Nonlinear MHD simulations of ELMs

Emiel van der Plas
Guido Huijsmans
Association Euratom-CEA Cadarache
Motivation : Edge Localised Modes

• «Edge Localised Modes» (ELMs) are MHD instabilities destabilised by the pressure gradient in the H-mode edge pedestal:
  – losses up to 10% plasma energy in several 100 micro-seconds
  – Major concern for the operation of ITER
    • ELM control is essential
    • Physics understanding of ELMs might be useful
ELMs

- The physics of the ELMs is still not well understood:
  - How to extrapolate to ITER parameters
- MHD stability limits are well known:
  - Ballooning modes (and peeling modes)
  - Good agreement with experiment
- Many open questions on the non-linear evolution:
  - What determines the size of an ELM?
  - How can a plasma become strongly unstable?
  - What is the relaxation mechanism?

**ELM physics studies by numerical simulation**

- Proposal ANR-CIS2006: project ASTER

*The objective of this project is the high resolution MHD simulation of a complete cycle of the ELM instability, from its onset, the highly non-linear phase and its decay.*
ANR-CIS projet ASTER: Adaptive mhd Simulation of Tokamak Elms for iteR

Rémi Abgrall (Prof, MAB, Univ. Bdx1)
Marina Bécoulet (CR, CEA/IRFM)
Olivier Coulaud (DR INRIA)
Pascal Hénon (CR INRIA)
Robin Huart (thesard, MdC, MAB, Univ Bdx1)
Guido Huysmans (CR, CEA/IRFM)
Boniface N'Konga (Prof, Univ. Nice/ MdC, Univ Bdx1)
Stanislas Pamela (thesard, CEA/IRFM)
Emiel van der Plas (postdoc, CEA/IRFM)
Pierre Ramet (MdC, Labri, Univ Bdx1)
Alessandro Liberati (postdoc, Labri, Univ. Bdx1)
Project ANR-CIS2006.001: ASTER

- Development of high-resolution non-linear MHD code(s) for the simulation of ELMs in ITER
  - Using the expertise of the fluid dynamics community to develop non-linear MHD simulation codes
    - Extension of the CFD code Fluidbox to MHD
  - Continued development of the code JOREK
    - Physics model, numerical methods
  - FLUIDBOX and JOREK use the same fully implicit methods leading to large sparse systems of equations:
    - Continued development of the PastiX sparse matrix library
  - Adaptive grid refinement

- Application to the study of the physics of ELMs

- Collaboration IRFM Cadarache & Labri, MAB et INRIA Bordeaux

- ASTER finances: postdoc Emiel van der Plas, PhD Stanislas Pamela
  1 postdoc (Labri) et 1 PhD (MAB)

site web: http://aster.gforge.inria.fr
JOREK2 nonlinear MHD code

• JOREK has been developed with the specific aim to simulate ELMs
  – non-linear MHD
  – domain with closed and open field lines
• New version: (JOREK2):
  – Discretisation:
    • Cubic finite elements in the poloidal plane
    • Fourier series in toroidal angle
  – Time stepping:
    • fully implicit Crank-Nicholson (no splitting)
  – Solver sparse matrices:
    • GMRES with ‘physics’ based preconditioner
    • Preconditioner per toroidal harmonic using PastiX
    • N(dof) ~ 5x10^6
  – Parallelisation using MPI (+threads, PastiX)
    • Distribution of elements, matrix construction
    • Matrix solution GMRES, PastiX
JOREK2 : time stepping

- Fully implicit time evolution allows large time steps:
  - time step independent of grid size
  - from 1-10 Alfven times for ELM simulations to 10,000 Alfven times for slow growing tearing modes

- Linearised Crank Nicholson scheme:

\[
\frac{\partial A(y)}{\partial t} = B(y)
\]

\[
\frac{\partial A}{\partial y} \delta y = \delta t B(y_n) + \frac{1}{2} \delta t \frac{\partial B(y_n)}{\partial y} \delta y
\]

\[
\Rightarrow B(y_n) \delta t = \left( \frac{\partial A(y_n)}{\partial y} - \frac{1}{2} \delta t \frac{\partial B(y_n)}{\partial y} \right) \delta y
\]

