Vector and scalar penalty-projection methods

for incompressible and variable density flows

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Motivations and objectives

Work focusing on the constraint of free divergence

- How to deal efficiently with the free-divergence constraint with fractional-step methods (prediction-correction steps) ?
- How to circumvent the major drawbacks of the usual projection methods including a scalar correction step for the Lagrange multiplier with solution of a Poisson-type equation ?

Example of fluid-type models with pressure as Lagrange multiplier

$\Rightarrow$ solution of unsteady incompressible Navier-Stokes equations in the primitive variables (velocity and pressure)

Outlines

1. *Projection methods for incompressible flows*
2. *Scalar penalty-projection methods*
3. *Vector penalty-projection methods*
4. *Conclusion and perspectives*

Outlines

1. ***Projection methods for incompressible flows***
   - Non-homogeneous incompressible flows
   - Semi-implicit method (linearly implicit)
   - Fractional-step and projection methods
2. ***Scalar penalty-projection methods***
3. ***Vector penalty-projection methods***
4. ***Conclusion and perspectives***

Navier-Stokes problem for incompressible flows

Incompressible and variable density flows of Newtonian fluids

Navier-Stokes equations with mixed boundary conditions :
Dirichlet on $\boldsymbol{\partial\Omega_D}$ and open (outflow) B.C. on $\boldsymbol{\partial\Omega_N}$

$$
\begin{cases}
\dfrac{\partial \varrho}{\partial t} + u \cdot \nabla \varrho = 0 & \text{in } \Omega \times]0, T[\\
\varrho \left[\dfrac{\partial u}{\partial t} + (u \cdot \nabla) u \right] - \nabla \cdot \tau(u) + \nabla p = f & \text{in } \Omega \times]0, T[\\
\nabla \cdot u = 0 & \text{in } \Omega \times]0, T[\\
u = u_D & \text{on } \partial\Omega_D \times]0, T[\\
-pn + \tau(u) \cdot n = f_N & \text{on } \partial\Omega_N \times]0, T[\\
\varrho = \varrho_0, \quad \text{and } u = u_0 & \text{in } \Omega \times \{0\}
\end{cases}
$$

$\nabla \cdot \tau(u) = \nabla \cdot [\mu(\nabla u + \nabla u^{\mathrm{T}})],$ or $\mu \Delta u$ (for a constant viscosity)

For an homogeneous fluid with constant density, we set $\varrho = 1$.

Semi-implicit method (linearly implicit)

Semi-discretization in time with low or high-order BDF schemes

for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$:

$$\begin{cases} \dfrac{D\bar{\varrho}^{n+1}}{\delta t} + \nabla \cdot \left(\bar{u}^{\star,n+1}\bar{\varrho}^{n+1}\right) = 0 & \text{in } \Omega \\ \bar{\varrho}^{n+1}\left(\dfrac{D\bar{u}^{n+1}}{\delta t} + (\bar{u}^{\star,n+1}\cdot\nabla)\bar{u}^{n+1}\right) & \\ \qquad -\nabla\cdot\tau(\bar{u}^{n+1}) + \nabla\bar{p}^{n+1} = f^{n+1} & \text{in } \Omega \\ \nabla\cdot\bar{u}^{n+1} = 0 & \text{in } \Omega \\ \bar{u}^{n+1} = u_{\mathrm{D}}^{n+1} & \text{on } \partial\Omega_{\mathrm{D}} \\ -\bar{p}^{n+1}n + \tau(\bar{u}^{n+1})\cdot n = f_{\mathrm{N}}^{n+1} & \text{on } \partial\Omega_{\mathrm{N}} \end{cases}$$

$\Rightarrow$ *Resolution of an elliptic problem in space (at each time step) by methods of finite elements, finite volumes, DGM...*

Semi-implicit method : coupled solver

Algebraic formulation of the Navier-Stokes system

$\Rightarrow$ ***inf-sup*** stable discretization in space or with stabilization...

$$\begin{cases} \dfrac{\beta_q}{\delta t}\mathbf{M}_{\varrho}\mathbf{U}^{n+1} + \mathbf{A}(\mathbf{U}^{n}, \mathbf{U}^{n-1})\mathbf{U}^{n+1} + \mathbf{B}^{\mathbf{T}}\mathbf{P}^{n+1} \\ \qquad\qquad\qquad = \mathbf{F}(\mathbf{U}^{n}, \mathbf{U}^{n-1}, f^{n+1}, f_{\mathbf{N}}^{n+1}, u_{\mathbf{D}}^{n+1}) \\ \mathbf{B}\mathbf{U}^{n+1} = \mathbf{G}(u_{\mathbf{D}}^{n+1}) \end{cases}$$

Semi-implicit method : coupled solver

Algebraic formulation of the Navier-Stokes system
$\Rightarrow$ ***inf-sup*** stable discretization in space or with stabilization...

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Saddle-point problem at each time step : efficient solver ?

