{\omega}+\overline{k_z u})}V_\perp A_x + \frac{I_n^{'}(i\lambda)\,\mathrm{e}^{\,\mathrm{i}n\theta}}{i(n-\overline{\omega}+\overline{k_z u})}V_\perp A_y\},$$

Solution

Pullback of current

Lab coord.

0^{th} order gyrocenter

$$\boldsymbol{j} = \int d^3v\boldsymbol{v}f(\boldsymbol{r},\boldsymbol{v},t) = \int d^6\bar{Z}\bar{\boldsymbol{V}}\bar{F}(\bar{Z},t)\delta(g_0^{-1}\bar{\boldsymbol{X}} - \boldsymbol{r}) = \int d^6\bar{Z}\bar{\boldsymbol{V}}[g_1^*F]\delta(g_0^{-1}\bar{\boldsymbol{X}} - \boldsymbol{r})$$

$$g_1^*F = F + L_GF = F - \frac{\boldsymbol{b}}{B_\parallel^\dagger}\times[\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0,t) + \boldsymbol{\nabla}S]\cdot\boldsymbol{\nabla}F - \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\frac{\partial S}{\partial u}\cdot\boldsymbol{\nabla}F$$
$$+\frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\cdot[\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0,t) + \boldsymbol{\nabla}S]\frac{\partial F}{\partial u_\parallel} + [\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0,t)\cdot\frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + \frac{\partial S}{\partial \theta}]\frac{\partial F}{\partial \mu}$$
$$-[\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0,t)\cdot\frac{\partial \boldsymbol{\rho}_0}{\partial \mu} + \frac{\partial S}{\partial \mu}]\frac{\partial F}{\partial \theta}$$

Pullback of g_1

$$\boldsymbol{j}_1 = \left\{e\int(\boldsymbol{v}_\perp + u\boldsymbol{b})[g_1^*(F_0+F_1)](Z)\delta(\boldsymbol{X}+\boldsymbol{\rho}_0 - \boldsymbol{r})d^6Z\right\}_1,$$
$$[g_1^*(F_0+F_1)]_1 = F_1 + \boldsymbol{b}\cdot[\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0) + \boldsymbol{\nabla}S]\frac{\partial F_0}{\partial u} + [\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0)\cdot\frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + \frac{\partial S}{\partial \theta}]\frac{\partial F_0}{\partial \mu}.$$

Pullback of current

$$\boldsymbol{j}_1(r) = e\int \delta(\boldsymbol{X}+\boldsymbol{\rho}_0-\boldsymbol{r})d^6Z(\boldsymbol{v}_\perp+u\boldsymbol{b})\Big\{A_z(\boldsymbol{X}+\boldsymbol{\rho}_0)\frac{\partial F_0}{\partial u}$$

$$+ik_zS^*\frac{\partial F_0}{\partial u}+[\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0)\cdot\frac{\partial \rho_0}{\partial \xi}+\frac{\partial S^*}{\partial \theta}]\frac{\partial F_0}{\partial \mu}\Big\}$$

$$= e\int d^3v\ \mathrm{e}^{-\boldsymbol{\rho}_0\cdot\boldsymbol{\nabla}}\ (\boldsymbol{v}_\perp+u_\parallel\boldsymbol{b})\Big\{A_z(\boldsymbol{X}+\boldsymbol{\rho}_0)\frac{\partial F_0}{\partial u}$$

$$+ik_zS^*\frac{\partial F_0}{\partial u}+[\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0)\cdot\frac{\partial \boldsymbol{\rho}_0}{\partial \theta}+\frac{\partial S^*}{\partial \theta}]\frac{\partial F_0}{\partial \mu}\Big\}_{X\mapsto r}$$

$$= -\frac{i\omega}{4\pi}\chi_p\cdot\begin{pmatrix}A_x\\A_y\\A_z\\\phi\end{pmatrix}$$

Gyrocenter gauge susceptibility – same as classical theory

$$\chi_p^{\parallel} = \frac{4\pi e^2}{i\omega m\Omega} \sum_{n=-\infty}^{n=\infty} 2\pi \int_0^{\infty} v_\perp dv_\perp \int_{-\infty}^{\infty} du \frac{1}{(n-\overline{\omega}+\overline{k_z u})} \frac{\partial F_0}{\partial u}$$

