A complete gas dynamics interpretation of radiation moment model

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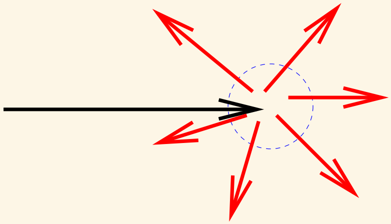
Numerical Flow Models for Controled Fusion

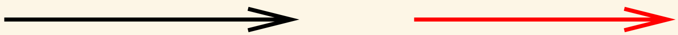
1. Presentation

• Radiation flows are fundamental in the context of ICF experiments. The goal of this talk is to give overview on a ongoing research about the discretization of Moment Models for radiation flows.

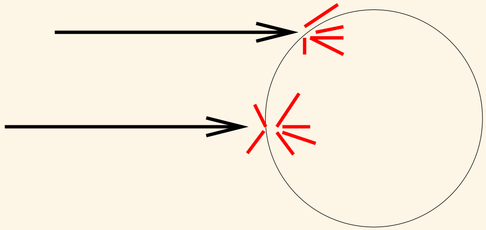
- Diffusion regime: $\sigma = O(1)$.



- Transport regime: $\sigma \approx 0$.



- ICF=Transport (anisotropic) +Diffusion



- We wonder wether the model M^1 grey moment model for radiative hydrodynamics in dimension $d = 1, 2, 3$ may be used for such problems

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon} \nabla . F = 0, \\ \partial_t F + \frac{1}{\varepsilon} \nabla . P = -\frac{\sigma}{\varepsilon^2} F. \end{cases}$$

We consider the entropy closure of Levermore,where the radiative intensity is given by the generalized Planckian

$$I(\nu, \mathbf{n}) = \frac{15}{4\pi^5} \frac{\nu^3}{\exp(\frac{\nu}{T} + \frac{\nu b.\mathbf{n}}{T}) - 1}, \quad |b| \leq 1, \quad |\mathbf{n}| = 1.$$

The radiative energy E, flux F and pressure P are

$$\begin{cases} E = \int\int I d\nu d\mathbf{n} = \frac{3 + |b|^2}{3(1 - |b|^2)^3} T^4 & \in \mathbb{R}, \\ F = \int\int \mathbf{n} I d\nu d\mathbf{n} = -\frac{4b}{3(1 - |b|^2)^3} T^4 & \in \mathbb{R}^d, \\ P = \int\int \mathbf{n} \otimes \mathbf{n} I d\nu d\mathbf{n} = \left(\frac{1 - \chi}{2} I + \frac{3\chi - 1}{2} \frac{f \otimes f}{|f|^2} \right) E & \in \mathbb{R}^{d \times d}. \end{cases}$$

The non dimensional radiation flux $f = \frac{F}{E}$. The Eddington factor is $\chi = \frac{3+4|f|^2}{5+2\sqrt{4-3|f|^2}}$.

- The radiation flux F models the anisotropy of radiation.
- Here ε is the ratio of the fluid sound velocity over the speed of light: $\varepsilon \leq 10^{-3}$ in ICF.
- The other coefficient is the opacity σ: typically $\sigma = 1$ or $\sigma \approx 0$.

1.1. The diffusion regime: $\sigma = 1$ and $\varepsilon \to 0$

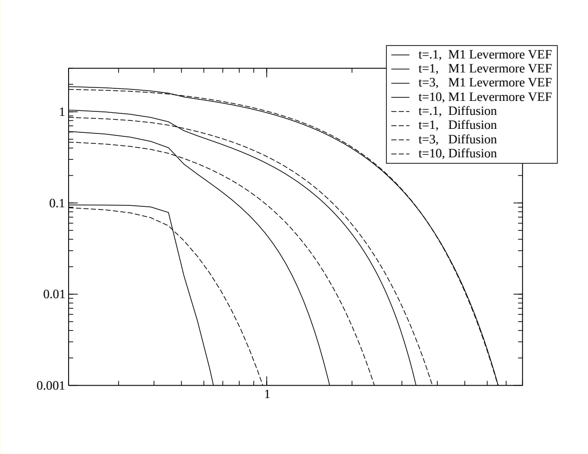
The limit

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\partial_x F = 0 \\ \partial_t F + \frac{1}{\varepsilon}\partial_x P = -\frac{1}{\varepsilon^2} F. \end{cases}$$