Fully-coupled solver with efficient multigrid preconditioner :
Kortas, PhD 1997 - Cihlář and Angot, 1999
Augmented Lagrangian method with iterative Uzawa algorithm :
Fortin and Glowinski, 1983 - Khadra et al., IJNMF 2000

$\Rightarrow$ *Projection methods using the Helmoltz-Hodge-Leray decomposition of* $L^2(\Omega)^d$

Fractional-step and projection-type methods

Projection scheme with scalar pressure-correction

[Chorin, 1968 - Temam, 1969 - Goda, 1979 - Van Kan, 1986]

See recent review : [Guermond, Minev, Shen, CMAME 2006]

Prediction step

$$\varrho^{n+1}\left(\frac{D\tilde{u}^{n+1}}{\delta t} + (u^{\star,n+1}\cdot\nabla)\tilde{u}^{n+1}\right) - \nabla\cdot\tau(\tilde{u}^{n+1}) + \nabla p^n = f^{n+1}$$

$$\tilde{u}^{n+1} = u_{\mathrm{D}}^{n+1} \text{ on } \partial\Omega_{\mathrm{D}}$$

$$-p^n n + \tau(\tilde{u}^{n+1})\cdot n = f_{\mathrm{N}}^{n+1} \text{ on } \partial\Omega_{\mathrm{N}}$$

Projection step

$$\beta_q \varrho^{n+1}\frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla\phi = 0$$

$\Rightarrow \nabla\phi\cdot n = 0$ on $\partial\Omega_{\mathrm{D}}$ (necessary since $u\cdot n = \tilde{u}\cdot n$)

$\phi = 0$ on $\partial\Omega_{\mathrm{N}}$ (sufficient by orthogonal projection onto H)

$$\nabla\cdot u^{n+1} = 0 \quad \Rightarrow \quad -\nabla\cdot\left(\frac{\delta t}{\varrho^{n+1}}\nabla\phi\right) = -\beta_q\nabla\cdot\tilde{u}^{n+1}$$

Pressure-correction step : ϕ pressure increment

$$p^{n+1} = p^n + \phi$$

Algebraic formulation for the Navier-Stokes system

$$\begin{cases} \dfrac{\boldsymbol{\beta_q}}{\boldsymbol{\delta t}}\mathbf{M}_{\boldsymbol{\varrho}}\tilde{\mathbf{U}}^{\boldsymbol{n+1}} + \mathbf{A}\tilde{\mathbf{U}}^{\boldsymbol{n+1}} + \mathbf{B}^{\mathbf{T}}\mathbf{P}^{\boldsymbol{n}} = \mathbf{F} \\ \mathbf{L}_{\boldsymbol{\varrho}}\boldsymbol{\Phi} = \dfrac{\boldsymbol{\beta_q}}{\boldsymbol{\delta t}}\left(\mathbf{B}\tilde{\mathbf{U}}^{\boldsymbol{n+1}} - \mathbf{G}\right) \\ \mathbf{P}^{\boldsymbol{n+1}} = \mathbf{P}^{\boldsymbol{n}} + \boldsymbol{\Phi} \\ \mathbf{M}_{\boldsymbol{\varrho}}\mathbf{U}^{\boldsymbol{n+1}} = \mathbf{M}_{\boldsymbol{\varrho}}\tilde{\mathbf{U}}^{\boldsymbol{n+1}} + \dfrac{\boldsymbol{\delta t}}{\boldsymbol{\beta_q}}\mathbf{B}^{\mathbf{T}}\boldsymbol{\Phi} \end{cases}$$

Fractional-step and projection-type methods

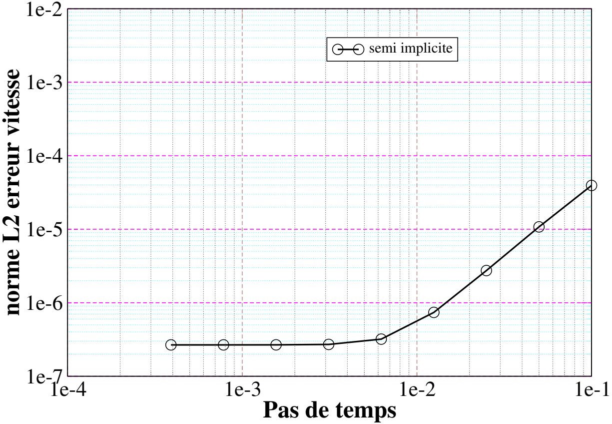
Major drawbacks of the incremental projection methods

- Time order of the splitting error ?
  i.e. error between the numerical solutions of the implicit (or semi-implicit) method and the fractional-step method
- $\boldsymbol{\nabla \phi \cdot n = 0}$ on $\boldsymbol{\partial \Omega_{\mathrm{D}}}$
  $\Rightarrow$ existence of an artificial pressure boundary layer in space
- $\boldsymbol{\phi = 0}$ on $\boldsymbol{\partial \Omega_{\mathrm{N}}}$
  $\Rightarrow$ convergence in time and space spoiled for outflow boundary conditions : splitting error varying like $\boldsymbol{\mathcal{O}(\delta t^{1/2})}$ (pressure) and no more negligible (for both velocity and pressure) with respect to the time and space discretization error
- Pressure-correction step strongly dependent on density and viscosity for non-homogeneous flows
  $\Rightarrow$ convergence very poor for large ratios of $\boldsymbol{\varrho \sim 1000}$.

