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Non-oscillatory Residual Distribution schemes for unsteady compressible MHD problems : very preliminary results

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Outline

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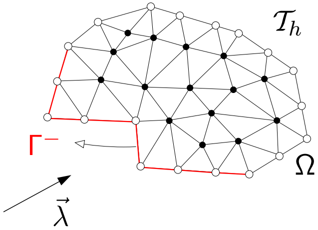
Framework for scalar $\mathcal{CL}$s

$$\nabla \cdot \mathcal{F}(u) = 0 \qquad \text{in} \quad \Omega$$

$$u = g \qquad \text{on} \quad \Gamma^-$$

$$\vec{\lambda}(u) = \frac{\partial \mathcal{F}}{\partial u}$$



Some notations . . .

- Consider $\mathcal{T}_h$ triangulation of Ω (can do with quads...)
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \mathcal{T}_h$ a given set of nodes (vertices +other dofs)
- Denote by u_h continuous piecewise polynomial interpolation (e.g. P^k Lagrange triangles) : $u_h = \sum_i \psi_i \, u_i$

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Residual Distribution ($\mathcal{RD}$), up to 2nd order

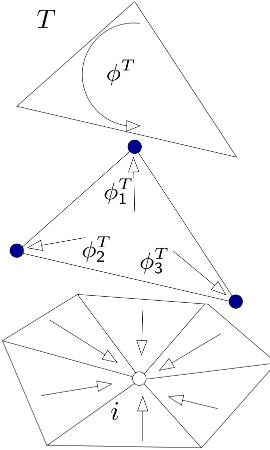
1. $\forall T \in \mathcal{T}_h$ compute : $\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h(u_h)$

2. Distribution : $\phi^T = \sum_{i \in T} \phi_i^T$

   Distribution coeff.s : $\phi_i^T = \beta_i^T \phi^T$

3. Compute nodal values : solve algebraic system

$$\sum_{T | i \in T} \phi_i^T = 0, \quad \forall i \in \mathcal{T}_h \qquad (1)$$



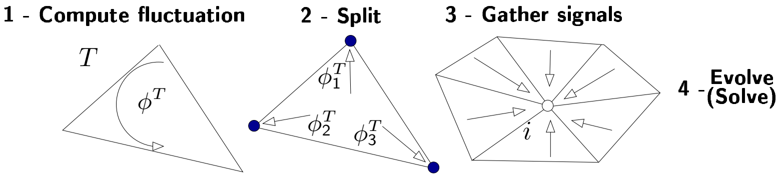
Simplified variant

Seek the limit $n \to \infty$ of

$$u_i^{n+1} = u_i^n - \omega_i \sum_{T|\, i \in T} \phi_i^T, \qquad \phi_i^T = \phi_i^T \left(\{u_j^n\}_{j \in T} \right) \tag{2}$$

The idea of Residual Distribution or Fluctuation Splitting

- Fluctuations & Signals (Roe, *Num.Meth.Fluid Dyn.*, 1982)
- Given an initial guess, nodal values evolve according to signals "proportional" to cell residuals (Roe's fluctuation)



Principle for unsteady problems

In principle, total fluctuation

$$\phi = \int_T \left(\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}(U)\right) dx$$

or

$$\phi = \int_{T \times [t_n, t_{n+1}]} \left(\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}(U)\right) dx$$

quite complex

Simple and more versatile solution

Replace

$$\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}(U) = 0$$

by
Given $U^0, \dots, U^n$,

$$\mathcal{D}(U^{n+1}, \dots U^0) + \nabla \cdot \boldsymbol{\mathcal{F}}(U^{n+1}) = 0$$

with $\mathcal{D}(U^{n+1}, \dots U^0) = \dfrac{\partial U}{\partial t} + \mathcal{O}(\Delta t^q)$

Here

$$\frac{3}{2}\frac{U^{n+1}-U^n}{\Delta t}-\frac{1}{2}\frac{U^n-U^{n-1}}{\Delta t}+\nabla\cdot\mathcal{F}(U^{n+1})=0$$

Amounts to solve a steady problem

$$\frac{3}{2\Delta t}U^{n+1}+\nabla\cdot\mathcal{F}(U^{n+1})+S=0$$
$$S=\frac{3}{2\Delta t}U^n-\frac{1}{2}\frac{U^n-U^{n-1}}{\Delta t}$$

Total residual

$$\alpha U + \nabla \cdot \boldsymbol{\mathcal{F}}(U) = S$$

$$\Phi = \int_T \Big(\alpha U + \nabla \cdot \boldsymbol{\mathcal{F}}(U) - S \Big) dx$$

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Design properties

Structural conditions, basic properties

Under which conditions on the ϕ_i^Ts we get

- Correct weak solutions (if convergent with h)
- Formal k^{th} order of accuracy
- Monotonicity (discrete max priciple)
- Convergence (with h, and with n !)