is the diffusion equation

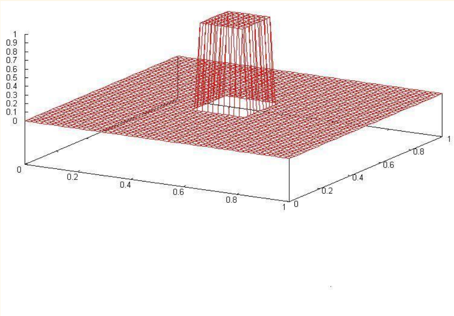
$$\partial_t E - \frac{1}{3}\partial_{xx} E = 0.$$

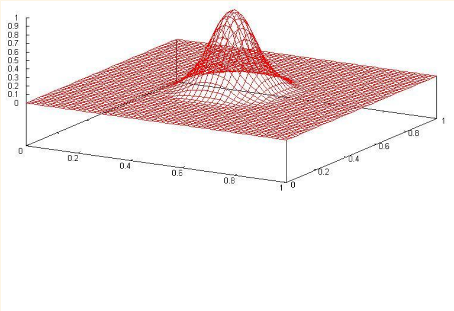
In the figure we solve the Moment model.

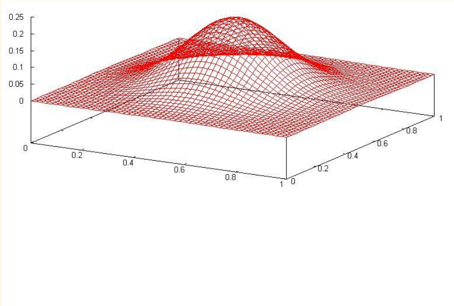


Radiative energy (log-scaled) versus the optical depth (log-scaled)

1.2. The diffusion regime in 2D: $\partial_t E - \frac{1}{3}\Delta E = 0$







- Results by Loi-Staudacher.
- A non physical diffusion coefficient is possible: $K = \frac{1}{3} + C\frac{\Delta x}{\varepsilon}$. The numerical issue is to have correct diffusion coefficient $C = 0$. Techniques available on Cartesan grids.
- No stability issue.
- A classical strategy is to solve this equation directly: K is correct by construction.

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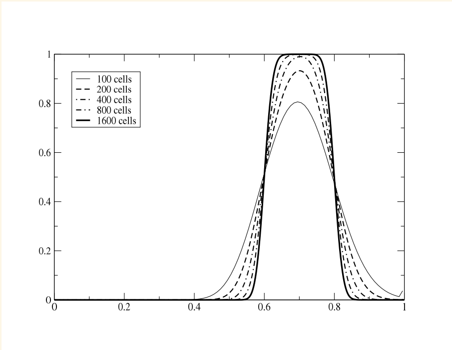
1.3. Streaming in 1D: $\sigma = 0$

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\partial_x F = 0 \\ \partial_t F + \frac{1}{\varepsilon}\partial_x P = 0 \end{cases}$$

Assume moreover that $\frac{F}{E} \equiv 1$ at $t = 0$. Then the equation is equivalent to pure transport of radiation because $P \equiv E$ for all $t > 0$.

- **A numerical experiment.** The initial data is $E = F = 1$ if $0.4 < x < 0.6$, $E = F = 0$ otherwise.



cells	100	200	400	800	1600
$\|\|e\|\|_{L^1}$	0.3	0.21	0.15	0.109	0.077

Radiation energy versus x. The order of convergence is approximatively 0.5.

- The numerical solution has been computed with the same code as for the Sue-Olson text case, with exactly the same implicit scheme.

