Gyro-kinetic models for the Vlasov-Maxwell equations

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Numerical flow models for the controlled fusion

Porquerolles, France, april 16-20 2007

Vlasov equation

$$\partial_t f_\pm + v_\pm(p) \cdot \nabla_x f_\pm + q_\pm (E + v_\pm(p) \wedge B) \cdot \nabla_p f_\pm = 0$$

Particle density : $f_\pm(t, x, p),\ (t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$

Electro-magnetic field : $(E(t, x), B(t, x)),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$

$(q_\pm, m_\pm)$: particle charge/mass (ions/electrons)

$$v_\pm(p) = \frac{p}{m_\pm} \quad \text{non relativistic case (NR)}$$

$$v_\pm(p) = \frac{p}{m_\pm}\left(1 + \frac{|p|^2}{m_\pm^2 c_0^2}\right)^{-\frac{1}{2}} \quad \text{relativistic case (R)}$$

Particle density

$f_{\pm}(t,x,p)$: ion/electron densities in the phase space

$\mathcal{N}_{\pm}(\mathcal{V}_x \times \mathcal{V}_p) =$ number of ions/electrons in $\mathcal{V}_x \times \mathcal{V}_p$ at time t

$$\mathcal{N}_{\pm}(\mathcal{V}_x \times \mathcal{V}_p) \approx \int_{\mathcal{V}_x}\int_{\mathcal{V}_p} f_{\pm}(t,x,p)\, dpdx$$

Transport equation

$$\begin{cases} \dfrac{d}{ds}X(s) = v(P(s)), \ \ \dfrac{d}{ds}P(s) = q\{E(s,X(s)) + v(P(s)) \wedge B(s,X(s))\} \\ X(s=t) = x, P(s=t) = p \end{cases}$$

$$\mathrm{div}_{(x,p)}(v(p), q(E(t,x) + v(p) \wedge B(t,x))) = 0$$

$$\Downarrow$$

$$\frac{d}{ds}\frac{\partial(X(s;t,x,p), P(s;t,x,p))}{\partial(x,p)} = 0$$

$$\Downarrow$$

$$\frac{d}{ds}\{f_\pm(s, X(s), P(s))\} = 0$$

$$\partial_t f_\pm + v_\pm(p)\cdot\nabla_x f_\pm + q_\pm(E(t,x) + v_\pm(p)\wedge B(t,x))\cdot\nabla_p f_\pm = 0$$

Maxwell equations

$$\partial_t E - c_0^2 \mathrm{rot}_x B = -\frac{1}{\varepsilon_0}\left(j_+(t,x) - j_-(t,x)\right), \quad \partial_t B + \mathrm{rot}_x E = 0$$

$$\mathrm{div}_x E = \frac{1}{\varepsilon_0}\left(\rho_+(t,x) - \rho_-(t,x)\right), \quad \mathrm{div}_x B = 0$$

c_0 light speed, ε_0 electric permittivity

$\rho_\pm / j_\pm$ charge/current densities

$$\rho_\pm = q_\pm \int_{\mathbb{R}^3} f_\pm \, dp, \quad j_\pm = q_\pm \int_{\mathbb{R}^3} v_\pm(p) f_\pm \, dp$$

Vlasov-Poisson system

$$\partial_t f + p \cdot \nabla_x f - eE(t,x) \cdot \nabla_p f = 0$$

$$E = -\nabla_x \phi, \quad -\Delta_x \phi = \frac{1}{\varepsilon_0}\left(\rho_{\text{ext}} - \int_{\mathbb{R}^3} f(t,x,p)\, dp\right)$$

$f(t,x,p)$ electron density

ρ_{ext} concentration of a neutralizing ion background

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \; dxdp = \int_{\mathbb{R}^3} \rho_{\text{ext}} \; dx$$