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Condition 1 : conservation

Lax-Wendroff theorem

(i)Technical assumptions, e.g. : continuity of ϕ_i^T, consistency of flux approximation .

(ii) If there is a $\boldsymbol{\mathcal{F}}_h$, continuous approximation of $\boldsymbol{\mathcal{F}}$ such that

$$\phi^T = \sum_{j\in T} \phi_j^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \hat{n} \tag{3}$$

then

If a bounded sequence u_h, solution of scheme (1), converges (with h) to $u \Longrightarrow u$ is a weak solution of the problem.

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Condition 1 : conservation

Remark. Conservation : 2 underlying conditions

1. Existence of continuous flux approximation $\boldsymbol{\mathcal{F}}_h$ such that

$$\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \hat{n}$$

for example $\boldsymbol{\mathcal{F}}_h = \boldsymbol{\mathcal{F}}(u_h)$, but also $\boldsymbol{\mathcal{F}}_h = \sum_i \psi_i \boldsymbol{\mathcal{F}}_i$!!

2. "Consistency" relation

$$\sum_{j \in T} \phi_j^T = \phi^T$$

Condition 2 : accuracy

Truncation error analysis

Error estimates built on variational formulation and stability analysis (coercivity) not available.

1. Given w_h discrete interpolation of nodal values of smooth exact solution w ;
2. Given φ a $C_0^1(\Omega)$ class function, and φ_h the discrete interpolation of $\{\varphi_i\}_{i\in\mathcal{T}_h}$, the nodal values of φ ;

Truncation error

$$\mathcal{E}(w_h) := \sum_{i\in\mathcal{T}_h} \varphi_i \Big(\sum_{T|\, i\in T} \phi_i^T(w_h) \Big)$$

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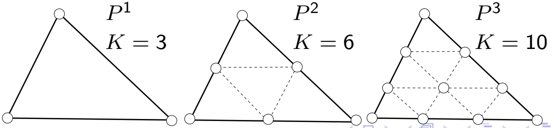
Condition 2 : accuracy

Guiding principle

Under which condition the $\mathcal{RD}$ scheme equivalent to the Galerkin scheme plus terms introducing and error (formally) within the one of the Galerkin approx.

$$\mathcal{E}(w_h) = \overbrace{\int_\Omega \varphi_h \nabla \cdot \boldsymbol{\mathcal{F}}_h}^{I \equiv \mathcal{E}^{\text{Galerkin}}} + \overbrace{\frac{1}{K} \sum_{T \in \mathcal{T}_h} \sum_{i,j \in T} (\varphi_i - \varphi_j)(\phi_i^T - \phi_i^{\text{Gal}})}^{II}$$

with ϕ_i^{Gal} elemental contribution of the standard (continuous) Galerkin discretization, and K the number of DoF per element.



Condition 2 : accuracy

- Final result

  If the (continuous) spatial approximations are $k+1^{\text{th}}$ order accurate (e.g. P^k Lagrange approximation), then one has the global estimate

$$|\mathcal{E}(w_h)| \leq C'(\mathcal{T}_h, w)\|\nabla\varphi\|_\infty\, h^{k+1}$$

  provided that (in 2D) $\forall i \in T$ and $\forall T \in \mathcal{T}_h$

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w)h^{k+2} = \mathcal{O}(h^{k+2})$$

- In dimension d

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w)h^{k+d}$$

Condition 2 : accuracy

- **Final result**

  If the (continuous) spatial approximations are $k+1^{\text{th}}$ order accurate (e.g. P^k Lagrange approximation), then one has the global estimate

$$|\mathcal{E}(w_h)| \leq C'(\mathcal{T}_h, w)\|\nabla\varphi\|_\infty\, h^{k+1}$$

provided that (in 2D) $\forall i \in T$ and $\forall T \in \mathcal{T}_h$

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w)h^{k+2} = \mathcal{O}(h^{k+2})$$

- **In dimension d**

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w)h^{k+d}$$

Condition 2 : accuracy

Linearity (Accuracy) preserving schemes

The condition $\phi_i^T(w_h) = \mathcal{O}(h^{k+2})$ gives a design criterion. In particular, since

$$\phi^T(w_h) = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h(w_h) \overbrace{=}^{\nabla\cdot\boldsymbol{\mathcal{F}}(w)=0} \int_T \nabla \cdot (F_h(w_h) - F(w)) =$$

$$\oint_{\partial T} (\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w)) \cdot \hat{n} = \mathcal{O}(\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w)) \times \mathcal{O}(|\partial T|)$$

$$\overbrace{=}^{k+1^{\text{th}}\ \text{order approx.}} \mathcal{O}(h^{k+1}) \times \mathcal{O}(h) = \mathcal{O}(h^{k+2})$$

