A Space-Time Expansion Discontinuous Galerkin Scheme with Local Time-Stepping for the Ideal and Viscous MHD Equations

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  - Principles
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General formulation

The Discontiniuous Galerkin (DG) method is an L_2 projection method allowing discontinuities at cell boundaries:

$$\|u - u_h\|_{L_2} \rightarrow min.$$

- Finite element context: Solution is initialized as a set of different base-functions and tested with space-dependent test-functions.
- Finite volume context: Discontinuities impose numerical flux at the element boundaries.

DG-Approach

- Hyperbolic PDE

$$u_t + f(u)_x = 0.$$

- Local polynomial Ansatz

$$u_i(x,t) := \sum_{j=1}^{p+1} \hat{u}_j^i(t)\phi_j^i(x) \text{ with the DOF } \hat{u}_j^i$$

- Multiplication with test function $\phi = \phi(x)$ and integration over cell Ω_i

$$\int_{\Omega_i} (u_t + f(u)_x) \cdot \phi \, dx = 0.$$

DG-Approach

- Integration by parts

$$\int_{\Omega_i} u_t\phi \; dx + \int_{\Omega_i} f(u)_x\phi \; dx = 0$$
$$\Rightarrow \int_{\Omega_i} u_t\phi \; dx + [f(u)\phi]_{\partial\Omega_i} - \int_{\Omega_i} f(u)\phi_x \; dx = 0.$$

- Approximate surface fluxes
- Perform high order time update (usually Runge-Kutta for transient problems) up to 4$^{\text{th}}$ order

STE-DG-Approach

- Multiplication with test function $\phi = \phi(x)$ and integration over an arbitrary space-time cell $\Omega_{i^n} := Q_i \times [t^n, t^{n+1}]$

$$\int_{\Omega_{i^n}} (u_t + f(u))\,\phi\ dxdt = 0$$

- Integration in space and in time from t_n to t_{n+1}
- Gauss quadrature in space **and** time

STE-DG-Approach

- For every time Gauss point the semi discrete DG operator (with appropriate numerical fluxes) has to be computed (for time order $p+1$, only $\frac{p+1}{2}$ Gauss points are needed!)
- Space time Taylor expansion at the barycenter (x_i, t_n)

$$\widetilde{u}(x,t) = \sum_{j=1}^{p+1} \frac{1}{j!} \left(x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \right)^j u_i \Big|_{(x_i, t_n)}$$

STE-DG-Approach

- Approximate pure time and mixed space-time derivatives with already known pure space derivatives using the governing equation: Cauchy-Kowalewskaya procedure

$$
\begin{aligned}
u_t &= -(f(u))_x \\
u_{tx} &= -(f(u))_{xx} \\
u_{txx} &= -(f(u))_{xxx} \\
\Rightarrow u_{tt} &= -(f(u))_{xt}
\end{aligned}
$$

STE-DG-Advantages

- Properties of the DG space discretization
  - Arbitrary high order accurate in space
  - Only direct neighbors are needed
  - High flexibility (hp adaptation)
  - Locally conservative
- Properties of the STE-DG scheme
  - All standard DG advantages and additionally
  - Arbitrary high order accurate in space *and* time
  - Explicit, preserves locality (parallelization)
  - **Natural** consistent local time stepping

Overview

- Current time discretizations:
  - All explicit schemes have a time step restriction to guarantee stability: $\Delta t \sim \frac{\Delta x}{p}$ for advection, $\Delta t \sim \left(\frac{\Delta x}{p}\right)^2$ for diffusion
  - Efficiency of explicit schemes with global time step decrease if locally varying meshes and/or polynomial orders are used (variations of x and p)
  - Implicit schemes are unconditional stable (arbitrary Δt)
  - But implicit schemes are lacking in locality (parallelization) and are usually only first/second order accurate in time

Overview

- Advantages of the STE-DG scheme in that matter:
  - Explicit
  - Arbitrary high order accurate in time
  - Due to the locality of the DG scheme (only direct neighbor data is needed) and the space-time character of the STE approach, we are able to run every element with its local time step restriction, using small time steps only where they are needed