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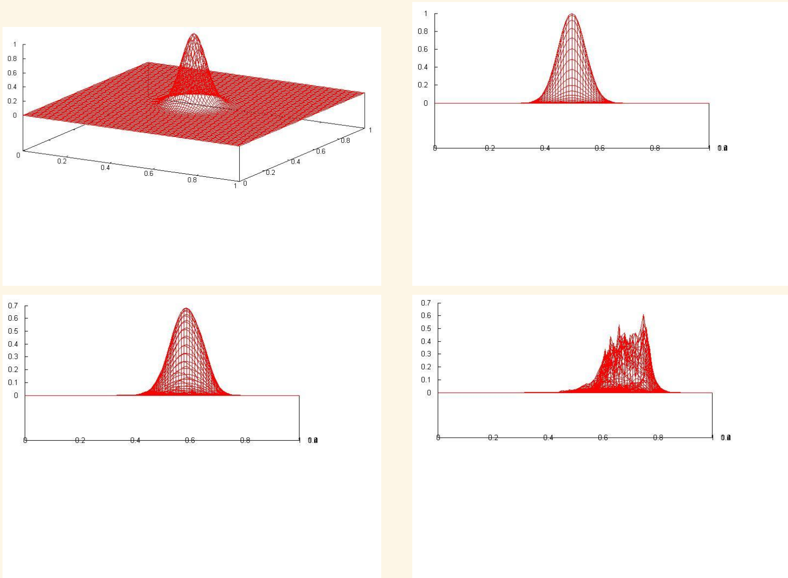
1.4. The streaming regime in 2D: $\frac{|F|}{E} = 1$ at $t = 0$



- Results by Loi-Staudacher.

- The numerical issue is to have stability for $\frac{|F|}{E} = 1$.

- In 1D, 2D and 3D, the PDE system is only weakly hyperbolic for $\frac{|F|}{E} = 1$.

- With the classical strategy (solving te diffusion directly) one has to limit the diffusion coefficient to mimic the streaming regime. This is a no end task.

1.5. Weak hyperbolicity of the Moment Model (in 1D)

- Set

$$f = \frac{F}{E} \in \mathbb{R} \text{ in 1D.}$$

The model is hyperbolic for $f^2 < 1$ and $1 < f^2 < \frac{4}{3}$. For $f^2 = \frac{4}{3}$ is non more differentiable.

- The Jacobian matrix of the flux is

$$\frac{\partial(F,P)}{\partial(E,F)} = \begin{pmatrix} 0 & 1 \\ \chi(f) - f\chi'(f) & \chi'(f) \end{pmatrix} \Rightarrow \frac{\partial(F,P)}{\partial(E,F)} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{for } f = 1.$$

The matrix is not diagonalisable: one eigenvector misses. Therefore the system is only weakly hyperbolic at $f = \pm 1$. We notice that $f = \pm 1$ corresponds to a strongly non isotropic radiation flux.

- Since the eigenvalues coincide for

$$f = \pm 1$$

it means the system is resonant for $|F| = E$. Resonnance implies weak L^2 stability around constants. Some open mathematical issues remain.

- Recall that

$$f = \frac{F}{E} = \frac{\int\int \mathbf{n} I d\nu d\mathbf{n}}{\int\int I d\nu d\mathbf{n}}$$

where $|\mathbf{n}| = 1$ is a direction of photons. Therefore

$$|f| \leq 1$$

is the domain in which the physical and mathematical solution must be seeked for.

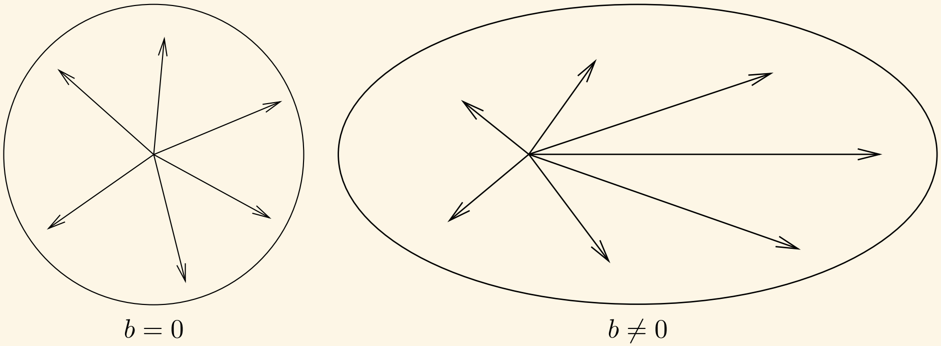
2. Main idea

• We propose to use a very strong similarity, at the PDE level, between radiation and gas dynamics. The anisotropic radiation is

$$E = \frac{3+|b|^2}{3(1-|b|^2)^3} T^4 \in \mathbb{R} \text{ and } F = -\frac{4b}{3(1-|b|^2)^3} T^4$$