Numerical tests

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

Velocity error (discrete $L^2(0,T;L^2(\Omega)^d)$ norm) versus time step δt



Time convergence in $\mathcal{O}(\delta t^2)$

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

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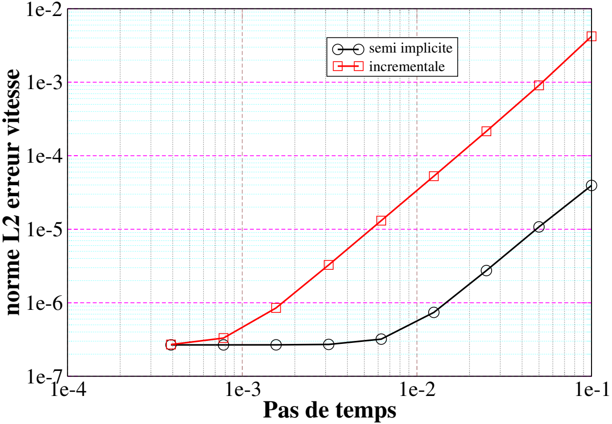

Time convergence in $\mathcal{O}(\delta t^2)$

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Outlines

Penalty-projection methods

[Jobelin et al., JCP 2006] ***: prediction step with augmented Lagrangian for $r > 0$ and consistent projection step by scalar pressure-correction***

- [Shen, 1992] : $\boldsymbol{r = 1/\delta t^2}$ with a different correction of pressure
- [Caltagirone and Breil, 1999] : $\boldsymbol{r > 0}$ with a singular projection operator...

Penalty-projection methods

Penalty-prediction step : augmentation parameter $r \geq 0$

$$\varrho^{n+1}\left(\frac{D\tilde{u}^{n+1}}{\delta t} + (u^{\star,n+1}\cdot\nabla)\tilde{u}^{n+1}\right) - \nabla\cdot\tau(\tilde{u}^{n+1}) - r\nabla\left(\nabla\cdot\tilde{u}^{n+1}\right) + \nabla p^{n} = f^{n+1}$$

$\tilde{u}^{n+1} = u_{\mathrm{D}}^{n+1}$ on $\partial\Omega_{\mathrm{D}}$

$-p^{n}n + \tau(\tilde{u}^{n+1})\cdot n = f_{\mathrm{N}}^{n+1}$ on $\partial\Omega_{\mathrm{N}}$

Projection step

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$\Rightarrow \nabla\phi\cdot n = 0$ on $\partial\Omega_{\mathrm{D}}$ (necessary since $u\cdot n = \tilde{u}\cdot n$)

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$$\nabla\cdot u^{n+1} = 0 \quad \Rightarrow \quad -\nabla\cdot\left(\frac{\delta t}{\varrho^{n+1}}\nabla\phi\right) = -\beta_q\nabla\cdot\tilde{u}^{n+1}$$

Pressure-correction step : ϕ consistent pressure increment

$$p^{n+1} = p^{n} - r\nabla\cdot\tilde{u}^{n+1} + \phi = \tilde{p}^{n+1} + \phi$$

Penalty-projection methods

Algebraic formulation for the Navier-Stokes system

$$\begin{cases} \dfrac{\beta_q}{\delta t}\mathbf{M}_\varrho\tilde{\mathbf{U}}^{n+1} + r\mathbf{B}^\mathrm{T}\mathbf{M}_{pl}^{-1}\left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G}\right) + \mathbf{B}^\mathrm{T}\mathbf{P}^n \\ \qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{A}\tilde{\mathbf{U}}^{n+1} = \mathbf{F} \\ \mathbf{L}_\varrho\boldsymbol{\Phi} = \dfrac{\beta_q}{\delta t}\left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G}\right) \\ \mathbf{P}^{n+1} = \mathbf{P}^n + r\mathbf{M}_{pl}^{-1}\left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G}\right) + \boldsymbol{\Phi} \\ \mathbf{M}_\varrho\mathbf{U}^{n+1} = \mathbf{M}_\varrho\tilde{\mathbf{U}}^{n+1} + \dfrac{\delta t}{\beta_q}\mathbf{B}^\mathrm{T}\boldsymbol{\Phi} \end{cases}$$

$\Rightarrow$ *Preconditioning the prediction step by one iteration of augmented Lagrangian and consistent scalar projection*
$\Rightarrow$ $r = 0$ *: incremental projection method*

