Is an unbiased estimate of an unbiased estimate still unbiased?

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Abstract

Yes.

1 Preliminaries

We investigate whether the method proposed by Popov et al. [1] produces unbiased estimates of the rendering equation. To simplify the analysis, we apply their sampling method to the 2D integral

$$I = \int_A \int_B f(x, y) \, \mathrm{d}y \, \mathrm{d}x, \tag{1}$$

where (roughly speaking) $x \in A$ corresponds to a light path and $y \in B$ corresponds to a camera path.

To compute a Monte Carlo estimate for this integral using the method in [1], we proceed as follows:

1. Sample a light path y with the probability density function (PDF) $p(y)$.
2. Sample M independent and identically distributed light paths $\mathcal{X} = \{x_1, \ldots, x_M\}$. That is, $p(\mathcal{X}) = \prod_{j=1}^{M} p(x_j)$.
3. Select K light paths $\{x_{\xi_1}, \ldots, x_{\xi_K}\}$ from $\mathcal{X}$ using weighted resampling with replacement. Crucially, as the probability of selecting a light path x_ξ depends on both the current set of light paths $\mathcal{X}$ and the camera path y, we write the probability mass function (PMF) as $p(\xi | y, \mathcal{X})$.
4. Estimate I as
$$\frac{1}{M} \frac{1}{K} \sum_{k=1}^{K} \frac{f(x_{\xi_k}, y)}{p(x_{\xi_k}) p(y) p(\xi_k | y, \mathcal{X})}. \tag{2}$$

2 Proof

We prove that the estimator in Equation 2 is unbiased by deriving its expected value

$$\hat{I} = E\left[\frac{1}{M} \frac{1}{K} \sum_{k=1}^{K} \frac{f(x_{\xi_k}, y)}{p(x_{\xi_k}) p(y) p(\xi_k | y, \mathcal{X})} \right] \tag{3}$$

with respect to the joint probability distribution $p(\mathcal{X}, y, \xi) = p(\mathcal{X}) p(y) p(\xi | y, \mathcal{X})$. Using the linearity of expectation, we see that

$$\hat{I} = \frac{1}{K} \frac{1}{M} \sum_{k=1}^{K} E\left[\frac{f(x_\xi, y)}{p(x_\xi) p(y) p(\xi | y, \mathcal{X})} \right] = \frac{1}{M} E\left[\frac{f(x_\xi, y)}{p(x_\xi) p(y) p(\xi | y, \mathcal{X})} \right]. \tag{4}$$

If we expand the expectation, we get

$$\hat{I} = \frac{1}{M}\int_B \int_{A^M} \sum_{\xi=1}^{M} \frac{f(x_\xi, y)}{p(x_\xi)p(y)p(\xi|y,\mathcal{X})} p(\mathcal{X})p(y)p(\xi|y,\mathcal{X})\,\mathrm{d}\mathcal{X}\,\mathrm{d}y \tag{5}$$

$$= \frac{1}{M}\int_B \int_{A^M} \sum_{\xi=1}^{M} \frac{f(x_\xi, y)}{p(x_\xi)p(y)p(\xi|y,\mathcal{X})} \left(\prod_{j=1}^{M} p(x_j)\right) p(y)p(\xi|y,\mathcal{X})\,\mathrm{d}\mathcal{X}\,\mathrm{d}y \tag{6}$$

$$= \frac{1}{M}\int_B \int_{A^M} \sum_{\xi=1}^{M} f(x_\xi, y) \prod_{j\neq\xi} p(x_j)\,\mathrm{d}\mathcal{X}\,\mathrm{d}y. \tag{7}$$

Let $\mathcal{X}_{\not\xi} = \{x_1, \ldots, x_{\xi-1}, x_{\xi+1}, \ldots, x_M\}$ be the subset of $\mathcal{X}$ that does not contain x_ξ. By rearranging the order of summation and integration, we see that

$$\hat{I} = \frac{1}{M}\sum_{\xi=1}^{M} \int_{A^{M-1}} \int_B \int_A f(x_\xi, y) \prod_{j\neq\xi} p(x_j)\,\mathrm{d}x_\xi\,\mathrm{d}y\,\mathrm{d}\mathcal{X}_{\not\xi} \tag{8}$$

$$= \frac{1}{M}\sum_{\xi=1}^{M} \int_{A^{M-1}} \prod_{j\neq\xi} p(x_j) \int_B \int_A f(x_\xi, y)\,\mathrm{d}x_\xi\,\mathrm{d}y\,\mathrm{d}\mathcal{X}_{\not\xi}. \tag{9}$$

Now, the innermost integral corresponds to our desired integral I in Equation 1. In other words,

$$\hat{I} = \frac{1}{M}\sum_{\xi=1}^{M} \int_{A^{M-1}} \prod_{j\neq\xi} p(x_j) I\,\mathrm{d}\mathcal{X}_{\not\xi} \tag{10}$$

$$= I\frac{1}{M}\sum_{\xi=1}^{M} \int_{A^{M-1}} \prod_{j\neq\xi} p(x_j)\,\mathrm{d}\mathcal{X}_{\not\xi} \tag{11}$$

$$= I\frac{1}{M}\sum_{\xi=1}^{M} 1 = I. \tag{12}$$

This concludes our proof. We have successfully shown that the expected value of the estimator described in Section 1 corresponds to the value of the desired integral I.

References

[1] Stefan Popov, Ravi Ramamoorthi, Frédo Durand and George Drettakis. Probabilistic connections for bidirectional path tracing. In *Computer Graphics Forum*, 34(1), 2015.