The intensity of radiation is

$$I(\nu, \mathbf{n}) = \frac{15}{4\pi^5} \frac{\nu^3}{\exp(\frac{\nu}{T} + \frac{\nu b.\mathbf{n}}{T}) - 1}, \quad |b| \le 1, \quad |\mathbf{n}| = 1.$$



There exists a moving reference frame in which radiation is isotropic.

The velocity in which radiation is isotropic is

$$u = -b \in \mathbb{R}^d.$$

It is compatible with Lorentz invariance.

• Define for convenience an artificial density $\rho \in \mathbb{R}$: $\partial_t \rho + \frac{1}{\varepsilon}\nabla.(\rho u) = 0$. The density depends of course of some artificial initial data. Let us define also a scalar pressure q

$$q = \frac{1}{3(1-|b|^2)^2} T^4 \in \mathbb{R} \quad .$$

• Using these relations we can rewrite the equation of radiation as a system which is formally close to the standard system of gas dynamics

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\nabla.(\rho u) = 0, \\ \partial_t \rho v + \frac{1}{\varepsilon}\nabla.(\rho u \otimes v) + \frac{1}{\varepsilon}\nabla q = -\frac{\sigma}{\varepsilon^2}\rho v, \\ \partial_t \rho e + \frac{1}{\varepsilon}\nabla.(\rho u e + q u) = 0, \\ \partial_r \rho s + \frac{1}{\varepsilon}\nabla.(\rho u s) \geq \cdots, \end{cases} \Longleftrightarrow \begin{cases} \partial_t E + \frac{1}{\varepsilon}\nabla.F = 0, \\ \partial_t F + \frac{1}{\varepsilon}\nabla.P = -\frac{\sigma}{\varepsilon^2}F, \\ + \text{ entropy law.} \end{cases}$$

where by definition $S = \rho s$, $F = \rho v$ and $E = \rho e$.

• q can be computed directly with respect to the main unknowns of this system. The system is closed since the scalar pressure is non singular.

• The vector v is different from the "velocity" u: $v \neq u$. But they are nevertheless colinear since

$$\rho v = \frac{4T^4}{3(1-|b|^2)^3} u = \left(\frac{4}{3+|b|^2}\rho e\right) u.$$

• The M^1 moment model for radiation is equivalent to compressible gas dynamics with friction.

• This is a system of conservation with a stiff right hand side.

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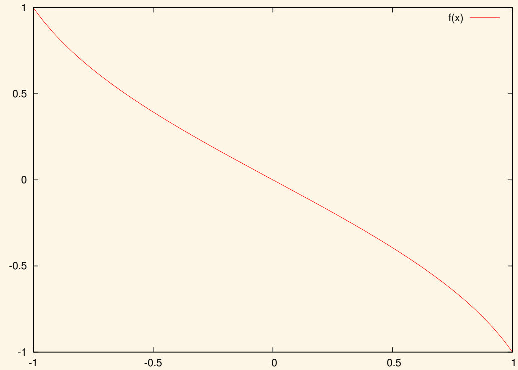
2.1. Connection with pressureless gas dynamics

The algebra yields

$$q = \frac{1 - |b|^2}{3 + |b|^2} \rho e.$$

Te vector of anisotropy is

$$b = -\frac{3f}{2 + \sqrt{4 - 3|f|^2}}, \quad f = \frac{|F|}{E}.$$



For $|f| = 1$ then $q = 0$. This is presureless gas dynamics.