Penalty-projection methods

Algebraic formulation for the Navier-Stokes system

$$\left\{\begin{array}{l} \dfrac{\beta_q}{\delta t}\mathbf{M}_\varrho\tilde{\mathbf{U}}^{n+1} + \mathbf{B}^{\mathbf{T}}\underbrace{\left(r\mathbf{M}_{pl}^{-1}\left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G}\right) + \mathbf{P}^n\right)}_{\tilde{\mathbf{P}}^{n+1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{A}\tilde{\mathbf{U}}^{n+1} = \mathbf{F} \\ \mathbf{L}_\varrho\mathbf{\Phi} = \dfrac{\beta_q}{\delta t}\left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G}\right) \\ \mathbf{P}^{n+1} = \underbrace{\mathbf{P}^n + r\mathbf{M}_{pl}^{-1}\left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G}\right)}_{\tilde{\mathbf{P}}^{n+1}} + \mathbf{\Phi} \\ \mathbf{M}_\varrho\mathbf{U}^{n+1} = \mathbf{M}_\varrho\tilde{\mathbf{U}}^{n+1} + \dfrac{\delta t}{\beta_q}\mathbf{B}^{\mathbf{T}}\mathbf{\Phi} \end{array}\right.$$

$\Rightarrow$ *Preconditioning the prediction step by one iteration of augmented Lagrangian and consistent scalar projection*
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Direct implementation with the algebraic formulation or not...
Why ?

Continuous term corresponding to the penalization $\boldsymbol{T_p} = \boldsymbol{r} \int_{\boldsymbol{\Omega}} \boldsymbol{\nabla} \cdot \boldsymbol{u} \boldsymbol{\nabla} \cdot \boldsymbol{v}$

But $\boldsymbol{T_p}$ does not generally vanish for a velocity field with zero discrete divergence !

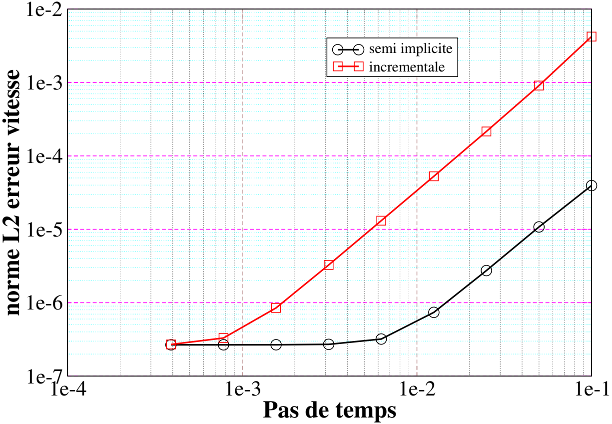
It depends on the spatial discretization...

Hence

- Introduction of an additional error due to the space discretization
- Error which increases with the augmentation parameter $\boldsymbol{r} > \boldsymbol{0}$

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

Velocity error (discrete $L^2(0,T;L^2(\Omega)^d)$ norm) versus time step δt



Time convergence in $\mathcal{O}(\delta t^2)$

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Numerical experiments

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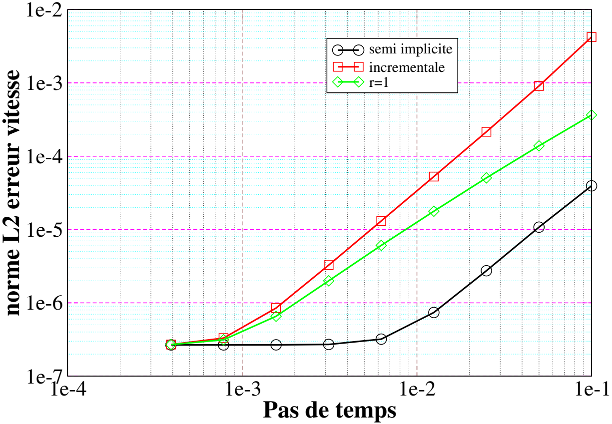

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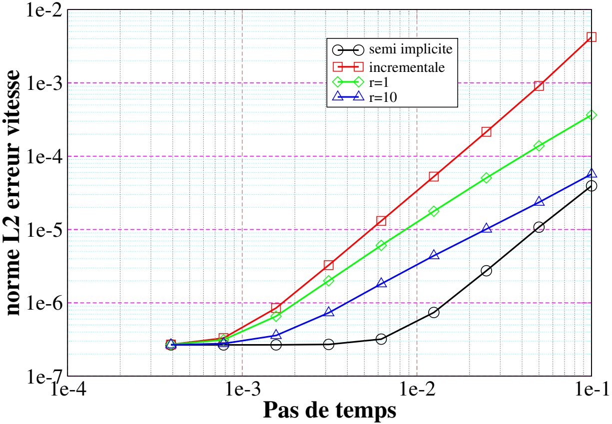