Reduced models for laser-plasma interaction

Particular geometry $\Rightarrow$ additional invariants

Laser wave direction of propagation : privileged direction Ox_1

$$\partial_{x_2} = \partial_{x_3} = 0, \quad B = \mathrm{rot} A, \quad E = -\partial_t A - \nabla_x \phi$$

$$B_2 = -\partial_{x_1} A_3, \quad B_3 = \partial_{x_1} A_2, \quad E_2 = -\partial_t A_2, \quad E_3 = -\partial_t A_3$$

$$\frac{d}{ds}\{P_2(s) - eA_2(s, X_1(s))\} = \frac{d}{ds}\{P_3(s) - eA_3(s, X_1(s))\} = 0$$

$$f(t,x,p)=\tilde{f}(t,x_1,p_1)\delta(p_2-eA_2(t,x_1))\otimes\delta(p_3-eA_3(t,x_1))$$

$$\partial_t\tilde{f}+\frac{p_1}{m\gamma}\partial_{x_1}\tilde{f}-e(E_1+\frac{e}{m\gamma}A_2\partial_{x_1}A_2+\frac{e}{m\gamma}A_3\partial_{x_1}A_3)\partial_{p_1}\tilde{f}=0$$

$$\partial_t^2A_k-c^2\partial_{x_1}^2A_k=-\frac{e^2}{m\varepsilon_0}\rho_\gamma(t,x_1)A_k(t,x_1),\quad k\in\{2,3\}$$

$$\partial_tE_1=\frac{e}{\varepsilon_0}j_1,\quad \partial_{x_1}E_1=\frac{e}{\varepsilon_0}(\rho_{ext}-\rho)$$

$$\{\rho,\rho_\gamma,j_1\}(t,x_1)=\int_{\mathbb{R}}\left\{1,\frac{1}{\gamma},\frac{p_1}{\gamma}\right\}\tilde{f}(t,x_1,p_1)dp_1$$

$$\gamma=\left(1+(p_1)^2/(mc)^2+e^2A_2(t,x_1)^2/(mc)^2+e^2A_3(t,x_1)^2/(mc)^2\right)^{\frac{1}{2}}$$

$$\partial_t f + \frac{p}{\gamma_1}\partial_x f - \left(E(t,x) + \frac{1}{\gamma_2} A(t,x)\partial_x A \right) \partial_p f = 0$$

$$\{\rho, \rho_{\gamma_2}, j\}(t,x) = \int_{\mathbb{R}} \{1, \frac{1}{\gamma_2}, \frac{p}{\gamma_1}\} f(t,x,p)\, dp$$

Non relativistic model (NR) $\gamma_1 = \gamma_2 = 1$

Quasi-relativistic model (QR) $\gamma_1 = (1+p^2)^{1/2}$, $\gamma_2 = 1$

Fully relativistic model (FR) $\gamma_1 = \gamma_2 = (1 + p^2 + A(t,x)^2)^{\frac{1}{2}}$

J.A. Carrillo, S. Labrunie '06 NR, QR

M.B. '06 FR

Gyro-kinetic models

High magnetic field limits

Gyration radius of particles around the magnetic field lines (Larmor radius)

Formal derivations based on Hilbert expansion method

Typical scales

$$p_{\rm th} = \sqrt{K_B T_{\rm th} m}$$ thermal impulsion

$$v_{\rm th} = \frac{p_{\rm th}}{m}$$ thermal velocity

$$\lambda_D = \sqrt{\frac{\varepsilon_0 K_B T_{\rm th}}{e^2 n}}$$ Debye length

$$\omega_p = \sqrt{\frac{e^2 n}{m\varepsilon_0}} = \frac{v_{\rm th}}{\lambda_D}$$ plasma frequency