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Condition 2 : accuracy

Linearity (Accuracy) preserving schemes

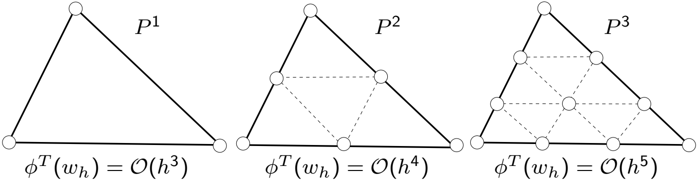
The condition $\phi_i^T(w_h) = \mathcal{O}(h^{k+2})$ gives a design criterion. In particular, since

$$\phi^T(w_h) = \mathcal{O}(h^{k+2})$$

schemes for which

$$\phi_i^T = \beta_i^T \phi^T$$

with β_i^T uniformly bounded distribution coeff.s, are formally $k+1^{\text{th}}$ order accurate (for $k+1^{\text{th}}$ order spatial interpolation)



Condition 3 : monotonicity

Scalar advection and positivity theory

$$\vec{\lambda}\cdot\nabla u = 0, \qquad \vec{\lambda} = \text{const}$$

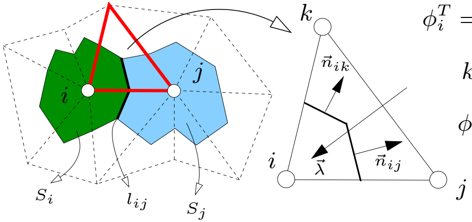
Construct schemes for which

$$\phi_i^T = \sum_{\substack{j\in T\\ j\neq i}} c_{ij}(u_i - u_j), \qquad c_{ij} \geq 0$$

Theory of positive coefficient schemes $\Rightarrow$ discrete max principle

$$u_i^{n+1} = u_i^n - \omega_i \sum_{T|\, i\in T}\sum_{\substack{j\in T\\ j\neq i}} c_{ij}(u_i^n - u_j^n) \underset{\omega_i\leq\omega_i^{\max}}{\overset{c_{ij}\geq 0}{\Longrightarrow}} \min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$$

Examples of positive schemes



$$\phi_i^T = -k_{ij}^-(u_i - u_j) - k_{ik}^-(u_i - u_k)$$

$$k_{ij} = \vec{\lambda} \cdot \vec{n}_{ij}, \quad k_{ik} = \vec{\lambda} \cdot \vec{n}_{ik}$$

$$\phi_i^T + \phi_j^T + \phi_k^T = \int_T \vec{\lambda} \cdot \nabla u_h$$

Positive schemes on P^1 meshes : the upwind FV scheme

$$|S_i| \frac{\Delta^n u_i}{\Delta t} = -\sum_j \int_{l_{ij}} H(u_i, u_j) \cdot \vec{n}_{ij} = -\sum_{T|i \in T} \phi_i^T$$

1. 1st order FV schemes have elmement-wise formulation
2. True also for nonlinear systems of $\mathcal{CL}$s
3. Upwind flux $\Longrightarrow$ positive and energy stable $\mathcal{RD}$

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Examples of positive schemes

Positive schemes : the Rusanov scheme (Local Lax Friedrichs)

Centered linear first order distribution :

$$\phi_i^{\text{Rv}} = \frac{1}{K}\phi^T + \frac{\alpha}{K}\sum_{\substack{j\in T\\ j\neq i}}(u_i - u_j), \quad \alpha \geq \max_{j\in T}\left|\int_T \vec{\lambda}\cdot\nabla\psi_j\right|$$

- K number of DoF per element
- ψ_j Lagrange basis fcn. relative to node j
- The Rv scheme is cheap and has general formulation
- The Rv scheme is positive (energy stable in the P^1 case)

Nonlinear higher order schemes

Nonlinear distribution strategies are needed to combine positivity and higher order of accuracy

Idea : construct generalizations of the successful PSI scheme of Struijs (Struijs, *PhD*, Delft U., 1994 ; Deconinck *et al.*, *Comp.Mech.* 11, 1993)

1. Starting point: a positive 1st order scheme (ϕ_i^{p})
2. Devise strategy to construct a splitting (ϕ_i^*) such that

$$\phi_i^* = \alpha_i \phi_i^{\mathrm{p}},\ \alpha_i \geq 0 \quad \stackrel{c_{ij}^p \geq 0}{\Longrightarrow} \quad c_{ij}^* \geq 0$$

and

$$\phi_i^* = \beta_i^* \phi^T, \quad \text{with } \beta_i^* \text{ uniformly bounded}$$

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Nonlinear higher order schemes

Generalizations of the PSI of Struijs (Struijs, *PhD*, Delft U., 1994 ; Deconinck *et al.*, *Comp.Mech.* 11, 1993)