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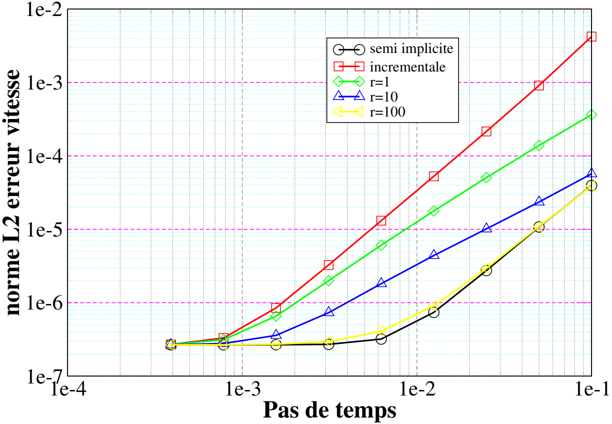


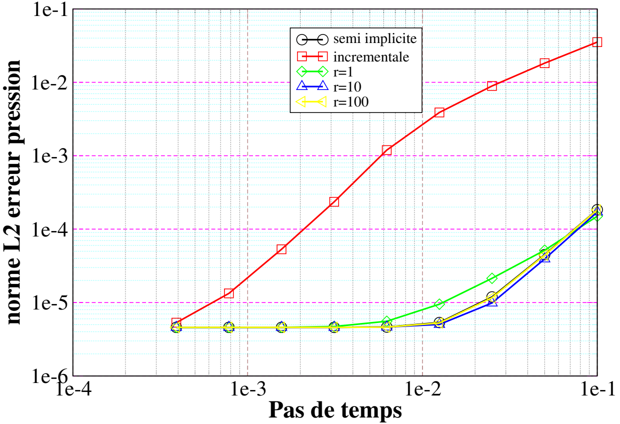
Time convergence in $\mathcal{O}(\delta t^2)$

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Numerical experiments

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

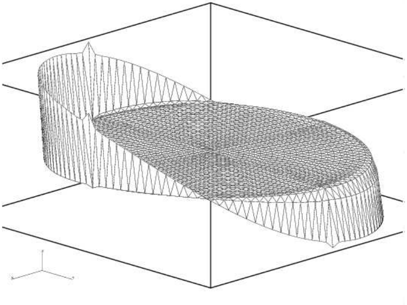
Pressure error (discrete $L^2(0,T;L^2(\Omega))$ norm) versus time step δt



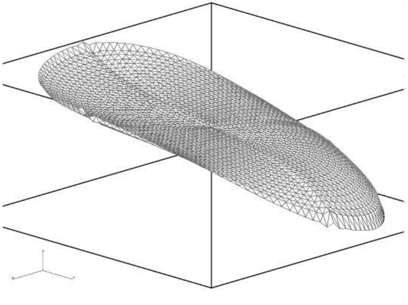
Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Numerical experiments

Artificial pressure boundary layer : Stokes with Dirichlet B.C. on a disk



incremental projection
$\boldsymbol{\|p_h - p\|_{L^\infty(\Omega)} = 1.5\ 10^{-2}}$



penalty-projection r=1
$\boldsymbol{\|p_h - p\|_{L^\infty(\Omega)} = 2.8\ 10^{-3}}$

Numerical experiments

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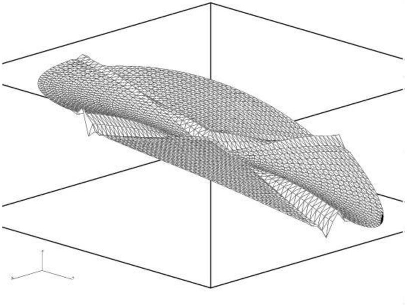
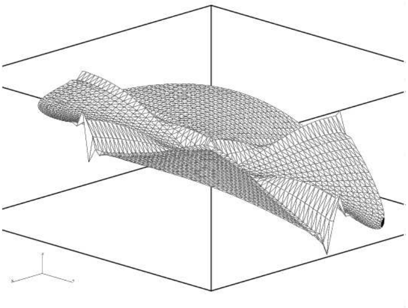