Guiding-center approximation

$$\omega_c = \frac{1}{T_c} = \frac{|q||B|}{m} \gg 1$$

Individual motion under constant fields

$$\frac{dX}{ds} = V(s), \quad \frac{dV}{ds} = \frac{q}{m}(E + V(s) \wedge B)$$

$$\frac{d}{ds}\left(V(s) \cdot \frac{B}{|B|}\right) = \frac{q}{m}\frac{E \cdot B}{|B|}$$

$$V(s) = \frac{E \wedge B}{|B|^2} + U(s), \quad \frac{dU}{ds} = \frac{q}{m}\left((E \cdot B)\frac{B}{|B|^2} + U(s) \wedge B\right)$$

$$U_\perp(s) = \left(\frac{B}{|B|} \wedge U(s)\right) \wedge \frac{B}{|B|}, \quad \frac{d^2}{ds^2} U_\perp + \omega_c^2 U_\perp(s) = 0$$

$$U_\perp(s) = \mathcal{R}(-\omega_c s) U_\perp(0) = \mathcal{R}(-\omega_c s) \left(V_\perp(0) - \frac{E \wedge B}{|B|^2} \right)$$

$$\begin{aligned} X_\perp(t) &= X_\perp(0) - \frac{1}{\omega_c} \mathcal{R}\left(\frac{\pi}{2}\right) U_\perp(0) + t \frac{E \wedge B}{|B|^2} \\ &+ \frac{1}{\omega_c} \mathcal{R}\left(-\omega_c t + \frac{\pi}{2}\right) U_\perp(0) \end{aligned}$$

$$B = \mathcal{O}(\varepsilon^{-1}), \quad \frac{E \wedge B}{|B|^2} = \mathcal{O}(\varepsilon), \quad t = \mathcal{O}(\varepsilon^{-1})$$

Observation units (non relativistic case)

$$L = \lambda_D, \quad T = \frac{T_p}{\varepsilon}, \quad P = p_{\rm th}$$

$$\frac{p_{\rm th}^2}{m_e} = K_B T_{\rm th}$$

$$t = Tt', \quad x = Lx', \quad p = p_{\rm th} p'$$

$$f = \frac{n_e}{p_{\rm th}^3} f'(\frac{t}{T}, \frac{x}{L}, \frac{p}{p_{\rm th}}), \quad E = \frac{U_{\rm th}}{L} E'(\frac{t}{T}, \frac{x}{L}), \quad B = \frac{1}{\varepsilon} \frac{m_e}{e T_p} B'(\frac{t}{T}, \frac{x}{L})$$

I Compactness method : uniform estimates w.r.t. $\varepsilon > 0$; provides weak convergence

II Modulated energy method : strong convergence if the solution of the limit system is smooth

plasma physics (Brenier '00, Brenier, Mauser, Puel '03, Golse, Saint-Raymond '03)

gaz dynamics (Saint-Raymond '03, Berthelin, Vasseur '05)

fluid-particles interaction (Goudon, Jabin, Vasseur '04)

Dimensionless system

$$\partial_t f + \frac{p}{\varepsilon} \cdot \nabla_x f - \frac{1}{\varepsilon}\left(E(t,x) + p \wedge \frac{B(t,x)}{\varepsilon}\right) \cdot \nabla_p f = 0$$

$$\partial_t E - \frac{1}{\varepsilon}\mathrm{curl}\left(\frac{B}{\varepsilon}\right) = \frac{1}{\varepsilon} j(t,x)$$

$$\partial_t \left(\frac{B}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathrm{curl} E = 0$$

$$\mathrm{div} E = 1 - \rho(t,x), \quad \mathrm{div} B = 0$$

Energy balance

$$\frac{d}{dt}\left\{\int\int\frac{|p|^2}{2}f_\varepsilon dpdx + \frac{1}{2}\int\left(|E_\varepsilon|^2 + \left|\frac{B_\varepsilon}{\varepsilon}\right|^2\right)dx\right\} = 0$$

$$\sup_{\varepsilon>0}\int\left|\frac{B_\varepsilon(0,x)}{\varepsilon}\right|^2 dx = +\infty$$