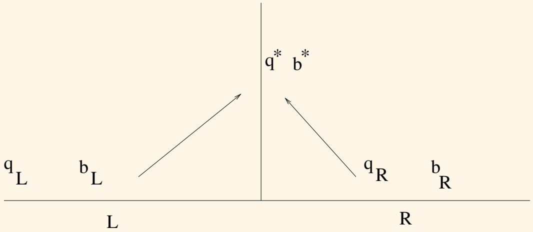
2.2. Numerical strategy: Adapt standard CFD schemes

• **Lagrangian step:** First we rewrite the system in quasi-lagrangian coordinates (standard gas dynamics with friction)

$$\begin{cases} \rho D_t \tau + \frac{1}{\varepsilon}\partial_x b = 0, & b \approx -u \\ \rho D_t v + \frac{1}{\varepsilon}\partial_x q = -\frac{\sigma}{\varepsilon^2}\rho v, & q \approx p \\ \rho D_t e - \frac{1}{\varepsilon}\partial_x (qb) = 0. & \end{cases}$$

-Acoustic solver: At the interfaces the linearized Riemann solver is the "piecewise steady approximation". The acoustic "well-balanced" Riemann solver between a L(eft) state and a R(ight) state $L = i$, $R = i+1$

$$\begin{cases} (q^* - q_R) + \frac{4}{\sqrt{3}}\frac{E_R}{3+|b_R|^2}(b^* - b_R) = \frac{\sigma}{2}\Delta x \rho_R v_R, \\ (q^* - q_L) - \frac{4}{\sqrt{3}}\frac{E_L}{3+|b_L|^2}(b^* - b_L) = -\frac{\sigma}{2}\Delta x \rho_L v_L. \end{cases}$$



• **Remap step** The remap step is standard for gas dynamics: just move the mesh. The velocity of the moving mesh is: $\widetilde{x^{n+1}_{j+\frac{1}{2}}} = x^n_{j+\frac{1}{2}} + \Delta t u^n_{j+\frac{1}{2}} = x^n_{j+\frac{1}{2}} - \Delta t b^n_{j+\frac{1}{2}}$. After that we project the numerical solution onto the old mesh in a conservative fashion.

- One could also take the **"modified upwind scheme"** of Jin and Levermore: $\rho v = -kb$ with $k = -\frac{T^4}{3(1-|b|^2)^3}$. The right hand side becomes

$$\begin{cases} (q^* - q_R) + \frac{4}{\sqrt{3}} \frac{E_R}{3+|b_R|^2}(b^* - b_R) = -\frac{\sigma}{2}\Delta x k_R b^*, \\ (q^* - q_L) - \frac{4}{\sqrt{3}} \frac{E_L}{3+|b_L|^2}(b^* - b_L) = \frac{\sigma}{2}\Delta x k_L b^*. \end{cases}$$

- The standard solver is

$$\begin{cases} (q^* - q_R) + \frac{4}{\sqrt{3}} \frac{E_R}{3+|b_R|^2}(b^* - b_R) = 0, \\ (q^* - q_L) - \frac{4}{\sqrt{3}} \frac{E_L}{3+|b_L|^2}(b^* - b_L) = 0. \end{cases}$$

Known to fail for such problems.

2.3. Theoretical result: correct diffusion regime

Property 1: *The asymptotic limit is the discrete diffusion equation*

$$\frac{d}{dt}E_j - \frac{E_{j+1} - 2E_j + E_{j-1}}{3\sigma\Delta x^2} = O_j^{weak}(\Delta x).$$

The right hand side is

$$O_j^{weak}(\Delta x) = \frac{O_{j+\frac{1}{2}}(\Delta x) - O_{j-\frac{1}{2}}(\Delta x)}{\Delta x}.$$

In the finite difference sense one has $O_j^{weak}(\Delta x) = O(1)$. But this term is consistent with the weak formulation of the heat equation because it is the difference of two $O(1)$ terms. That is

$$O^{weak}(\Delta x) = O(\Delta x) \text{ in the finite volume sense.}$$

In other words

$$O^{weak}(\Delta x) \to 0 \text{ as } \Delta x \to 0$$

in the weak sense.