1. Starting point: a positive 1st order scheme (ϕ_i^{p})
2. If $\phi^T = 0$, set $\phi_i^* = 0 \ \forall i \in T$
3. Otherwise, compute $\beta_i^p = \phi_i^p / \phi^T \quad \forall i \in T$ and map them onto bounded coefficients verifying

$$\beta_i^* \beta_i^p \geq 0 \text{ (equivalent to } \alpha_i \geq 0) \quad \text{and} \quad \sum_{j \in T} \beta_j^* = 1$$

4. For example take

$$\beta_i^* = \frac{\max(0, \beta_i^p)}{\sum_{j \in T} \max(0, \beta_j^p)} = \frac{\max(0, \phi_i^p / \phi^T)}{\sum_{j \in T} \max(0, \phi_i^p / \phi^T)}$$

Limited Rv (LRv) scheme

Summarizing

1. $\forall T \in \mathcal{T}_h$:
   (a) Compute ϕ^T (for ex. use P^k interpolation for flux)
   (b) Compute Rv distribution ϕ_i^{Rv}, $\forall i \in T$
   (c) Compute Rv distribution coeff.s and map them $\Rightarrow \phi_i^{\text{Rv}*} = \beta_i^{\text{Rv}*} \phi^T$, $\forall i \in T$
2. Evolve nodal values : $u_i^{n+1} = u_i^n - \omega_i \sum_{T|i \in T} \phi_i^{\text{Rv}*}$

Apply the mapping to the Rv scheme$\Rightarrow$ Limited Rv scheme

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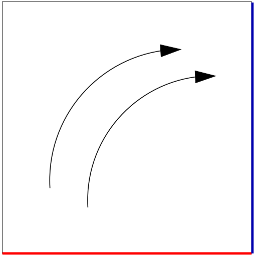
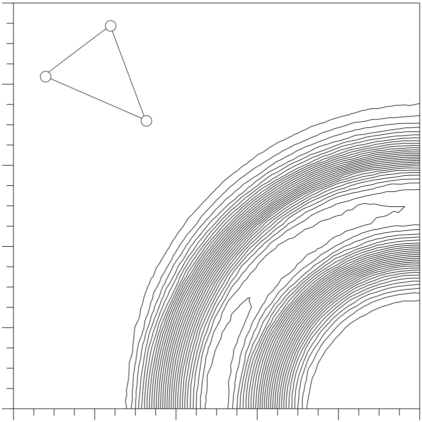
Numerical example : rotation



$u_{\text{inlet}} = \cos^2(2\pi x)$
$0.25 \le x \le 0.75$



LRvS scheme, P^1 interpolation

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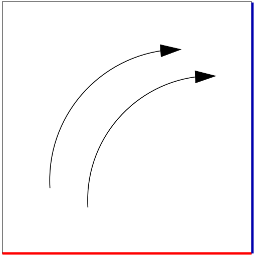

$u_{\text{inlet}} = \cos^2(2\pi x)$
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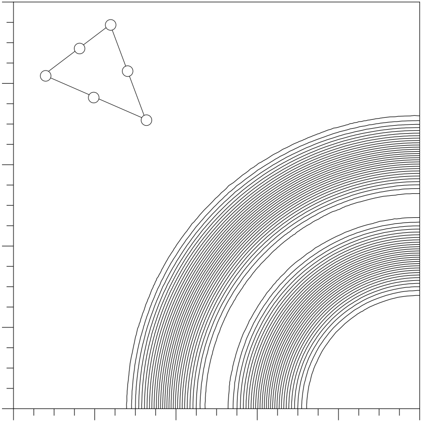
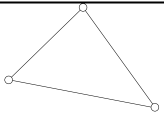
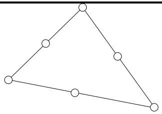
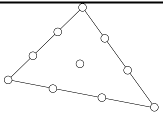
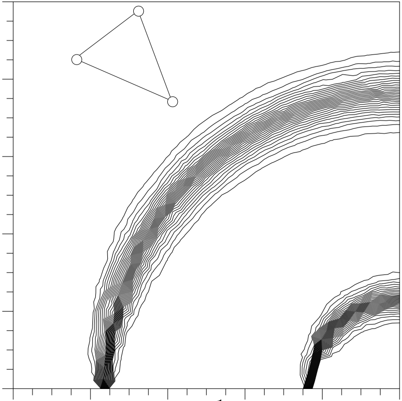