$$\partial_t E_\varepsilon - \frac{1}{\varepsilon}\mathrm{curl}\left(\frac{B_\varepsilon - B_0}{\varepsilon}\right) = \frac{1}{\varepsilon}j^\varepsilon,\ \ \partial_t\left(\frac{B_\varepsilon - B_0}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathrm{curl}E_\varepsilon = 0$$

$$\frac{d}{dt}\left\{\int\int\frac{|p|^2}{2}f_\varepsilon dpdx+\frac{1}{2}\int\left(|E_\varepsilon|^2+\left|\frac{B_\varepsilon-B_0}{\varepsilon}\right|^2\right)dx\right\}=0$$

$$\sup_{\varepsilon>0}\int\int\frac{|p|^2}{2}f_\varepsilon^0 \quad + \quad \frac{1}{2}\int|E_\varepsilon^0|^2+\frac{1}{2}\int\left|\frac{B_\varepsilon^0-B_0}{\varepsilon}\right|^2<+\infty$$

$$\Downarrow$$

$$\sup_{\varepsilon>0,t>0}\int\int\frac{|p|^2}{2}f^\varepsilon \quad + \quad \frac{1}{2}\int|E_\varepsilon|^2+\frac{1}{2}\int\left|\frac{B_\varepsilon-B_0}{\varepsilon}\right|^2<+\infty$$

$$\frac{B_\varepsilon}{\varepsilon}=\frac{B_\varepsilon-B_0}{\varepsilon}+\frac{B_0}{\varepsilon}$$

Two dimensional case

$$f^\varepsilon = f^\varepsilon(t,x,p),\ \ (t,x,p) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2,\ \ (E_\varepsilon, B_\varepsilon) = (E_1^\varepsilon, E_2^\varepsilon, 0, 0, 0, B_3^\varepsilon)$$

$$F^\varepsilon(t,x,u) = \varepsilon^2 f^\varepsilon(t,x,\varepsilon u),\ \ (t,x,u) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2,\ \ p = \varepsilon u$$

$$\rho^\varepsilon = \int f^\varepsilon \, dp = \int F^\varepsilon \, du,\ \ j^\varepsilon = \int p f^\varepsilon \, dp = \varepsilon \int u F^\varepsilon \, du = \varepsilon J^\varepsilon$$

$$\partial_t F^\varepsilon + u \cdot \nabla_x F^\varepsilon - \frac{1}{\varepsilon^2}(E_\varepsilon + {}^\perp u \, B_3^\varepsilon) \cdot \nabla_u F^\varepsilon = 0$$

$${}^\perp v = (v_2, -v_1),\ \ \forall\, v = (v_1, v_2)$$

Formal analysis

$$\sup_{\varepsilon>0,t>0} \varepsilon^2 \int\int \frac{|u|^2}{2} F^\varepsilon du dx + \frac{1}{2}\int |E_\varepsilon(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2 dx < \infty$$

Mass balance

$$\partial_t \rho^\varepsilon + \mathrm{div}_x J^\varepsilon = 0$$

Momentum balance

$$\varepsilon^2 \partial_t \int\int u F^\varepsilon \, du + \varepsilon^2 \mathrm{div}_x \int\int (u \otimes u) F^\varepsilon \, du + \rho^\varepsilon E_\varepsilon + B_3^\varepsilon {}^\perp J^\varepsilon = 0$$

$$\rho(t,x)E(t,x) + B_{0,3}{}^{\perp}J(t,x) = 0 \quad \Rightarrow J = \rho \frac{{}^{\perp}E}{B_{0,3}}$$

$$\partial_t \left(\frac{B_3^{\varepsilon} - B_{0,3}}{\varepsilon} \right) + \frac{1}{\varepsilon}(\partial_{x_1} E_2^{\varepsilon} - \partial_{x_2} E_1^{\varepsilon}) = 0 \ \Rightarrow \ \partial_{x_1} E_2 - \partial_{x_2} E_1 = 0$$