2.4. Theoretical result: stability for the streaming regime

Property 2: *The semi-discrete (for sake of simplicity) lagrangian scheme with source term is entropic*

$$s_j'(t) \geq 0.$$

• Let us discuss the consequences of this property on the maximum principle. One has the formula

$$\Delta m_j s_j = \Delta x_j S_j = \left[\Delta x_j \frac{4}{3} \left(\frac{3}{3 + |b_j|^2} \right)^{\frac{3}{4}} \right] E_j^{\frac{3}{4}} \left(1 - |b_j|^2\right)^{\frac{1}{4}}.$$

Assume for simplicity the energy in the cell is positive and

$$|b| = \left| -\frac{3f}{2 + \sqrt{4 - 3|f|^2}} \right| < 1 \text{ at } t = 0.$$

Then $s_j(0) > 0$. So $s_j(t) > 0$. Therefore the product

$$\Delta x_j E_j^{\frac{3}{4}} \left(1 - |b_j|^2\right)^{\frac{1}{4}} > 0$$

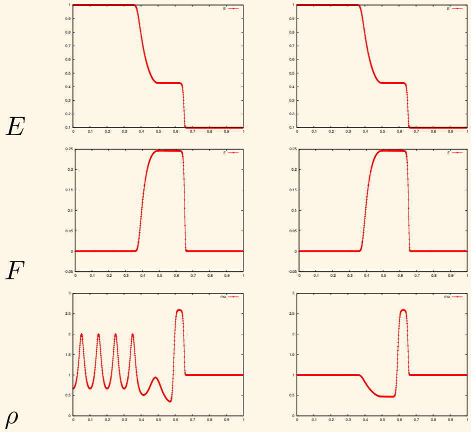
is positive. The maximum principle can be seen as a consequence of this inequality.

• If the energy E_j is positive and the cell is non degenerate $0 < \Delta x_j < \infty$ then $|b_j| < 1$. By continuity the energy can not vanish. The only case where $|b| = 1$ is possible is if the mesh degenerates $\Delta x_j = \infty$. This is not possible in finite time since the size of the cell is a continuous function of the interface velocities $u_{j+\frac{1}{2}} = b^*_{j+\frac{1}{2}}$ and $u_{j-\frac{1}{2}} = b^*_{j-\frac{1}{2}}$.

3. Numerical results

3.1. Test case 1: radiative Riemann problem

We consider a Riemann problem. The coefficients are $\sigma = 0$ everywhere and $\varepsilon = 1$. The initial values are $(E, F_1) = (1, 0)$ for $x < 0.5$, and $(E, F_1) = (0.1, 0)$ for $0.5 < x$. The second component of the radiative flux is zero $F_2 \equiv 0$. The solution consists in a mathematical rarefaction fan on the left and a shock on the right.



- E, F and ρ at $t = 0.2$. Results in the first and second columns have been computed with different densities but same radiative energy and radiative flux.

- The results are independent of the initial value of the density ρ.

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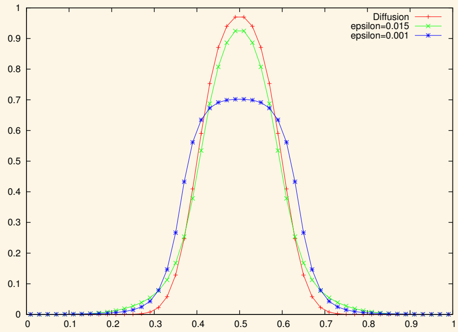
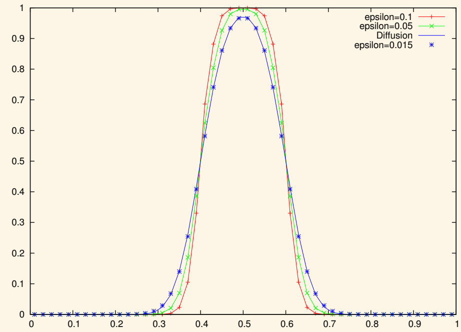
3.2. Test case 2: diffusion limit $\sigma = 1$ and $10^{-4} \leq \varepsilon \leq 10^{-1}$

The initial values are $(E, F_1, F_2) = (1, 0, 0)$ for $0.4 < x < 0.6$, and $(E, F_1, F_2) = (10^{-6}, 0, 0)$ elsewhere.