LRvS scheme, P^2 interpolation

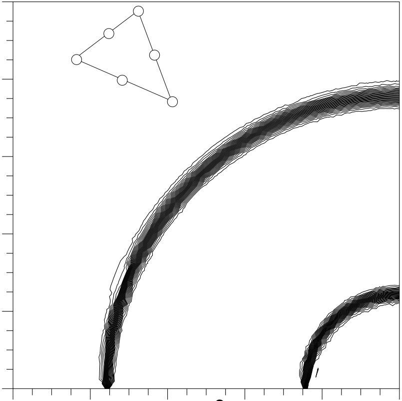
Grid convergence

			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
1/25	0.50493E-02	0.32612E-04	0.12071E-05
1/50	0.14684E-02	0.48741E-05	0.90642E-07
1/75	0.74684E-03	0.13334E-05	0.16245E-07
1/100	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\text{ls}}_{L^2}$ =**1.790**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**2.848**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**3.920**

Rotation of a top hat



LRvS scheme, P^1 interpolation



LRvS scheme, P^2 interpolation

Contact in spread on same numer of DoF (fewer cells in P^2 case)

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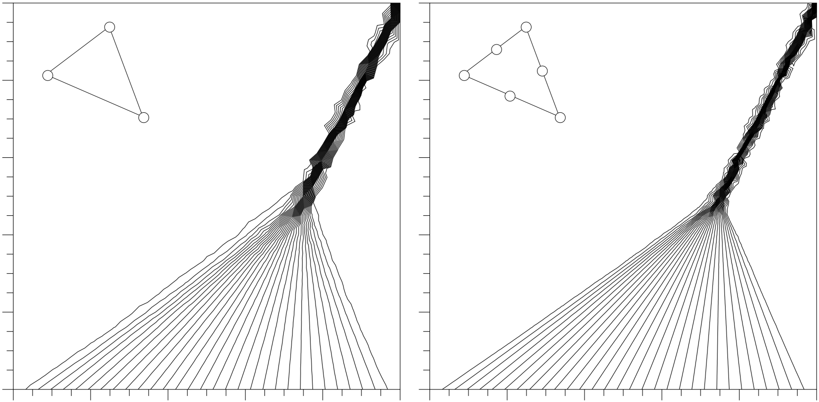
Numerical example : Burger's eq.n



LRvS scheme, P^1 interpolation — LRvS scheme, P^2 interpolation

Shock captured in 1 or 2 cells (more DoF in P^2 case)

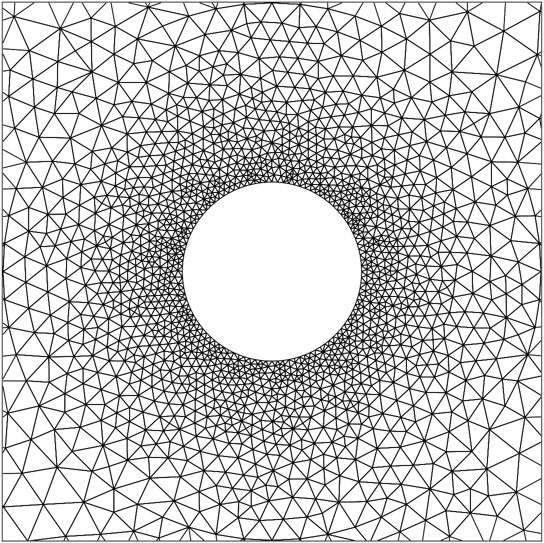
Extension to systems

$$\nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{u}) = 0$$

- Schemes formally identical to scalar case
- Nonlinear mapping on scalar residuals obtained by locally projecting on Eigenvector basis
- + other technical details

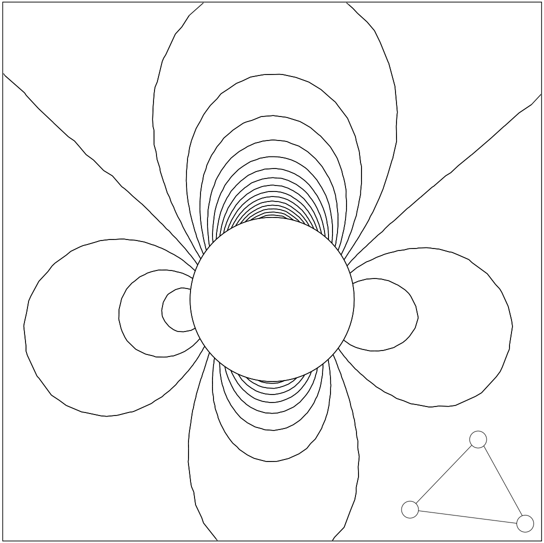
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
Mesh :
2719 nodes
5308 elements
100 nodes
on cylinder