- Leads to very large systems of equations to be solved at every time step
JOREREK Boundary Conditions

• Wall parallel to field lines:
  – all variables constant in time

• Wall crossing field lines:
  – Free outflow of density and temperature
  – Mach 1 parallel flow

\[ v_\parallel = c_s = \sqrt{\gamma T_{\text{divertor}}} \]

• Generalised to include poloidal flow component:

\[ v_\parallel \cdot n + v_E \cdot n = \frac{v_\parallel \cdot n}{|v_\parallel|} c_s \]
**OREK Reduced MHD**

- Results JOREK2  
  (Huysmans, Nuc.Fus. 2007)

- Reduced MHD:

  \[ \mathbf{v} = -R \nabla u(t) \times \mathbf{e}_\phi + v_{\parallel}(t) \mathbf{B}, \]

  \[ \mathbf{B} = \frac{F_0}{R} \mathbf{e}_\phi + \frac{1}{R} \nabla \psi(t) \times \mathbf{e}_\phi \]

- Formation of density filaments sheared off by induced n=0 flow

**density**  
**flow (iso-lines of \( \phi \))**
JOREK development: full MHD

⇒ comparison with experiment (growth rates etc)!

- Introduces new physics
  ⇒ compressibility, (fast) magneto-acoustic waves
  ⇒ effects of parallel flow
  ⇒ parallel vs perpendicular physics untangled

The full MHD equations:

\[ \partial_t \rho = - \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (D \nabla \rho) + S_\rho, \]

\[ \rho \partial_t \mathbf{v} = - \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla (\rho T) + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}, \]

\[ \partial_t T = - \mathbf{v} \cdot \nabla T - (\gamma - 1) T \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T) + S_T, \]

\[ \partial_t \mathbf{A} = \mathbf{v} \times (\nabla \times \mathbf{A}) - \eta (\nabla \times \nabla \times \mathbf{A}). \]

- Vector equations
Vector basis

- Use the local coordinates:
  \[ a_1 = \mathcal{J} (\nabla t \times \nabla \phi), \]
  \[ a_2 = \mathcal{J} (\nabla \phi \times \nabla s), \]
  \[ a_3 = \mathcal{J} (\nabla s \times \nabla t), \]

- Then metric tensor
  \[ g_{11} = \mathcal{J}^2 |\nabla \phi|^2 |\nabla t|^2, \quad g^{11} = |\nabla s|^2, \]
  \[ g_{22} = \mathcal{J}^2 |\nabla \phi|^2 |\nabla s|^2, \quad g^{22} = |\nabla t|^2, \]
  \[ g_{12} = g_{21} = -\mathcal{J}^2 |\nabla \phi|^2 \nabla s \cdot \nabla t, \quad g^{12} = g^{21} = \nabla s \cdot \nabla t, \]
  \[ g_{33} = \mathcal{J}^2 |\nabla s|^2 |\nabla t|^2 = |\nabla \phi|^{-2} = R^2 \quad g^{33} = \frac{1}{R^2}. \]

- Christoffel symbols
  \[ \Gamma^k_{ij} \equiv \frac{1}{2} g^{mk} (\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}). \]

- But how to obtain
  \[
  \begin{pmatrix}
  s_x & t_x \\
  s_y & t_y 
  \end{pmatrix}
  = \frac{1}{J_{2D}}
  \begin{pmatrix}
  y_t & -y_s \\
  -x_t & x_s 
  \end{pmatrix}
  \]
Example: JxB force

• Choose $\mathbf{v}$ covariant, $\mathbf{A}$ contravariant, so $\mathbf{A} = A_1 a^1 + A_2 a^2 + A_3 a^3$

• Then $\mathbf{B} = B^1 a_1 + B^2 a_2 + B^3 a_3$
  
  $$= \frac{1}{J} \left\{ \left( \partial_2 A_3 - \partial_3 A_2 \right) a_1 + \left( \partial_3 A_1 - \partial_1 A_3 \right) a_2 + \left( \partial_1 A_2 - \partial_2 A_1 \right) a_3 \right\}$$