Limit system

$$J = \rho \frac{{}^{\perp}E}{B_{0,3}}, \ \ \mathrm{div}^{\perp}E = 0$$

$$\partial_t \rho + \mathrm{div} \left(\rho \frac{{}^{\perp}E}{B_{0,3}} \right) = 0, \ \ \mathrm{div} E = 1 - \rho(t,x)$$

Modulated energy

$$\mathcal{H}_\varepsilon(t) = \varepsilon^2 \int\int \frac{|u-D|^2}{2} F^\varepsilon(t,x,u)\, dudx + \frac{1}{2}\int \left\{ |E_\varepsilon(t,x) - E(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon} \right)^2 \right\} dx$$

$$D(t,x) = \frac{J(t,x)}{\rho(t,x)} = \frac{{}^\perp E(t,x)}{B_{0,3}}$$

Modulated energy balance

If $D \in W^{1,\infty}(]0,T[\times \mathbb{T}^2)^2$, then $\exists\ C > 0$ s.t.

$$\frac{d}{dt}\mathcal{H}_\varepsilon(t) \le C\varepsilon^2 + C\mathcal{H}_\varepsilon(t),\ \ \forall\, t \in [0,T]$$

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- H2) $(E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon) \in L^2(\mathbb{T}^2)^3$, $\mathrm{div} E_\varepsilon^0 = 1 - \rho_0^\varepsilon$, $\rho_0^\varepsilon = \int f_\varepsilon^0\,dp$

Hypotheses

- H1) $f_\varepsilon^0 \geq 0, \quad \int\int f_\varepsilon^0(x,p)\, dpdx = 1$
- H2) $(E^\varepsilon_{0,1}, E^\varepsilon_{0,2}, B^\varepsilon_{0,3}) \in L^2(\mathbb{T}^2)^3$, $\mathrm{div} E_\varepsilon^0 = 1 - \rho_0^\varepsilon$, $\rho_0^\varepsilon = \int f_\varepsilon^0 \, dp$
- H3) $\lim_{\varepsilon \searrow 0} \int\int \frac{|p|^2}{2} f_\varepsilon^0(x,p)\, dpdx = 0$

Hypotheses

- H1) $f_\varepsilon^0 \geq 0, \quad \int\!\!\int f_\varepsilon^0(x,p)\, dpdx = 1$
- H2) $(E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon) \in L^2(\mathbb{T}^2)^3$, $\operatorname{div} E_\varepsilon^0 = 1 - \rho_0^\varepsilon$, $\rho_0^\varepsilon = \int f_\varepsilon^0\, dp$
- H3) $\lim_{\varepsilon \searrow 0} \int\!\!\int \frac{|p|^2}{2} f_\varepsilon^0(x,p)\, dpdx = 0$
- H4) $\exists\ E_0 = (E_{0,1}, E_{0,2}) \in L^2(\mathbb{T}^2)^2, \quad \partial_{x_1} E_{0,2} - \partial_{x_2} E_{0,1} = 0$ and $B_0 = (0,0,B_{0,3}),\ B_{0,3} \neq 0$ s.t.

$$\lim_{\varepsilon \searrow 0} \left\{ \frac{1}{2} \int |E_\varepsilon^0(x) - E_0(x)|^2 dx + \frac{1}{2} \int \left(\frac{B_{0,3}^\varepsilon(x) - B_{0,3}}{\varepsilon} \right)^2 dx \right\} = 0$$