penalty-projection r=100
$\|p_h - p\|_{L^\infty(\Omega)} = 2.8\ 10^{-4}$

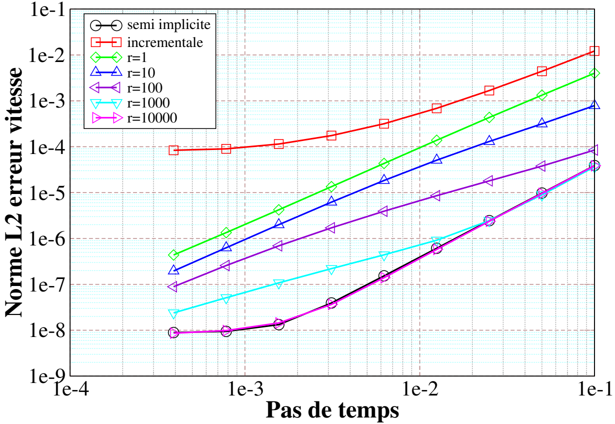


implicit scheme
$\|p_h - p\|_{L^\infty(\Omega)} = 1.8\ 10^{-4}$

Numerical experiments

Stokes with open boundary condition at a channel outflow

Velocity error (discrete $L^2(0,T;L^2(\Omega)^d)$ norm) versus time step δt

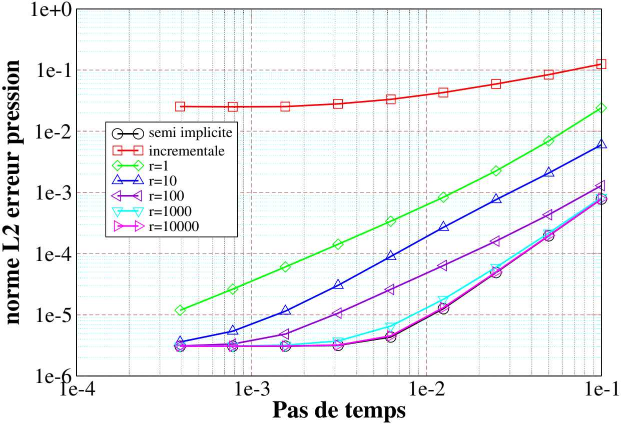


Time convergence in $\mathcal{O}(\delta t^2)$

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Stokes with open boundary condition at a channel outflow

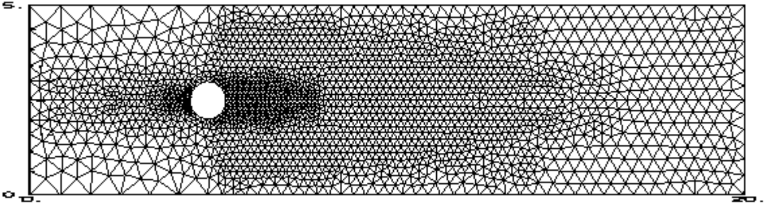
Pressure error (discrete $L^2(0,T;L^2(\Omega))$ norm) versus time step δt



Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Numerical experiments

Navier-Stokes open flow around a circular cylinder

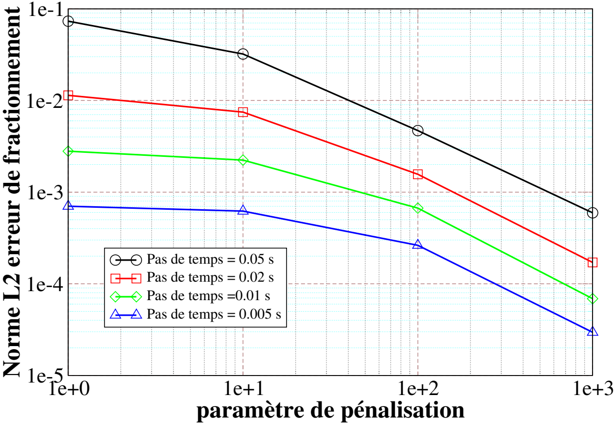


Taylor-Hood P2-P1 finite elements
Scheme 2nd order in time
Reynolds number : Re $= \mathbf{100}$ ($\boldsymbol{\mu} = \mathbf{0.01}$)
Unstructured mesh with 4200 elements
Complete study of this configuration : see Khadra et al., IJNMF 2000
with FVM (MAC mesh) and iterative augmented Lagrangian algorithm.

Splitting error varying as $\mathcal{O}(\frac{1}{r})$

Numerical experiments

Navier-Stokes open flow around a circular cylinder



Splitting error varying as $\mathcal{O}(\frac{1}{r})$

Unsteady homogeneous model problem with $\varrho = 1$

Ω connected and bounded open domain of $\mathbb{R}^d$, Lipschitz boundary $\partial\Omega$, $T > 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = f & \text{dans } \Omega \times]0, T[\\ \nabla \cdot u = 0 & \text{dans } \Omega \times]0, T[\\ u = 0 & \text{sur } \partial\Omega \times]0, T[\\ u(x, 0) = u_0(x) & \text{dans } \Omega \end{cases}$$

u denotes the velocity vector, p the pressure field.

For $f \in L^2(0, T; L^2(\Omega)^d)$ and $u_0 \in V$ given, there exists a unique solution $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ and $p \in L^2(0, T; L_0^2(\Omega))$
See [Temam, 1979 - Girault and Raviart, 1986]

Theoretical analysis for Stokes-Dirichlet problem

[Shen, 1992-95 - Guermond, 1996] : standard projection (pressure-correction form)
[Guermond and Shen, 2003] : standard projection (velocity-correction)
[Guermond and Shen, 2004] : rotational variant of [Timmermans et al., 1996]
[Angot, Jobelin, Latché, Preprint 2006] ***: penalty-projection***

Analysis for small values of the penalty parameter r

Theorem (Splitting error - fully discrete case in time and space)

Energy estimates of splitting errors compared to Euler implicit scheme there exists $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{\Omega}, \boldsymbol{T}, \boldsymbol{f}, \boldsymbol{u_0}, \boldsymbol{h}) > \mathbf{0}$ *such that : for* $\mathbf{1} \leq \boldsymbol{n} \leq \boldsymbol{N}$,