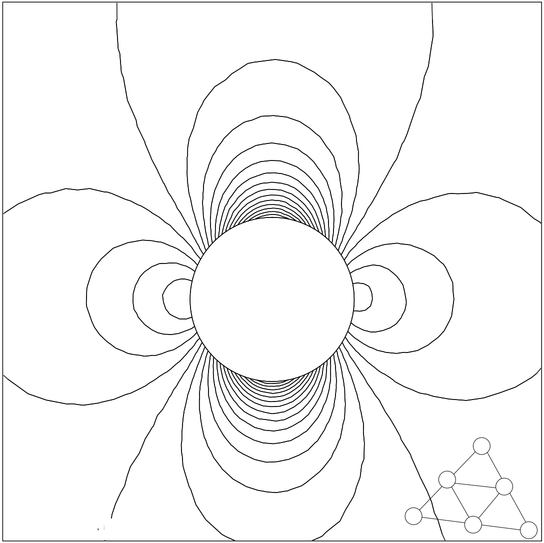
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements :
pressure



Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements
P^2 conformal
sub-triangulation :
pressure



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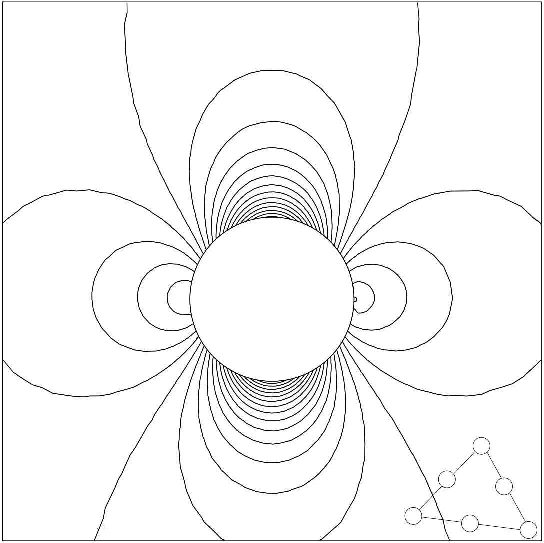
$Ma = 0.35$ flow on cylinder LRvS scheme P^2 elements : **pressure** (linear boundary representation)



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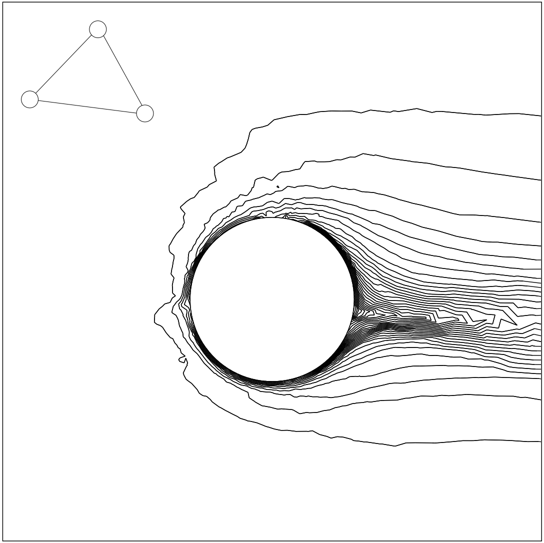
$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements :
entropy



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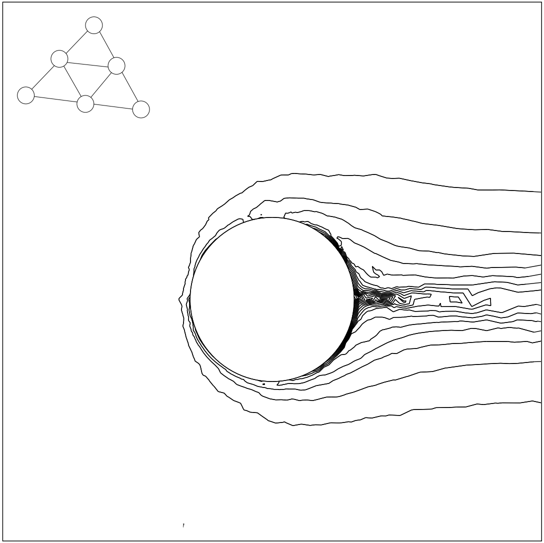
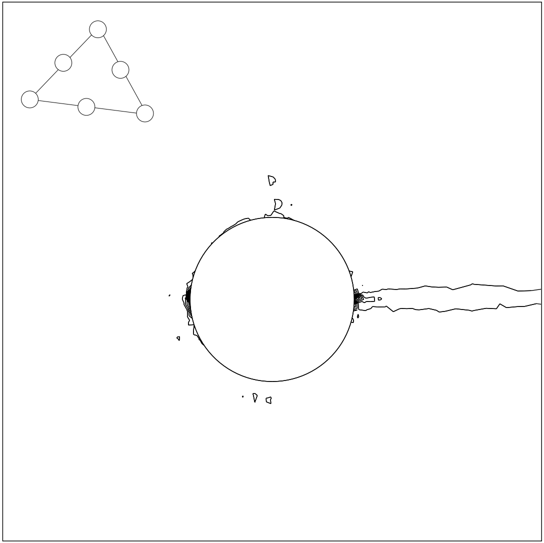
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements
P^2 conformal
sub-triangulation :
entropy