Theorem

$$\int\!\!\int \frac{|p|^2}{2} f_\varepsilon^0 \, dpdx + \frac{1}{2}\int |E_\varepsilon^0 - E_0|^2 + \left(\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon}\right)^2 \leq C\varepsilon^2$$

$$\Downarrow$$

$$\int\!\!\int \frac{|p|^2}{2} f^\varepsilon(t,x,p)\, dpdx + \frac{1}{2}\int |E_\varepsilon(t,x) - E(t,x)|^2 \, dx$$
$$+ \quad \frac{1}{2}\int \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2 dx \leq C\varepsilon^2$$

$$\lim_{\varepsilon \searrow 0} \rho^\varepsilon = \rho, \quad \lim_{\varepsilon \searrow 0} J^\varepsilon = \lim_{\varepsilon \searrow 0} \frac{j^\varepsilon}{\varepsilon} = J \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)$$

Higher order approximation

$$F^{\varepsilon} = F + \varepsilon F^{(1)} + \varepsilon^2 F^{(2)} + \dots$$

$$E_{\varepsilon} = E + \varepsilon E^{(1)} + \varepsilon^2 E^{(2)} + \dots$$

$$B_{\varepsilon} = B_0 + \varepsilon^2 A + \varepsilon^3 A^{(1)} + \dots$$

Remark There is no first order term in B_{ε} since

$$\lim_{\varepsilon \searrow 0} \frac{B_{\varepsilon} - B_0}{\varepsilon} = 0, \quad \text{in} \quad L^2$$

$$\varepsilon^0 \ : \ -(E + B_0 {}^\perp u) \cdot \nabla_u F = 0 \Rightarrow J = \rho \frac{{}^\perp E}{B_0}$$

$$\varepsilon^1 \ : \ -(E + B_0 {}^\perp u) \cdot \nabla_u F^{(1)} - E^{(1)} \cdot \nabla_u F = 0 \Rightarrow J^{(1)} = \frac{\rho^{(1)} {}^\perp E + \rho {}^\perp E^{(1)}}{B_0}$$

$$(\partial_t + u \cdot \nabla_x) F - (E + B_0 {}^\perp u) \cdot \nabla_u F^{(2)} - E^{(1)} \cdot \nabla_u F^{(1)} - (E^{(2)} + A {}^\perp u) \cdot \nabla_u F = 0$$

Limit systems

$$J = \rho \frac{{}^\perp E}{B_0}, \;\; \partial_t E -^\perp \nabla A = J, \;\; \mathrm{div}^\perp E = 0, \;\; \mathrm{div} E = 1 - \rho$$

$$J^{(1)} = \frac{\rho^{(1)\perp} E + \rho^\perp E^{(1)}}{B_0}, \;\; \partial_t E^{(1)} -^\perp \nabla A^{(1)} = J^{(1)}$$
$$\mathrm{div}^\perp E^{(1)} = 0, \;\; \mathrm{div} E^{(1)} = -\rho^{(1)}$$

$$B_0 \partial_t \rho^{(2)} +^\perp E \cdot \nabla \rho^{(2)} +^\perp E^{(2)} \cdot \nabla \rho = \mathcal{F}(\rho, J, E, A, \rho^{(1)}, J^{(1)})$$
$$\partial_t A + \mathrm{div}^\perp E^{(2)} = 0, \;\; \mathrm{div} E^{(2)} = -\rho^{(2)}$$

$$B_0 \partial_t \rho^{(1)} + \nabla_x \rho^{(1)} \cdot {}^\perp E + \nabla_x \rho \cdot {}^\perp E^{(1)} = 0, \;\; \mathrm{div}^\perp E^{(1)} = 0, \;\; \mathrm{div} E^{(1)} = -\rho^{(1)}$$

$$\begin{aligned} \mathcal{F}(\rho, J, E, A, \rho^{(1)}, E^{(1)}) &= -\partial_t {}^\perp J + \frac{1}{(B_0)^2} (\nabla_x \otimes \nabla_x) : (\rho^\perp E \otimes E) \\ &- {}^\perp E^{(1)} \cdot \nabla_x \rho^{(1)} + \mathrm{div}(AJ) + \rho \partial_t A \end{aligned}$$

