Kameny Polynomial 1

kameny.1.7

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Ring: $\mathbb{C}[x]$
Equations:

$(x-3c^2)^2 + icx^7, \quad 0 < c << 1, \quad (c = 10^{-6})$

Polynomial from the REDUCE test suite, supplied by Stan Kameny. Added to the list by D.A. Bini (Oct. 1996).

1 Characteristics

- Degree 7, real and imaginary parts of the coefficients are rational.
- The polynomial has two simple complex roots with extremely small real separation (33 common digits) and imaginary parts close to zero:

$$\begin{array}{l} 2.999999999999999999999999999999997 \cdot 10^{-12} + \\ \qquad 3.306811152757290432566333500085327 \cdot 10^{-44}\mathrm{i}, \\ 3.000000000000000000000000000000003 \cdot 10^{-12} - \\ \qquad 3.306811152757290432566333500085321 \cdot 10^{-44}\mathrm{i}, \end{array}$$

  The remaining roots are well separated:

- Define the condition number of the simple root α of the polynomial $p(x) = \sum_{j=0}^{n} p_j x^j$ as $cond(\alpha) = \frac{\sum_{j=0}^{n} |p_j| \cdot |\alpha|^j}{|\alpha| \cdot |p'(\alpha)|}$, in this way a relative perturbation of modulus ϵ in the coefficients p_j generates a relative change in the root α of modulus at most $\epsilon \cdot cond + O(\epsilon^2)$. Then, for the above pair of roots we have $cond > 10^{35}$, that is, in order to separate the two above roots by using a floating point computation, at least 68 digits of working precision are needed.
- More difficult situations can be obtained with smaller values of the parameter c. For $c = 10^{-20}$ we have a pair of roots with 113 common digits and nonzero imaginary part of modulus less than 10^{-149}. For $c = 10^{-70}$ we have a pair of roots with 385 common digits and nonzero imaginary part of modulus less than 10^{-524}.

2 Problem

Newton-isolate all the roots.

Newton-isolate all the roots of modulus greater than 1.

Newton isolating the roots means to compute approximations $\tilde{x}_i$ to the roots x_i, $i = 1, \ldots, n$ such that $|\tilde{x}_i - x_i| < \frac{1}{3(n-1)}|\tilde{x}_i - x_j|$ for any $i \neq j$. In this way Newton's iteration applied to $\tilde{x}_i$ converges quadratically right from the start to x_i, i.e., for the approximation $z_i^{(k)}$ obtained after k Newton's steps with $z^{(0)} = \tilde{x}_i$, it holds $|z_i^{(k)} - x_i| \leq \left(\frac{1}{2}\right)^{2^k-1} |\tilde{x}_i - x_i|$ (compare P. Tilli, Convergence conditions of some methods for the simultaneous computation of polynomial zeros, *Calcolo* 1996).

2.1 Solution

Algorithm based on simultaneous iterations (Aberth's method) with cluster analysis (see [**?**]).

Program in Fortran 90 based on the multiprecision package MPFUN, by D. Bailey.

Machine used: IBM Risc 6000.

Time for Newton-isolating all the roots: for $c = 10^{-6}$, 0.6 seconds; for $c = 10^{-20}$, 1.3 seconds; for $c = 10^{-70}$, 4.5 seconds.

Time for Newton-isolating all the roots of modulus greater than 1: for $c = 10^{-6}$, 0.28 seconds; for $c = 10^{-20}$, 0.28 seconds; for $c = 10^{-70}$, 0.45 seconds.

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References

[1] D. Bini. Numerical computation of polynomial zeros by means of aberth's method. *Numerical Algorithms*, 1997. (to appear).

[2] D. Bini and B. Mourrain. Polynomials test suite. 1996.