Technique

- We give up the assumption that every grid cell runs with the same time step
  $\Rightarrow$ no common time levels t_n!
- We introduce local time levels $t^i_{n_i}$ and time step $\Delta t^{n_i}_i$ for element Q_i
- We use the STE approach, this time with $t_n \longrightarrow t^i_{n_i+1}$ instead of $t_n \longrightarrow t_{n+1}$
- Volume integral could be integrated with Gauss quadrature as before (because depends only on data in the element)

Technique

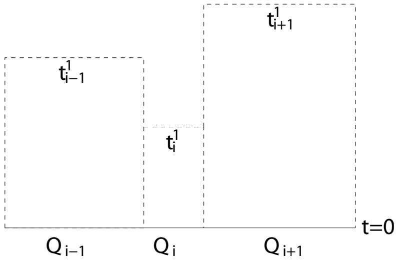
- Surface integrals need neighbor data and therefore have to be treated a little bit more carefully
- To calculate the integrals, the element Q_i at its local time level $t^i_{n_i}$ has to satisfy the **evolve condition**

$$t^i_{n_i+1} \leq \min\{t^{i-1}_{n_{i-1}+1}, t^{i+1}_{n_{i+1}+1}\}$$

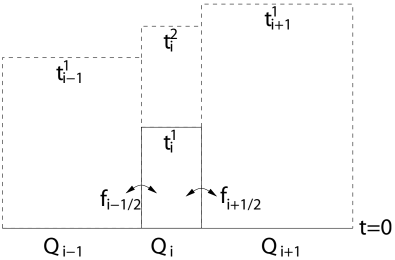
  This ensures that all data needed to perform the element update is available

Technique

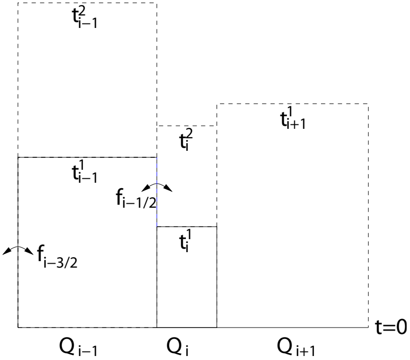
Functionality



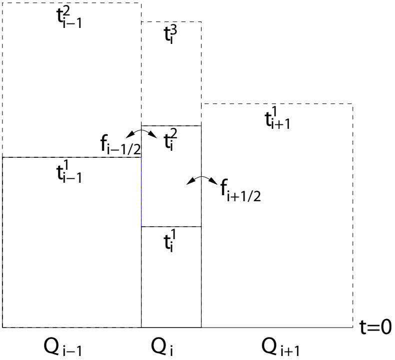
Technique



Technique



Technique



Animation

- Visualization of the local time step algorithm
  - 1D Euler equation
  - irregular grid cells
  - time step depends on the solution

(3D) Ideal MHD equation System

- Implemented within the STE-DG framework

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{v}) \\
\frac{\partial (\rho \mathbf{v})}{\partial t} &= -\nabla \cdot \left(\rho \mathbf{v}\mathbf{v}^t - \mathbf{B}\mathbf{B}^t + \left(p + \tfrac{1}{2}|\mathbf{B}|^2 \right) I \right) \\
\frac{\partial E}{\partial t} &= -\nabla \cdot \left((E + p)\, \mathbf{v} + \left(\tfrac{1}{2}|\mathbf{B}|^2 I - \mathbf{B}\mathbf{B}^t \right) \cdot \mathbf{v} \right) \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times (\mathbf{B} \times \mathbf{v})
\end{aligned}
$$

with

$$
p = (\gamma - 1) \cdot \left(\rho E - \frac{1}{2} \rho \left(|\mathbf{v}^2| - |\mathbf{B}^2| \right) \right)
$$