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{\frac{1}{2}} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{\frac{1}{2}} \leq c \min(\delta t^2, \frac{\delta t^{3/2}}{r^{1/2}})$$

$$\left[\sum_{k=0}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]^{\frac{1}{2}} + \left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{\frac{1}{2}} \leq c \max(1, \frac{1}{r^{1/2}}) \delta t^{3/2}.$$

Analysis for large values of the penalty parameter r

Theorem (Splitting error - fully discrete case in time and space)

Energy estimates of splitting errors compared to Euler implicit scheme there exists $\boldsymbol{c} = \boldsymbol{c}(\Omega, T, f, u_0, h) > 0$ *such that : for* $1 \leq n \leq N$,

$$\|e^n\|_0 + \|\tilde{e}^n\|_0 + \left[\sum_{k=0}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]^{\frac{1}{2}} \leq c\frac{\delta t^{\frac{1}{2}}}{r}$$

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{\frac{1}{2}} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{\frac{1}{2}} \leq c\frac{\delta t}{r}$$

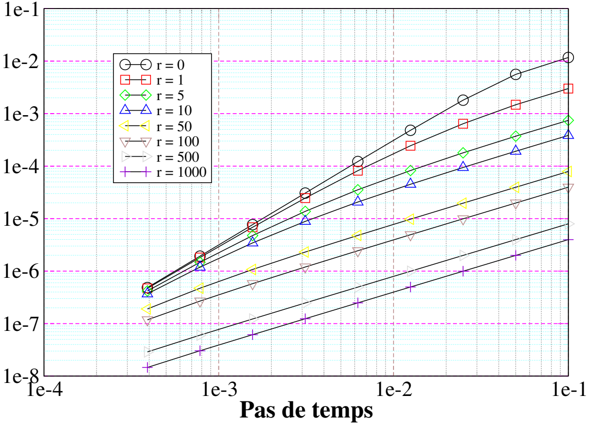
$$\|\epsilon^n\|_h \leq c \, \frac{1}{r^{1/2}}$$

$$\left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{\frac{1}{2}} \leq c \, \frac{1}{r}.$$

Theoretical analysis for Stokes-Dirichlet problem

$\left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{\frac{1}{2}}$ *: velocity splitting error* $\tilde{e}^n = \bar{u}^n - \tilde{u}^n$

discrete $L^2(0,T;L^2(\Omega)^d)$ norm versus time step δt



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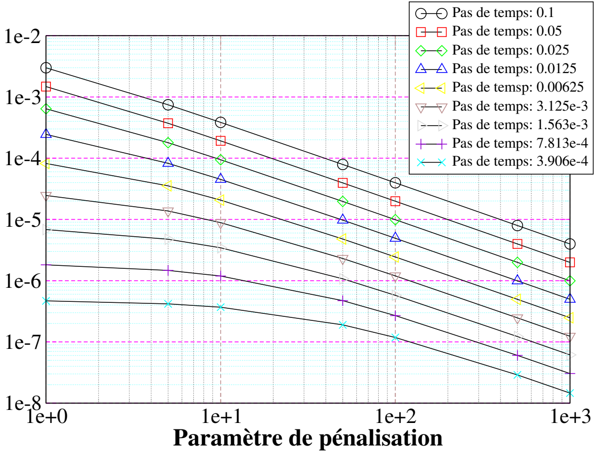
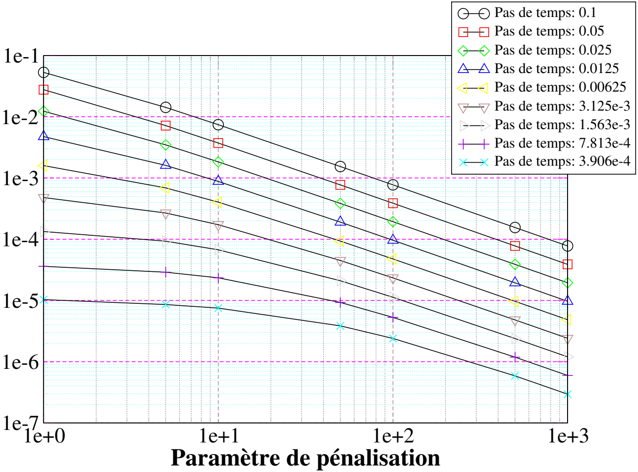
discrete $L^2(0,T;L^2(\Omega)^d)$ norm versus penalty parameter $r > 0$



Theoretical analysis for Stokes-Dirichlet problem

$\left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2 \right]^{\frac{1}{2}}$ *: pressure splitting error* $\epsilon^n = \bar{p}^n - p^n$

discrete $L^2(0,T; L^2(\Omega))$ norm versus penalty parameter $r > 0$



Outlines

1. *Projection methods for incompressible flows*
2. *Scalar penalty-projection methods*
3. ***Vector penalty-projection methods***
   - New class of vector penalty-projection methods
   - Analysis of the vector penalty-projection method for the Stokes problem
4. *Conclusion and perspectives*