• and $\mathbf{J} \times \mathbf{B} \equiv (\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2} \nabla|\mathbf{B}|^2$

$$(\mathbf{B} \cdot \nabla)\mathbf{B}^1 = B^1 \partial_1 B^1 + B^2 \partial_2 B^1 + B^3 \partial_3 B^1 + B^1 B^1 \Gamma^1_{11} + 2B^1 B^2 \Gamma^1_{12} + B^2 B^2 \Gamma^1_{22} + B^3 B^3 \Gamma^1_{33}$$

$$\partial_1 |\mathbf{B}|^2 = \partial_1 (g_{11} B^1 B^1 + 2g_{12} B^1 B^2 + g_{22} B^2 B^2 + g_{33} B^3 B^3)$$

• so $$(\mathbf{B} \cdot \nabla)\mathbf{B}^1(A_1) = \frac{1}{J} (\partial_3 A_1) \partial_2 B_0^1 + \frac{1}{J} (-\partial_2 A_1) \partial_3 B_0^1$$
  
  $$+ 2 \left( \frac{1}{J} (\partial_3 A_1) \left( B_0^1 \Gamma^1_{12} + B_0^2 \Gamma^1_{22} \right) + \frac{1}{J} (-\partial_2 A_1) B_0^3 \Gamma^1_{33} \right)$$

$$\partial_1 |\mathbf{B}|^2(A_1) = 2 \left( \frac{1}{J} \partial_3 A_1 \partial_1 g_{12} B_0^1 + 2 g_{12} B_0^1 \partial_1 (\frac{1}{J} \partial_3 A_1) \right)$$
  
  $$+ 2 \left( \frac{1}{J} \partial_3 A_1 \partial_1 g_{22} B_0^2 + 2 g_{22} B_0^2 \partial_1 (\frac{1}{J} \partial_3 A_1) \right)$$
  
  $$+ 2 \left( -\frac{1}{J} \partial_2 A_1 \partial_1 g_{33} B_0^3 + 2 g_{33} B_0^3 \partial_1 (-\frac{1}{J} \partial_2 A_1) \right)$$
Beziane finite elements

• Bezier curves in general

\[ \mathbf{B}(t) = \sum_{k=0}^{N} \mathbf{P}_k \frac{N!}{k!(N-k)!} t^k (1-t)^{N-k} \]

• 3rd order:

\[ \mathbf{B}(t) = \mathbf{P}_0 (1-t)^3 + 3\mathbf{P}_1 t(1-t)^2 + 3\mathbf{P}_2 t^2(1-t) + \mathbf{P}_3 t^3 \]

• defined by 4 points or 2 vectors

\[ \mathbf{u}_0 = \frac{3}{2} (\mathbf{P}_1 - \mathbf{P}_0), \quad \mathbf{u}_3 = \frac{3}{2} (\mathbf{P}_3 - \mathbf{P}_2) \]

• Continuity up to first order choosing value of variable and first derivative
Beziers finite elements

- 2D Bezier patches on \{s,t\}
  - 16 points, or
  - 1 value and 3 vectors per node

- iso-parametric: space & variable

\[ R_i(s, t) = \sum_{k=1}^{4} \sum_{j=1}^{4} R_{ijk} H_j(s, t) \]

\[ Z_i(s, t) = \sum_{k=1}^{4} \sum_{j=1}^{4} Z_{ijk} H_j(s, t) \]

- flexible discretisation
- easily (h) refinable: AMR!
- basis for vector formalism

\[ u_1 = (P_{10} - P_{00}) / h_{u_1} \]

\[ v_1 = (P_{01} - P_{00}) / h_{v_1} \]

\[ w_1 = (P_{11} + P_{00} - P_{01} - P_{10}) / h_{u_1} h_{v_1} \]
Rectangular grid

• Advantages:
  – no (near) singularity at axis

• Disadvantages
  – Need very high resolution to resolve features on resonant surface
  – higher order finite elements lose validity
Results (square)