• The diffusion limit is not captured with the classical Riemann solver. The solver becomes pathological as $\varepsilon \longrightarrow 0$.



Classical fluxes



New fluxes

Non convergence towards the solution of the heat equation as $\varepsilon \longrightarrow 0$. The curve for $\varepsilon = 0.015$ is completely different from the solution of the diffusion (heat) equation. Final time $T = 0.003$.

• With the new Riemann solver, the stability and convergence of the algorithm is evident. The curve for $\varepsilon = 0.015$ is not distinguishable from the solution of the diffusion (heat) equation.

3.3. Diffusion limit: the error

• We show the relative error (in the L^∞ norm) between the discrete solution of the heat equation and the discrete solution of the moment model

$$(\Delta x, \varepsilon) \mapsto \frac{\|u_{\Delta x}^{\text{moment model},\varepsilon} - u_{\Delta x}^{\text{diffusion equation}}\|_{L^\infty}}{\|u_{\Delta x}^{\text{diffusion equation}}\|_{L^\infty}}, \qquad T = 0.003.$$

This error is made of two contributions: ε is the model error, Δx is the discretization error.

• In first and second columns the model error is dominant and this is why the error increases as Δx tends to zero.

• In the fourth and fifth columns, the discretization error is dominant so the error decreases as Δx tends to zero. The third column is somehow in between, the model error is of the same order than the discretization error.

• The behavior on lines is monotone and illustrates the theoretical result of convergence.

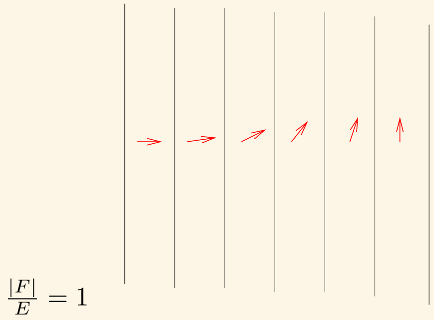
	$\varepsilon = 10^{-1}$	$\varepsilon = 5.10^{-2}$	$\varepsilon = 1.5\ 10^{-2}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$
$\Delta x = 1/50$	0.15	0.061	0.012	0.017	0.025
$\Delta x = 1/100$	0.17	0.080	0.009	0.010	0.014
$\Delta x = 1/200$	0.20	0.106	0.009	0.066	0.008
$\Delta x = 1/400$	0.24	0.130	0.012	0.004	0.004

3.4. Test case 3: streaming regime $\sigma = 0$ $\varepsilon = 1$

- The initial values are

$$(E, F_1, F_2) = (1, \frac{0.7 - x}{0.4}, \sqrt{1 - F_1^2}), \quad 0.3 < x < 0.7,$$

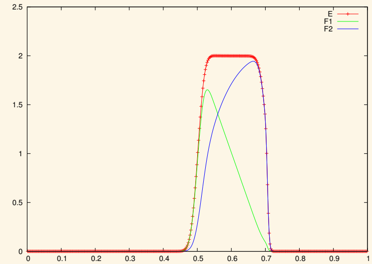
and $(E, F_1, F_2) = (0, 0, 0)$ elsewhere.



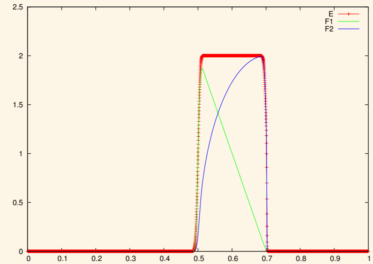
- At $t < 0.4$ the analytical solution is

$$(E, F_1, F_2) = (\frac{0.4}{0.4 - t}, \frac{0.4(0.7 - x)}{(0.4 - t)^2}, \sqrt{1 - F_1^2}) \text{ for } 0.3 + t < x < 0.7,$$

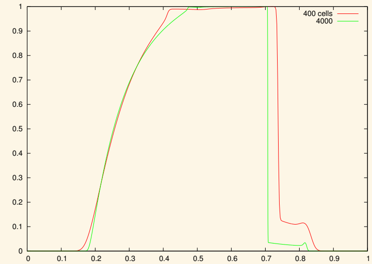
and $(E, F_1, F_2) = (0, 0, 0)$ elsewhere.