Modulated energy revisited

$$\tilde{\mathcal{H}}_\varepsilon(t) = \varepsilon^2 \int\int \frac{1}{2} \left| u - \frac{{}^\perp(E + \varepsilon E^{(1)})}{B_0} \right|^2 F^\varepsilon(t, x, u)\, dudx$$
$$+ \frac{1}{2} \int \left\{ |E_\varepsilon - E - \varepsilon E^{(1)}|^2 + \left(\frac{B_\varepsilon - B_0}{\varepsilon} - \varepsilon A \right)^2 \right\} dx$$

Proposition For any $T \in \mathbb{R}_+$ there is $C > 0$ s.t.

$$\frac{d}{dt} \tilde{\mathcal{H}}_\varepsilon(t) \le C\varepsilon^4 + C\tilde{\mathcal{H}}_\varepsilon(t), \quad t \in [0, T]$$

Proposition If $\sup_{\varepsilon>0}\{\varepsilon^{-4}\tilde{\mathcal{H}}_\varepsilon(0)\} < +\infty$ then for any $T \in \mathbb{R}_+$ there is $C > 0$ s.t.

$$\tilde{\mathcal{H}}_\varepsilon(t) \leq C\varepsilon^4, \quad t \in [0, T]$$

In particular

$$\|E_\varepsilon - E - \varepsilon E^{(1)}\|_{L^2} + \left\|\frac{B_\varepsilon - B_0}{\varepsilon} - \varepsilon A\right\|_{L^2} \leq C\varepsilon^2$$

$$\|\rho^\varepsilon - \rho - \varepsilon\rho^{(1)}\|_{\mathcal{D}'} + \|J^\varepsilon - J - \varepsilon J^{(1)}\|_{\mathcal{D}'} \leq C\varepsilon^2$$

Other models

Mono-kinetic distribution $\quad F^\varepsilon(t,x,u) = \rho^\varepsilon(t,x)\delta(u - u^\varepsilon(t,x))$

$$\partial_t \rho^\varepsilon + \mathrm{div}_x(\rho^\varepsilon u^\varepsilon) = 0$$

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}_x(\rho^\varepsilon(u^\varepsilon \otimes u^\varepsilon)) + \frac{1}{\varepsilon^2}\rho^\varepsilon(E_\varepsilon + B_\varepsilon\, {}^\perp u^\varepsilon) = 0$$

$$\partial_t E_\varepsilon - \frac{1}{\varepsilon}{}^\perp\nabla_x \left(\frac{B_\varepsilon}{\varepsilon}\right) = \rho^\varepsilon u^\varepsilon, \;\; \partial_t \left(\frac{B_\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathrm{div}_x^\perp E_\varepsilon = 0$$

$$\mathrm{div}_x E_\varepsilon = 1 - \rho^\varepsilon$$

Total energy conservation

$$\frac{d}{dt}\left\{\int_{\mathbb{T}^2}\frac{\varepsilon^2}{2}|u^\varepsilon|^2\rho^\varepsilon\,dx+\frac{1}{2}\int_{\mathbb{T}^2}\left(|E_\varepsilon|^2+\left(\frac{B_\varepsilon-B_0}{\varepsilon}\right)^2\right)\,dx\right\}=0$$

Modulated energy

$$\tilde{\tilde{\mathcal{H}}}_\varepsilon(t)=\int_{\mathbb{T}^2}\frac{\varepsilon^2}{2}|u^\varepsilon-u|^2\rho^\varepsilon\,dx+\frac{1}{2}\int_{\mathbb{T}^2}\left\{|E_\varepsilon-E|^2+\left(\frac{B_\varepsilon-B_0}{\varepsilon}\right)^2\right\}\,dx$$