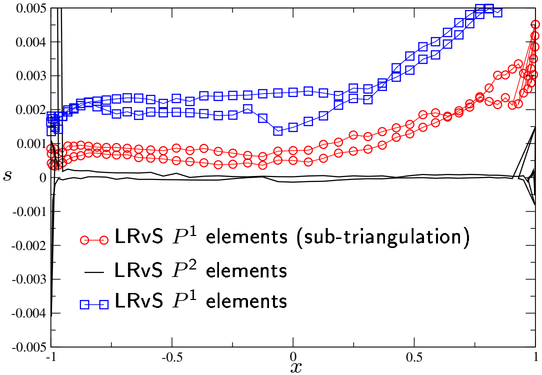
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^2 elements :
entropy
(linear boundary representation)



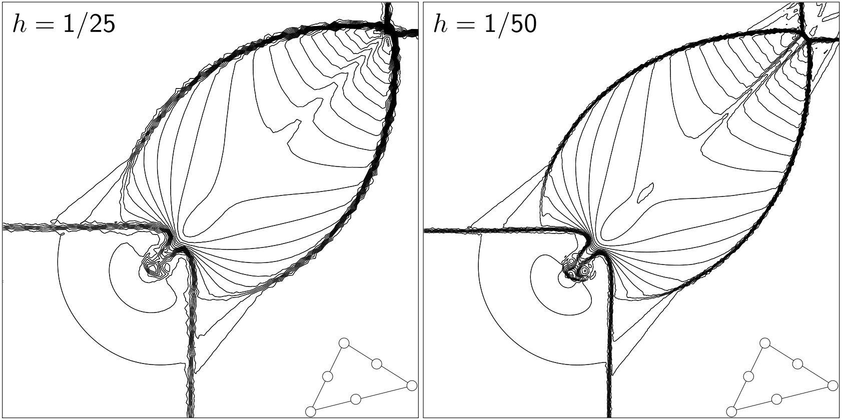
$Ma = 0.35$ cylinder flow : entropy distribution

$Ma = 0.35$
flow on cylinder
LRvS scheme :
entropy
on the cylinder



Euler eq.s : 2D RP

From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)
2D Riemann Problem, configuration 12



Density contours
LRvS scheme, P^2 elements

Euler eq.s : 2D RP

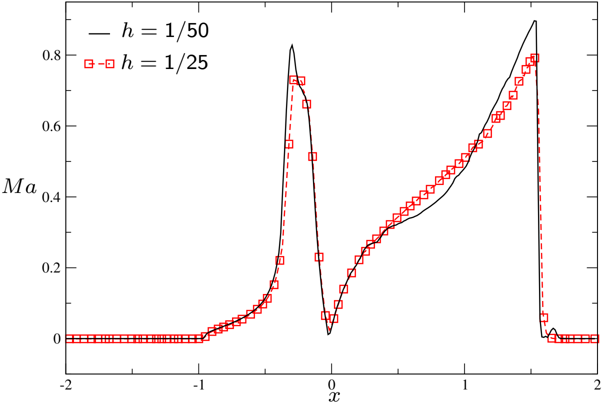
From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)

2D Riemann Problem, configuration 12

Density contours

LRvS scheme, P^2 elements

Euler eq.s : 2D RP, data on $y = x$



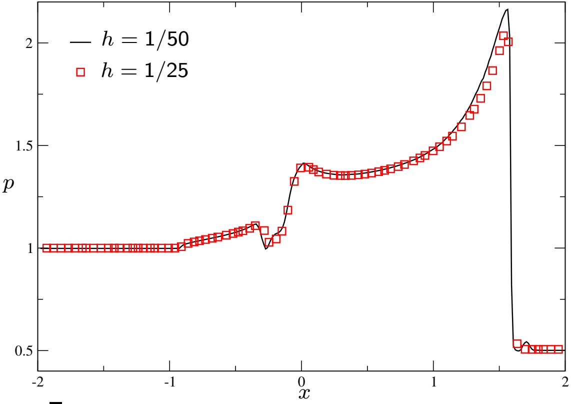
Mach number

From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)

2D Riemann Problem, configuration 12

LRvS scheme, P^2 elements

Euler eq.s : 2D RP, data on $y = x$



Pressure

From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)

2D Riemann Problem, configuration 12

LRvS scheme, P^2 elements

The model : ideal MHD

$$U = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ E \\ \mathbf{B} \end{pmatrix}, \qquad \boldsymbol{\mathcal{F}}(U) = \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}\mathbf{B}^2\right)\mathrm{Id} - \mathbf{B}\mathbf{B}^T \\ \mathbf{u}\left(E + p + \frac{1}{2}\mathbf{B}^2\right) - \mathbf{u}\cdot\mathbf{B}\mathbf{B} \\ \mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \end{pmatrix}$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}(U) = 0$$

with

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\left(\rho \mathbf{u}^2 + \mathbf{B}^2\right)$$

and $\nabla \cdot \mathbf{B} = 0$.