$$\times \begin{pmatrix} \frac{n^2 J_n^2(\lambda) k_z}{\lambda^2 c} v_\perp^2 & \frac{in J_n(\lambda) J_n'(\lambda) k_z}{\lambda c} v_\perp^2 & \frac{-n J_n^2(\lambda)(n-\overline{\omega})\Omega}{\lambda c} v_\perp & \frac{-n J_n^2(\lambda) k_z \Omega}{k_x} \\ -\frac{in J_n(\lambda) J_n'(\lambda) k_z}{\lambda c} v_\perp^2 & \frac{J_n'^2(\lambda) k_z}{c} v_\perp^2 & \frac{J_n(\lambda) J_n'(\lambda)(n-\overline{\omega})\Omega}{c} v_\perp & i J_n(\lambda) J_n'(\lambda) k_z v_\perp \\ \frac{n J_n^2(\lambda) k_z}{\lambda c} v_{\perp V\parallel} & \frac{i J_n(\lambda) J_n'(\lambda) k_z}{c} v_{\perp V\parallel} & \frac{-J_n^2(\lambda)(n-\overline{\omega})\Omega}{c} u & -J_n^2(\lambda) k_z u \end{pmatrix},$$

$$\chi_p^{\perp} = \frac{4\pi e^2}{i\omega m\Omega} \sum_{n=-\infty}^{n=\infty} 2\pi \int_0^{\infty} v_\perp dv_\perp \int_{-\infty}^{\infty} du \frac{1}{(n-\overline{\omega}+\overline{k_z u})} \frac{\partial F_0}{\partial v_\perp}$$

$$\times \begin{pmatrix} \frac{n^2 J_n^2(\lambda)(\omega - k_z u)}{\lambda^2 c} v_\perp & \frac{in J_n(\lambda) J_n'(\lambda)(\omega - k_z u)}{\lambda c} v_\perp & \frac{n^2 J_n^2(\lambda)\Omega}{\lambda c} u & \frac{-n^2 J_n^2(\lambda)\Omega}{\lambda} \\ \frac{-in J_n(\lambda) J_n'(\lambda)(\omega - k_z u)}{\lambda c} v_\perp & \frac{J_n'^2(\lambda)(\omega - k_z u)}{c} v_\perp & \frac{-in J_n(\lambda) J_n'(\lambda)\Omega}{c} u & in J_n(\lambda) J_n'(\lambda)\Omega \\ \frac{n J_n^2(\lambda)(\omega - k_z u)}{\lambda c} u & \frac{i J_n(\lambda) J_n'(\lambda)(\omega - k_z u)}{c} u & \frac{n J_n^2(\lambda)\Omega u^2}{c v_\perp} & \frac{-n J_n^2(\lambda)\Omega u_\parallel}{v_\perp} \end{pmatrix}.$$

Gyrocenter gauge algorithm

Gyrokinetic eq.

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathrm{x}} \langle F \rangle + \frac{du}{dt} \frac{\partial \langle F \rangle}{\partial u} = 0\,,$$

Gyrocenter gauge

$$\frac{\partial S}{\partial t} + \dot{\boldsymbol{X}} \cdot \frac{\partial S}{\partial \boldsymbol{X}} + u \frac{\partial S}{\partial u} + \dot{\theta} \frac{\partial S}{\partial \theta} = \tilde{\phi}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) - \widetilde{\boldsymbol{v} \cdot \boldsymbol{A}}(\boldsymbol{X} + \boldsymbol{\rho}_0, t)$$