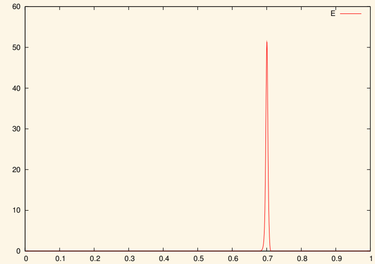
$t = 0.2$ 400 cells



$t = 0.2$ 4000 cells



$f = \frac{\sqrt{F_1^2 + F_2^2}}{E}$



E at $t = 0.4$

- This is also the exact solution of free transport.

- Impossible with a diffusion equation, whatever is the diffusion coefficient. Because diffusion equations satisfy the maximum principle.

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3.5. Other models

Consider ($\alpha \in \mathbb{R}$ is a parameter)

$$\begin{cases} S = \frac{\alpha}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}} T^{\alpha+1}, \\ E = \frac{(\alpha+1)-b^2}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}-1} T^{\alpha+1} \\ F = -\frac{\alpha}{\alpha+1} b(1-b^2)^{\frac{\alpha}{2}-1} T^{\alpha+1}, \\ q = -\frac{1}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}} T^{\alpha+1}, \\ P = \frac{(\alpha+1)b^2-1}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}} T^{\alpha+1}, \end{cases}$$

Then

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\nabla.(\rho u) = 0, \\ \partial_t \rho v + \frac{1}{\varepsilon}\nabla.(\rho u \otimes v) + \frac{1}{\varepsilon}\nabla q = -\frac{\sigma}{\varepsilon^2}\rho v, \\ \partial_t \rho e + \frac{1}{\varepsilon}\nabla.(\rho u e + q u) = 0, \\ \partial_r \rho s + \frac{1}{\varepsilon}\nabla.(\rho u s) \geq 0, \end{cases} \iff \begin{cases} \partial_t E + \frac{1}{\varepsilon}\nabla.F = 0, \\ \partial_t F + \frac{1}{\varepsilon}\nabla.P = -\frac{\sigma}{\varepsilon^2}F, \\ \\ + \text{ entropy law.} \end{cases}$$

- Radiation is $\alpha = 4$.

- The use of such methods for other plasma physics models is open.
One may think about the diffusion of electrons.

4. Conclusion

• It is possible ro rewrite the Moment Model for radiation as the standard gas dynamic Euler system, using a convenient choice of unknowns.

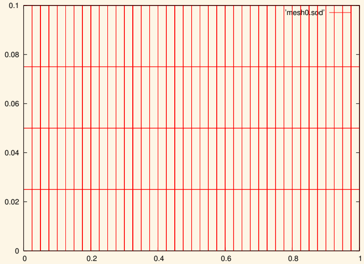
Numerical results show the resulting scheme with ad-hoc Riemann solvers is

1) Asymptotic preserving (AP)

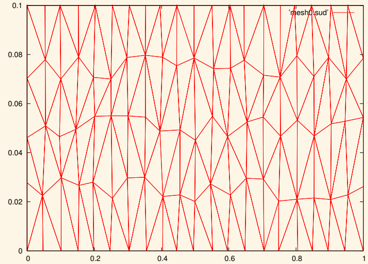
2) Positive $\frac{|F|}{E} \leq 1$.

One may think about using other techniques (DG, nodal fluxes, ...).

• At the level of principles, Cartesian structured meshes and Lagrangian unstructured meshes are taken into account. This is an important progress with respect to the theory on Cartesian meshes (see Dubroca, Berthon and al, our previous work).



Cartesian structured mesh



Lagrangian unstructured non regular mesh

• It may give ideas for other models (in plasma physics).