Limit model

$$u(t,x)=\frac{{}^\perp E(t,x)}{B_0},\quad \mathrm{div}_x^\perp E=0,\quad \partial_t\rho+\frac{{}^\perp E}{B_0}\cdot\nabla_x\rho=0,\quad \mathrm{div}_x E=1-\rho$$

Collisions

Fokker-Planck collision operator

$$L_{FP}(f)(p) = \frac{1}{\tau}\mathrm{div}_p(pf + mK_B T_{\mathrm{th}} \nabla_p f)$$

$$\tau = \text{ relaxation time} \quad l = \tau\sqrt{\frac{K_B T_{\mathrm{th}}}{m}} = \text{ mean free path}$$

Observation units

$$T = \frac{1}{\varepsilon} T_p, \quad L = \lambda_D, \quad p = p_{\mathrm{th}}, \quad \tau = \varepsilon T_p$$

Dimensionless Fokker-Planck equation (2D)

$$\partial_t f^\varepsilon + \frac{p}{\varepsilon} \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(E_\varepsilon + {}^\perp p \frac{B_\varepsilon}{\varepsilon}\right) \cdot \nabla_p f^\varepsilon = \frac{1}{\varepsilon^2} \mathrm{div}_p (p f^\varepsilon + \nabla_p f^\varepsilon)$$

Multiplication by $1 + \ln f^\varepsilon + \frac{|p|^2}{2}$

$$\iint \mathrm{div}_p (p f^\varepsilon + \nabla_p f^\varepsilon)\left(1 + \ln f^\varepsilon + \frac{|p|^2}{2}\right) = -\iint \left| \frac{p f^\varepsilon + \nabla_p f^\varepsilon}{\sqrt{f^\varepsilon}} \right|^2 \leq 0$$

Free energy balance with entropy production term

$$\frac{d}{dt}\left\{ \iint f^\varepsilon \left(\ln f^\varepsilon + \frac{|p|^2}{2}\right) + \int \left(|E_\varepsilon|^2 + \left(\frac{B_\varepsilon - B_0}{\varepsilon}\right)^2 \right) \right\}$$
$$+ \quad \frac{1}{\varepsilon^2} \iint \left| \frac{p f^\varepsilon + \nabla_p f^\varepsilon}{\sqrt{f^\varepsilon}} \right|^2 = 0$$

Formal analysis

Mass balance

$$\partial_t \int f^\varepsilon dp + \mathrm{div}_x \int \frac{p}{\varepsilon} f^\varepsilon dp = 0$$

Momentum balance

$$\varepsilon \partial_t \int p f^\varepsilon dp + \mathrm{div}_x \int (p \otimes p) f^\varepsilon dp + \rho^\varepsilon E_\varepsilon + ^\perp \int \frac{p}{\varepsilon} f^\varepsilon dp B_\varepsilon = - \int \frac{p}{\varepsilon} f^\varepsilon dp$$

$$\rho^\varepsilon = \int f^\varepsilon dp \to \rho, \quad \int \frac{p}{\varepsilon} f^\varepsilon dp \to J, \quad E_\varepsilon \to E, \quad B_\varepsilon \to B_0$$

$$\int (p \otimes p) f^\varepsilon dp \to \int (p \otimes p) \rho(t,x) M(p) dp = \rho I_2, \quad M(p) = \frac{1}{2\pi} e^{-\frac{|p|^2}{2}}$$

Limit model

$$\begin{cases} \partial_t \rho + \mathrm{div}_x J = 0 \\ \nabla_x \rho + \rho E + {B_0}^{\perp} J + J = 0 \\ {\mathrm{div}_x}^{\perp} E = 0, \quad \mathrm{div}_x E = 1 - \rho \end{cases}$$

$$J = J(\rho, E, B_0)$$

$$\Downarrow$$

$$(1 + (B_0)^2)\partial_t \rho + {B_0}^{\perp} E \cdot \nabla_x \rho - \mathrm{div}_x(\nabla_x \rho + \rho E) = 0$$