- Test: $m=1, n=1$ internal kink mode ($q(0) = 0.75$)
- The flow and perturbed vector potential $A_3$ for quadrangular grid:
Polar grid

- Advantages:
  - orthogonality

- Disadvantage:
  - as in square grid: convergence may fail for modes around specific flux surface
Results (polar)

- Test: $m=1, n=1$ internal kink mode ($q(0) = 0.64$)
- The flow and perturbed vector potential $A_3$ for polar grid:
Flux surface grid

- Advantages:
  - resolves thin layers around flux surfaces
  - open & closed field lines
Results: growth rate $m=1, n=1$ internal kink

- The linear growth rate is in good correspondence with e.g. CASTOR
- scales as $\sim \eta^{1/3}$

$$\gamma_{\text{kink}} = \frac{(qB/\sqrt{\mu_0 \rho})^{2/3}}{r_1^{1/3}} \frac{1}{(\mu_0 r^2/\eta)^{1/3}}$$

- Error from grid size: $\sim h^5$

- $\gamma \sim \eta^{0.36 \pm 0.03}$

$\sim h^{6.4}$
Development

- Regularisation on axis of grid
  - on axis, metric becomes singular: \( \partial_t = 0, \Gamma_{ij}^k = 0 \)
  - Add constraining equation to momentum balance (as b.c.)
    \[
    \n    + \alpha (v_j^n - v_j^{n+1})
    \]

- Then flux surface grid with x-point, edge pressure gradient, ELM

\[ v^1(n = 0) \]
JOREK Collaborations

• UKAEA, Ian Chapman: resistive wall modes
  – Implementation of a resistive wall
  – Coupling with the magnetic field outside vessel

• FOM, Dirk Hoving (University Eindhoven/FOM): divertor model
  – Interaction plasma-neutrals (fluid model)

• FOM, Egbert Westerhof:
  – Tearing Modes, stabilisation, application TEXTOR

• Longer term subjects (2009-2012):
  – Disruptions
  – Fast particles (FOM, J.W. Blokland)
  – Divertor model
• Renewal ANR on MHD (2010?)
Conclusion

• project **ASTER**: model full ELM cycle

• JOREK2 in RMHD works ok

• JOREK: full MHD
  – in quadrangular/polar grid quasilinear kink mode
    • growth rate
    • scaling under $\eta$
  – under development: full nonlinear time evolution
    • vector representation at center

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*Ce travail a bénéficié d'une aide de l'Agence Nationale de la Recherche portant (référence ANR-06- CIS6-001).*
Equilibrium, flows

• Non-linear MHD simulation typically start from an equilibrium

• JOREK recipe:
  – Static Grad-Shafranov equation
  – Align grid to flux surfaces
  – Static Grad-Shafranov equation
  – Evolve n=0 only to obtain a stationary equilibrium
  – Add other toroidal harmonics
Equilibrium flow transitions (S. Pamela)

Unexpected transitions in equilibrium flow pattern/amplitude
- during decaying (diffusing) density profile
- transition from $m=1$ to $m=0$ pattern

Flow amplitude [a.u.]

![Graph showing flow amplitude over time](image)

![Density and poloidal velocity plots](image)
Implementation MHD (non-reduced)

- MHD Model:

\[
\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (D \nabla \rho) + S_\rho, \\
\rho \partial_t \mathbf{v} = -\rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla (\rho T) + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}, \\
\partial_t T = -\mathbf{v} \cdot \nabla T + (\gamma - 1)T \nabla \cdot \mathbf{v} + \nabla \cdot (\nabla T) + S_T, \\
\partial_t \mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}).
\]

- Requires a vector basis aligned to magnetic field
  - problem at x-point
  - Use the vectors defining the Bezier finite Elements.