(3D) Viscous MHD equation System

- Also implemented within the STE-DG framework

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{v}) \\
\frac{\partial (\rho \mathbf{v})}{\partial t} &= -\nabla \cdot \left(\rho \mathbf{v}\mathbf{v}^t - \mathbf{B}\mathbf{B}^t + \left(p + \frac{1}{2}|\mathbf{B}|^2 \right) I - \tau \right) \\
\frac{\partial E}{\partial t} &= -\nabla \cdot \left((E + p)\, \mathbf{v} + \left(\frac{1}{2}|\mathbf{B}|^2 I - \mathbf{B}\mathbf{B}^t \right) \cdot \mathbf{v} \right. \\
&\quad \left. - \mathbf{v}\tau + \eta \left(\mathbf{B} \cdot \nabla \mathbf{B} - \nabla \left(\frac{1}{2}|\mathbf{B}|^2 \right) \right) - \mu \frac{1}{Pr} \nabla T \right) \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \left(\mathbf{B} \times \mathbf{v} + \eta \nabla \times \mathbf{B} \right)
\end{aligned}
$$

and

$$
\tau = \mu \left(\partial_j v_i + \partial_i v_j \right) - \frac{2}{3} \nabla \cdot \mathbf{v} \delta_{ij}
$$

(3D) Viscous MHD equation System

- Implementation of viscous terms is done according to **Warburton and Karniadakis**, *Journal of Computational Physics* 152, 608-641 (1999)
- For the ideal part, the HLLC riemann-solver is used at the moment, the HLLD riemann solver will be implemented soon
- For the viscous part, the approach (inluding the generalized diffusive riemann problem (dGRP)) as described in **Gassner et al.**, *Journal of Computational Physics* in press (2006) is left unchanged

Convergence Test Case

- Dissipation of torsional Alfven waves under different angles to the mesh
- The direction of wave propagation is along the unit vector

$$\hat{n} = n_x\hat{i} + n_y\hat{j} = \frac{1}{\sqrt{r^2+1}}\hat{i} + \frac{r}{\sqrt{r^2+1}}\hat{j}.$$

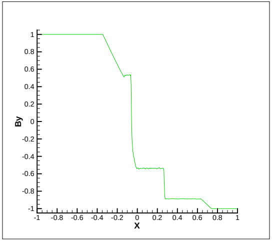
- Useful convergence test since no additional source terms are necessary

MHD Alfven Wave Decay Convergence Test
t=0.1, Bz, exact boundaries

Order	7	STE-DG MHD	
Zellen	DOF's	L2	L2 order
2	112	7,68E-04	
4	448	6,52E-06	6,9
8	1792	5,28E-08	6,9
16	7168	4,58E-10	6,8
32	28672	4,06E-12	6,8
Order	8	STE-DG MHD	
Zellen	DOF's	L2	L2 order
1	36	1,23E-02	
2	144	1,09E-04	6,8
4	576	6,68E-07	7,3
8	2304	3,16E-09	7,7
16	9216	1,56E-11	7,7

1D Test Case

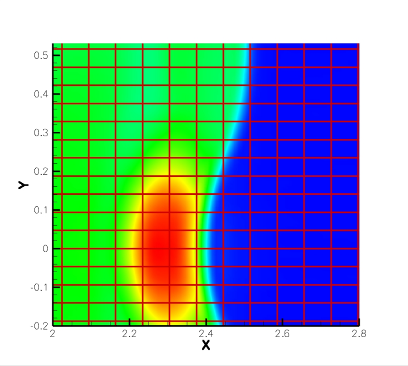
- The ideal MHD compound shocks test-case
- Special shock capturing strategy used that adds artificial viscosity to smear the shock profile. See **Persson and Perraire**, *Proc. of the 44th AIAA Aerospace Sciences Meeting and Exhibit*, January 2006



2D Test Case

- Interaction of a shock with a magnetic cloud
- Persson shock-capturing strategy used again

- Zoom into a 5^{th} order calculation on a very coarse grid (too coarse for small rool-up vortices)



- Animation of the 2D density plot

Conclusions

- A high order numerical scheme in both space and time
- High order accurate local timestepping to dramatically increase efficiency
- Code ready for parallelization
- Ideal and viscous MHD equations implemented
- Code ready for h/p-adaptation and shock capturing techniques
- Implementation of astrophysical test-cases imminent

Outlook

- Extension to 3D and parallelization necessary (with local timestepping)
- Extensive testing of the viscous MHD equtions
- Implementation of other non-ideal MHD terms
- Extensive stability analysis (under way)
- Incorporation of more test-cases (not only in the context of astrophysics)

Questions?