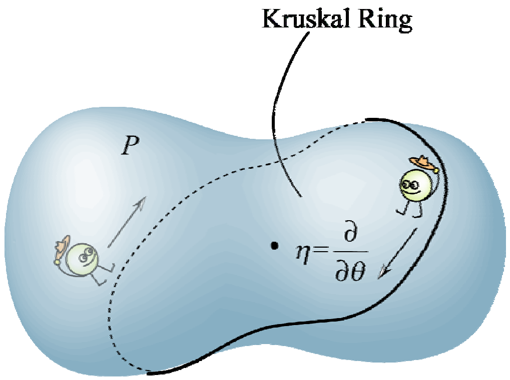
Pullback

$$f(z) = g^* \left[F(Z) \right] = F\left(g(z) \right).$$

Maxwell's

$$d * dA = 4\pi \int_{\pi^{-1}(x)} f \Omega$$

Gyrocenter gauge data structure – Kruskal ring



replace particle → Kruskal ring

Kruskal ring:
- · species
- · X
- · u
- · μ
- · $s[n]$

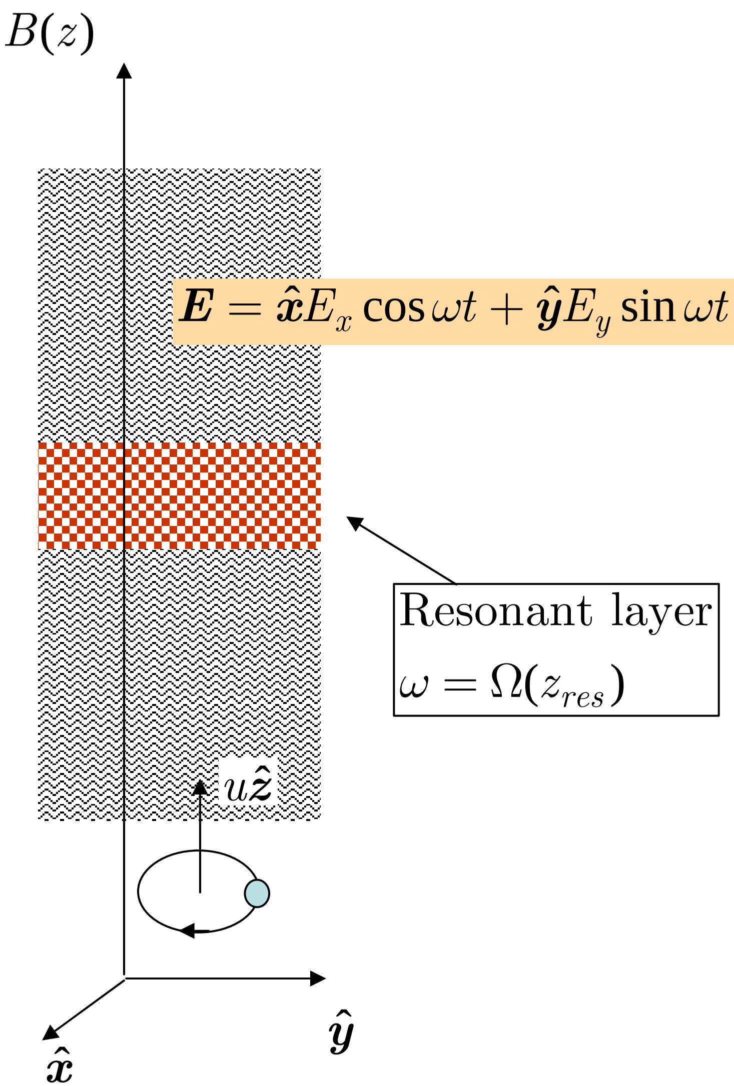
gyrocenter → X, u, μ

replace gyrophase → $s[n]$

gyrocenter gauge → $s[n]$

SciDAC proposal

Example – stochastic resonant heating



Gyrocenter gauge

$$\frac{dS}{dt} = \tilde{\phi} \approx -\boldsymbol{\rho} \cdot \boldsymbol{E}$$

Gyrocenter

$$z = z_0 + ut, \quad \frac{d}{dt}\left(\mu + G_{1\mu}\right) = 0, \;\; \dot{\theta} = \Omega$$

$$\frac{d\mu}{dt} = -\frac{d}{dt}G_{1\mu} = -\frac{d}{dt}\frac{\partial S}{\partial \theta}$$

$$= -\frac{\partial}{\partial \theta}\frac{dS}{dt} = \frac{\partial \boldsymbol{\rho}}{\partial \theta} \cdot \boldsymbol{E} = \sqrt{\frac{2\mu}{B}}\hat{\boldsymbol{v}}_\perp \cdot \boldsymbol{E}$$

Stochastic resonant heating rate

$$\frac{1}{\sqrt{2\mu}}\frac{d\mu}{dt} = \left[-\frac{E_x}{\sqrt{B}}\sin\theta(t)\cos\omega t - \frac{E_y}{\sqrt{B}}\cos\theta(t)\sin\omega t\right]$$

$$\sqrt{2\mu} = \sqrt{2\mu}\,|_{t=0} + \int_{t=0}^{t}\left[-\frac{E_x}{\sqrt{B}}\sin\theta(t)\cos\omega t - \frac{E_y}{\sqrt{B}}\cos\theta(t)\sin\omega t\right]$$

$$\approx \sqrt{2\mu}\,|_{t=0} + \frac{\left(E_y - E_x\right)}{\sqrt{B}}\sin\left(\theta_{res} - \omega t_{res}\right)\sqrt{\frac{2\pi}{\Omega'}}\frac{\sqrt{2}}{2}$$

$$\int_{t=0}^{t}\sin\theta(t)\cos\omega t \approx -\sin\left(\theta_{res} - \omega t_{res}\right)\sqrt{\frac{2\pi}{\Omega'}}\cos\left(\frac{\pi}{4}\right)$$

$$\int_{t=0}^{t}\cos\theta(t)\sin\omega t \approx +\sin\left(\theta_{res} - \omega t_{res}\right)\sqrt{\frac{2\pi}{\Omega'}}\cos\left(\frac{\pi}{4}\right)$$

Asymptotic for large Ω'

$$\Delta\left\langle\frac{v_\perp^2}{2}\right\rangle = \frac{\pi}{4}\frac{\left(E_y - E_x\right)^2}{\Omega'}$$

average over random phase

Conclusions