- In progress:
  - Derivation of equation completed (non-orthogonal basis)
  - Equations programmed, begin debugged

Emiel van der Plas
JOREK Ongoing activities (2008-2009)

• ELM physics studies (S. Pamela)

• 2-fluid MHD, Te, Ti, diamagnetic flows (S. Pamela)

• Resonant magnetic field perturbations (M. Becoulet)

• Full MHD implementation (E. van der Plas)

• Fluid neutral model (D. Hoving)

• Implementation external magnetic field (G. Huysmans)

• Adaptive grid refinement (postdoc Bordeaux)
JOREK Equations reduced MHD

- Formulation using electric and magnetic potentials:

\[ \mathbf{v} = -R \nabla u(t) \times \mathbf{e}_\phi + v_\parallel(t) \mathbf{B}, \quad \mathbf{B} = \frac{F_0}{R} \mathbf{e}_\phi + \frac{1}{R} \nabla \psi(t) \times \mathbf{e}_\phi \]

Poloidal flux

\[ \frac{1}{R^2} \frac{\partial \psi}{\partial t} = +\eta \nabla \cdot \left( \frac{1}{R^2} \nabla \psi \right) - \frac{1}{R} [u, \psi] - \frac{F_0}{R^2} \partial_\phi u \]

Parallel momentum

\[ \mathbf{B} \cdot \left( \rho \frac{\partial \mathbf{v}}{\partial t} = -\rho \left( \mathbf{v} \cdot \nabla \right) \mathbf{v} - \nabla \left( \rho T \right) + \mathbf{J} \times \mathbf{B} + \mu \Delta \mathbf{v} \right) \]

Poloidal momentum

\[ \mathbf{e}_\phi \cdot \nabla \times \left( \rho \frac{\partial \mathbf{v}}{\partial t} = -\rho \left( \mathbf{v} \cdot \nabla \right) \mathbf{v} - \nabla \left( \rho T \right) + \mathbf{J} \times \mathbf{B} + \mu \Delta \mathbf{v} \right) \]

Temperature

\[ \rho \frac{\partial T}{\partial t} = -\rho \mathbf{v} \cdot \nabla T - (\gamma - 1) \rho T \nabla \cdot \mathbf{v} + \nabla \cdot \left( K_{\perp} \nabla_{\perp} T + K_{\parallel} \nabla_{\parallel} T \right) + S_T \]

Density

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (D_{\perp} \nabla_{\perp} \rho) + S \]
JOREK Time stepping

• Implementation of new time evolution scheme in JOREK2:
  – Fully implicit, using iterative solution (GMRES)
    • Matrix for each toroidal harmonic used as preconditioner
      – Solved using PastiX or MUMPS
        (using N MPI groups, new option in PastiX)
      – Preconditioning only recalculated when GMRES requires too many iterations
      – Reduce time step when number of iterations still too large
    • Much improved scaling parallelisation/memory use
      – First tests on CCRT platine (up to 512 cores)
      – reasonable scaling when \( \frac{n_{\text{tor}}}{n_{\text{cpu}}} \) constant
  • To be compared with PastiX iterative/direct solution
    – Alternative : partially explicit scheme:
      • Each toroidal harmonic implicit (ignoring coupling between harmonics)
      • Coupling terms explicit
      • Scheme as in SFELES CFD code
JOREK2 Non-linear MHD ELM Simulations

- First simulations including parallel flow and Mach one boundary conditions at the divertor:
  - $N_{tor}=0-21$ (periodicity 3)
  - $N_{cpu}=128$ (Norma)
Non-linear MHD ELM Simulations

- The non-linear MHD simulations (ballooning modes) are, qualitatively, in agreement with experimental observations:
  - Formation of filaments
  - Density profile perturbations (“density depression”)
  - Fine Structure in the power deposited at the divertor target
Power deposition profiles

• Fine Structures (~cm) observed in the JOREK simulations are similar to the structures observed in the AUG divertor.
‘Turbulence’

- First tests of possibility to do turbulence with JOREK
  - Resistive ballooning turbulence (n=0-48,6), (n_r=41, n_p=128)

Density

Vorticity

Temperature

Current density
JOREK2 : real or complex

- JOREK version using complex variables
  \[ u(\varphi) = \sum u_n e^{in\varphi} + u_n^* e^{-in\varphi} \]
  - Tearing mode test cases diverges after $10^5$ Alfvén times (in steady state with island)
  - Ballooning mode cases diverge in the early non-linear phase