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Problem

- variables : density, momentum, total energy, magnetic field
- all located at the same points

Fulfilling the $\nabla \cdot \mathbf{B} = 0$ contraints

- A. Dedner, F. Kemm, D. Kröner, C.-D. Munz, T. Schneitzer and W. Wenseberg, *Hyperbolic divergence cleaning for the MHD equations*, J. Comp. Phys, **175**, 2002.
- "originate" from C.D. Munz, P. Omnes, R. Schneider and E. Sonnendrücker, *Divergence correction techniques for maxwell solvers based on a hyperbolic model*, J. Comp. Phys., **161**(2), 2000
- Artificial compressibility,
- ...

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Modified system

$$U = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ E \\ \mathbf{B} \\ \psi \end{pmatrix}, \qquad \boldsymbol{\mathcal{F}}(U) = \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}\mathbf{B}^2\right)\mathsf{Id} - \mathbf{B}\mathbf{B}^T \\ \mathbf{u}\left(E + p + \frac{1}{2}\mathbf{B}^2\right) - \mathbf{u}\cdot\mathbf{B}\mathbf{B} \\ \mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u} + \psi\mathsf{Id} \\ c_h^2 \mathbf{B} \end{pmatrix}$$

$$S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\alpha\psi \end{pmatrix} \qquad \frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}(U) = S$$

Properties

- “Old” and “new” systems are hyperbolic (6 waves/7 waves)
- If $\alpha > 0$ “new” system is also dissipative
- The new system can be made Galilean invariant

but need of extensive test to tune the right parameters c_h and α.

$c_h >$ max(eigenvalues of MHD system)

- $\gamma = 5/3$
- Initialisation

$$\begin{pmatrix} \rho \\ \mathbf{u} \\ p \\ B_x, B_y \\ \psi \end{pmatrix} = \begin{cases} (1, 0, 0, 1, \frac{5}{2\sqrt{\pi}}, \frac{5}{2\sqrt{\pi}}, 0)^T & \text{if } \sqrt{x^2+y^2} \\ (0.125, 0, 0, 0.1, \frac{5}{2\sqrt{\pi}}, \frac{5}{2\sqrt{\pi}}, 0)^T & \text{else} \end{cases}$$

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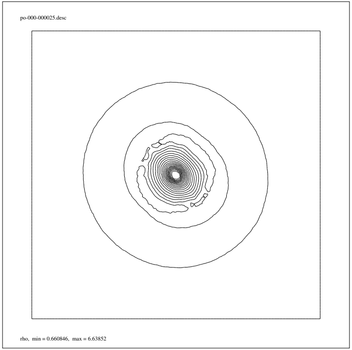
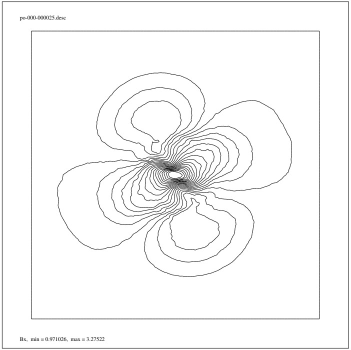
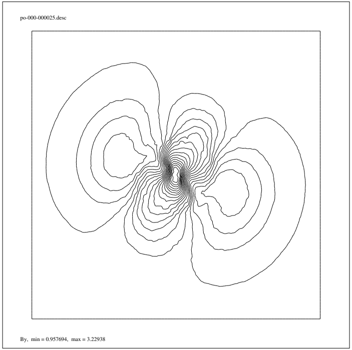
Density



Magnetic field



B_x



B_y

$\mathcal{RD}$ for compressible MHD flows : preliminary results

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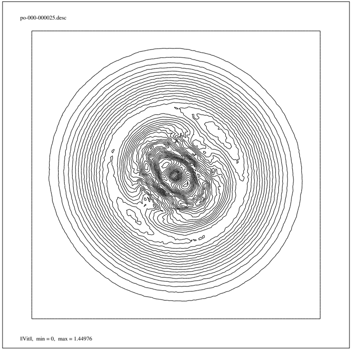
Velocity



Conclusions and perspectives

Conclusions

- Convergent higher order non-oscillatory $\mathcal{RD}$ schemes
- General procedure
- Efficient discretizations (fewer DoF and op.s w.r.t. DG)
- For systems less matrix algebra than with upwind schemes

Problems and perspectives

- Does–it work ?
- Iter–like simulations, ELM instabilities
- High order
- h–p adaptation ?

Mesh