New class of vector penalty-projection methods

Work in progress with CALTAGIRONE AND FABRIE
A two-parameter family with vector projection step

Penalty-prediction step : augmentation parameter $r \geq 0$

$$\varrho^{n+1}\left(\frac{D\tilde{u}^{n+1}}{\delta t} + (u^{\star,n+1}\cdot\nabla)\tilde{u}^{n+1}\right) - \nabla\cdot\tau(\tilde{u}^{n+1}) - r\nabla\left(\nabla\cdot\tilde{u}^{n+1}\right) + \nabla p^{n} = f^{n+1}$$

$$\tilde{u}^{n+1} = u_{\mathbf{D}}^{n+1} \text{ on } \partial\Omega_{\mathbf{D}}, \text{ and } -p^{n}n + \tau(\tilde{u}^{n+1})\cdot n = f_{\mathbf{N}}^{n+1} \text{ on } \partial\Omega_{\mathbf{N}}$$

$$\tilde{p}^{n+1} = p^{n} - r\nabla\cdot\tilde{u}^{n+1}$$

Vector projection step : penalty parameter $0 < \eta \ll \delta t$

$$\eta\left[\varrho^{n+1}\left(\frac{\beta_q}{\delta t}\hat{u}^{n+1} + (u^{\star,n+1}\cdot\nabla)\hat{u}^{n+1}\right) - \nabla\cdot\tau(\hat{u}^{n+1})\right] - \nabla\left(\nabla\cdot\hat{u}^{n+1}\right) = \nabla\left(\nabla\cdot\tilde{u}^{n+1}\right)$$

$$\hat{u}^{n+1} = 0 \text{ on } \partial\Omega_{\mathbf{D}}, \text{ and } -(\tilde{p}^{n+1} - p^{n})n + \tau(\hat{u}^{n+1})\cdot n = 0 \text{ on } \partial\Omega_{\mathbf{N}}$$

Correction step :

$$u^{n+1} = \tilde{u}^{n+1} + \hat{u}^{n+1}, \quad \text{and} \quad p^{n+1} = \tilde{p}^{n+1} - \frac{1}{\eta}\nabla\cdot u^{n+1}$$

Why does it work well ?

- Very nice conditioning properties of the vector penalty-correction step as $\eta \longrightarrow 0$ since the right-hand side becomes more and more "adapted" to the left-hand side operator :
  only a few iterations (2 or 3) of MILU0-BiCGStab solver whatever the mesh step h !
- Original boundary conditions not spoiled through a scalar projection step
- Penalty-correction step quasi-independent on the density or viscosity when $\eta \longrightarrow 0$ for non-homogeneous incompressible flows

It is only an approximate projection scheme since $\nabla \cdot u^{n+1} \neq 0$

But both the velocity divergence and the splitting error can be made negligible with respect to the time and space discretization errors when $\eta \rightarrow 0$
cheap method for small values of $r < 1$ and $\eta \simeq 10^{-15}$ (zero machine)

Error estimates of the one-parameter family : $r = 0$ and $0 < \eta \ll \delta t$

Theorem (Splitting error in the semi-discrete case)

Energy estimates of splitting errors w.r. to Euler implicit scheme there exists $C = C(\Omega, T, f, u_0) > 0$ such that : for $1 \leq n \leq N$,

$$\|e^n\|_0 + \left[\sum_{k=0}^{n} \delta t \, \|\nabla e^k\|_0^2\right]^{\frac{1}{2}} \leq C \, \eta \sqrt{\delta t}$$

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{\frac{1}{2}} \leq C \, \eta \delta t$$

$$\sqrt{\eta} \, \|\epsilon^n\|_0 + \left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{\frac{1}{2}} \leq C \, \eta$$

$$\|\nabla \cdot u^n\|_0 = \|\nabla \cdot e^n\|_0 \leq C \, \max\left(1, \frac{\sqrt{\eta}}{\delta t}\right) \eta \delta t.$$

Outlines

Conclusion

Scalar penality-projection methods for incompressible flows

- Reduce the splitting error up to make it negligible
- Suppress pressure boundary layers for moderate values of $\boldsymbol{r} > \mathbf{0}$
- Optimal convergence for $\boldsymbol{r} > \mathbf{0}$ with open boundary conditions
- Generalization to dilatable and low Mach number flows :
  see [Jobelin et al., Preprint 2006]
- Require efficient preconditioning for large values of $\boldsymbol{r}$
  $\Rightarrow$ multi-level preconditioner for FVM with MAC mesh :
  see [Kortas, PhD 1997].

Vector penality-projection methods for incompressible and non-homogeneous flows

- Small values of $r < 1$ are sufficient to get a good pressure field
- Approximate projection with a vector correction step all the cheaper than $\eta \rightarrow 0$
- Same convergence properties as before
- Vector correction step all the less dependent on density or viscosity than $\eta \rightarrow 0$
- Other numerical experiments (in progress)
- Error estimates : two-parameter class, Navier-Stokes, outflow B.C., variable density flows...