- using real variables
  \[ u(\varphi) = \sum u_{n,c} \cos(n\varphi) + u_{n,s} \sin(n\varphi) \]
  - Stable for both tearing and ballooning mode cases
  - But slower for matrix construction/solution, larger memory requirements
Implicit n=0 equation

- Implicit complex scheme cannot be reduced to remove conjugate harmonics from the equation for n=0:

$$\delta \psi = \partial_t \left( F + \frac{1}{2} F' \delta \psi \right)$$

$$\left( 1 - \frac{1}{2} \delta t F' \right) \delta \psi = \delta t F$$

$$\begin{pmatrix}
1 - \frac{1}{2} \delta t f_0 & -\frac{1}{2} \delta t f_n^* & -\frac{1}{2} \delta t f_n \\
-\frac{1}{2} \delta t f_n & 1 - \frac{1}{2} \delta t f_0 & 0 \\
-\frac{1}{2} \delta t f_n^* & 0 & 1 - \frac{1}{2} \delta t f_0
\end{pmatrix} \begin{pmatrix}
\delta \psi_0 \\
\delta \psi_n \\
\delta \psi_n^*
\end{pmatrix} = \begin{pmatrix}
\delta t F_0 \\
\delta t F_n \\
\delta t F_n^*
\end{pmatrix}$$

$$F' \delta \psi = \left( f_0 + f_n e^{+i\varphi} + f_n^* e^{-i\varphi} \right) \left( \delta \psi_0 + \delta \psi_n e^{+i\varphi} + \delta \psi_n^* e^{-i\varphi} \right)$$

= $$\left( f_0 \delta \psi_0 + f_n \delta \psi_n^* + f_n \delta \psi_n \right) + \left( f_0 \delta \psi_n + f_n \delta \psi_0 \right) e^{i\varphi} + \left( f_0 \delta \psi_n^* + f_n \delta \psi_0 \right) e^{-i\varphi}$$

- but conjugate variables not available

- Implicit real scheme keeps all terms:

$$F' \delta \psi = \left( f_0 + f_c \cos (\varphi) + f_s \sin (\varphi) \right) \left( \delta \psi_0 + \delta \psi_c \cos (\varphi) + \delta \psi_s \sin (\varphi) \right)$$

$$\begin{pmatrix}
1 - \frac{1}{2} \delta t f_0 & -\frac{1}{4} \delta t f_c & -\frac{1}{4} \delta t f_s \\
-\frac{1}{4} \delta t f_c & \frac{1}{2} \left( 1 - \frac{1}{2} \delta t f_0 \right) & 0 \\
-\frac{1}{4} \delta t f_s & 0 & \frac{1}{2} \left( 1 - \frac{1}{2} \delta t f_0 \right)
\end{pmatrix} \begin{pmatrix}
\delta \psi_0 \\
\delta \psi_c \\
\delta \psi_s
\end{pmatrix} = \begin{pmatrix}
\delta t F_0 \\
\frac{1}{2} \delta t F_c \\
\frac{1}{2} \delta t F_s
\end{pmatrix}$$

$$\begin{pmatrix}
\psi_n = \frac{1}{2} \psi_c - \frac{1}{2} i \psi_s \\
\psi_n^* = \frac{1}{2} \psi_c + \frac{1}{2} i \psi_s \\
\psi_c = \left( \psi_n + \psi_n^* \right) \\
\psi_s = i \left( \psi_n - \psi_n^* \right)
\end{pmatrix}$$
• Isoparametric bicubic Bezier finite element grids:
  – space is represented with the same finite elements as the variables

\[ R_i(s,t) = \sum_{k=1}^{4} \sum_{j=1}^{4} R_{ijk} H_j(s,t) \]

\[ Z_i(s,t) = \sum_{k=1}^{4} \sum_{j=1}^{4} Z_{ijk} H_j(s,t) \]
Bezier finite elements

• Modified Cubic Hermite finite elements, defined as Bezier patches:
  – Variables described by 4 values per node
    • value, 2 derivatives, 1 cross derivative
  – Continuity of value and gradients
  – Compatible with local h-refinement

\[ \text{O. Czarny and G. Huysmans JCP 2008} \]