Additional Material: Formulas for energy computation

We present here several expressions which we use in the computation of energy for modes.

The instant energy of a mode is given by:

$$\int_{t_1}^{t_2} \left(\sin(\omega x)e^{-\alpha x}\right)^2 dx = \frac{1}{4}\frac{e^{-2\alpha t_1}\left(\alpha^2+\omega^2-\alpha^2\cos(2\omega t_1)+\alpha\omega\sin(2\omega t_1)\right)}{\alpha(\alpha^2+\omega^2)} - \frac{1}{4}\frac{e^{-2\alpha t_2}\left(\alpha^2+\omega^2-\alpha^2\cos(2\omega t_2)+\alpha\omega\sin(2\omega t_2)\right)}{\alpha(\alpha^2+\omega^2)} \quad (20)$$

To compute the total energy of two modes as $\|s\|^2 = <\sum_i a_i f_i, \sum_j a_j f_j> = \sum_i \sum_j a_i a_j < f_i, f_j >$, the expression $< f_i, f_j >$ is given below, using Eq. 20:

$$\int_0^{\infty} \left(\sin(\omega_1 x)e^{-\alpha_1 x}\right).\left(\sin(\omega_2 x)e^{-\alpha_2 x}\right) dx = \frac{2\omega_1\omega_2(\alpha_1+\alpha_2)}{\left((\alpha_1+\alpha_2)^2+(\omega_1-\omega_2)^2\right)\left((\alpha_1+\alpha_2)^2+(\omega_1+\omega_2)^2\right)} \quad (21)$$

Similarly, the scalar product of two modes in a given interval (t,t+dt) is given as follows (we substitute α_2 by $\alpha_2+\alpha_1$ and ω_2 by $\omega_2+\omega_1$):

$$\begin{aligned}
&\int_t^{t+dt} \left(\sin(\omega_1 x)e^{-\alpha_1 x}\right).\left(\sin(\omega_2 x)e^{-\alpha_2 x}\right) dx = \\
&(e^{-\alpha_2\,(t+dt)}\,((\omega_2^3-2\,\omega_1\,\omega_2^2+ \\
&\alpha_2^2\,\omega_2-2\,\alpha_2^2\,\omega_1)\,\sin((t+dt)\,\omega_2+(-2t-2\,dt)\,\omega_1)+ \\
&(-\alpha_2\,\omega_2^2-\alpha_2^3)\,\cos((t+dt)\,\omega_2+(-2t-2\,dt)\,\omega_1)+ \\
&e^{\alpha_2\,dt}\,((-\omega_2^3+2\,\omega_1\,\omega_2^2-\alpha_2^2\,\omega_2+2\,\alpha_2^2\,\omega_1)\,\sin(t\,\omega_2-2\,t\,\omega_1)+ \\
&(\alpha_2\,\omega_2^2+\alpha_2^3)\,\cos(t\,\omega_2-2\,t\,\omega_1)+(\omega_2^3-4\,\omega_1\,\omega_2^2+ \\
&(4\,\omega_1^2+\alpha_2^2)\,\omega_2)\,\sin(t\,\omega_2)+ \\
&(-\alpha_2\,\omega_2^2+4\,\alpha_2\,\omega_1\,\omega_2-4\,\alpha_2\,\omega_1^2-\alpha_2^3)\,\cos(t\,\omega_2))+ \\
&(-\omega_2^3+4\,\omega_1\,\omega_2^2+(-4\,\omega_1^2-\alpha_2^2)\,\omega_2)\,\sin((t+dt)\,\omega_2)+ \\
&(\alpha_2\,\omega_2^2-4\,\alpha_2\,\omega_1\,\omega_2+4\,\alpha_2\,\omega_1^2+\alpha_2^3)\,\cos((t+dt)\,\omega_2))) \\
&/(2\,\omega_2^4-8\,\omega_1\,\omega_2^3 \\
&+(8\,\omega_1^2+4\,\alpha_2^2)\,\omega_2^2-8\,\alpha_2^2\,\omega_1\,\omega_2+8\,\alpha_2^2\,\omega_1^2+2\,\alpha_2^4) \quad (22)
\end{aligned}$$

This expression can be computed, after appropriate factorization using 17 additions, 24 multiplications, 8 cosine/sine operations, 2 exponentials and 1 division.