Algebraic methods
and
Arithmetic filtering
for
Exact predicates on circle arcs

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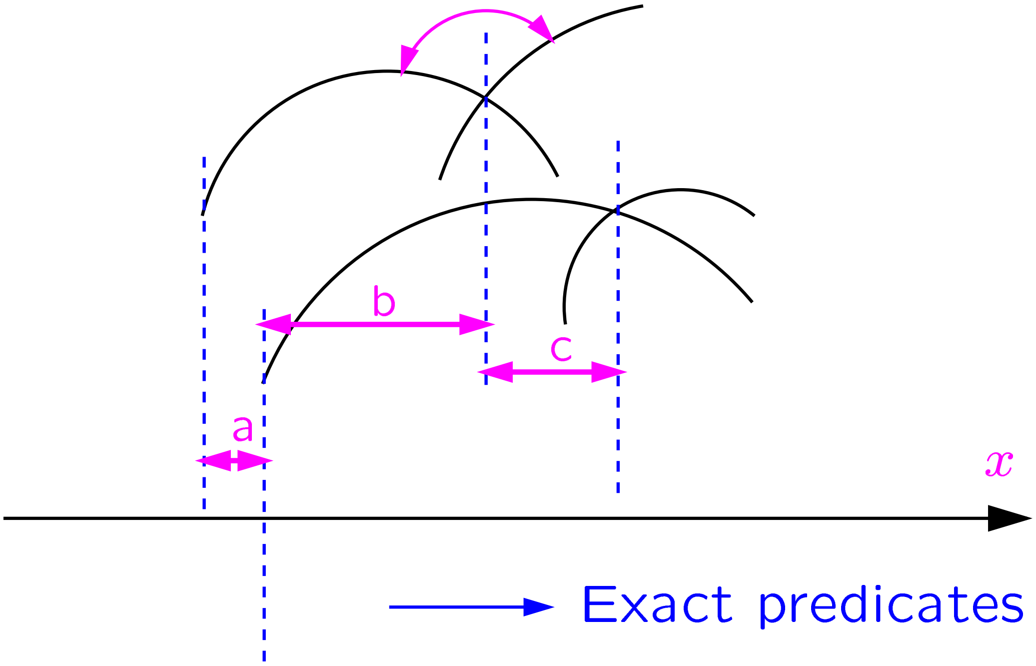
Sweep algorithm and robustness

Arrangement of circle arcs



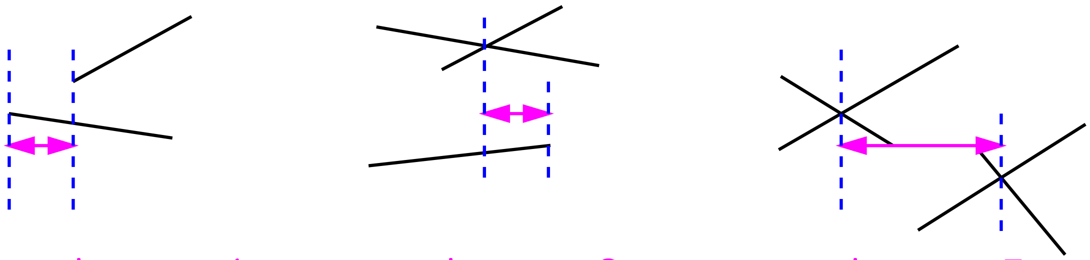
Exact predicates

2 approaches

Algorithms using "cheap" predicates

[Boissonnat *et al.*]

data = coordinates of endpoints



degree 1 degree 3 degree 5

Reducing the cost of "expensive" predicates

x-comparison of intersections

- Representation of circle arcs
- Resultants and x-comparison
- Invariants and simplification
- Strategy
- Filters
- Experimental results

Representation

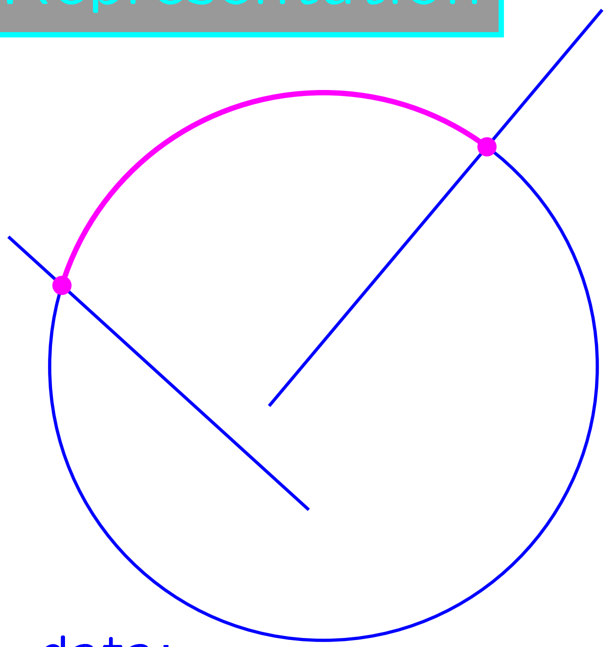
Result can be used for further computations

→ Same representation for

output arcs

input arcs

Representation



circle

center (α, β)

square radius r

line

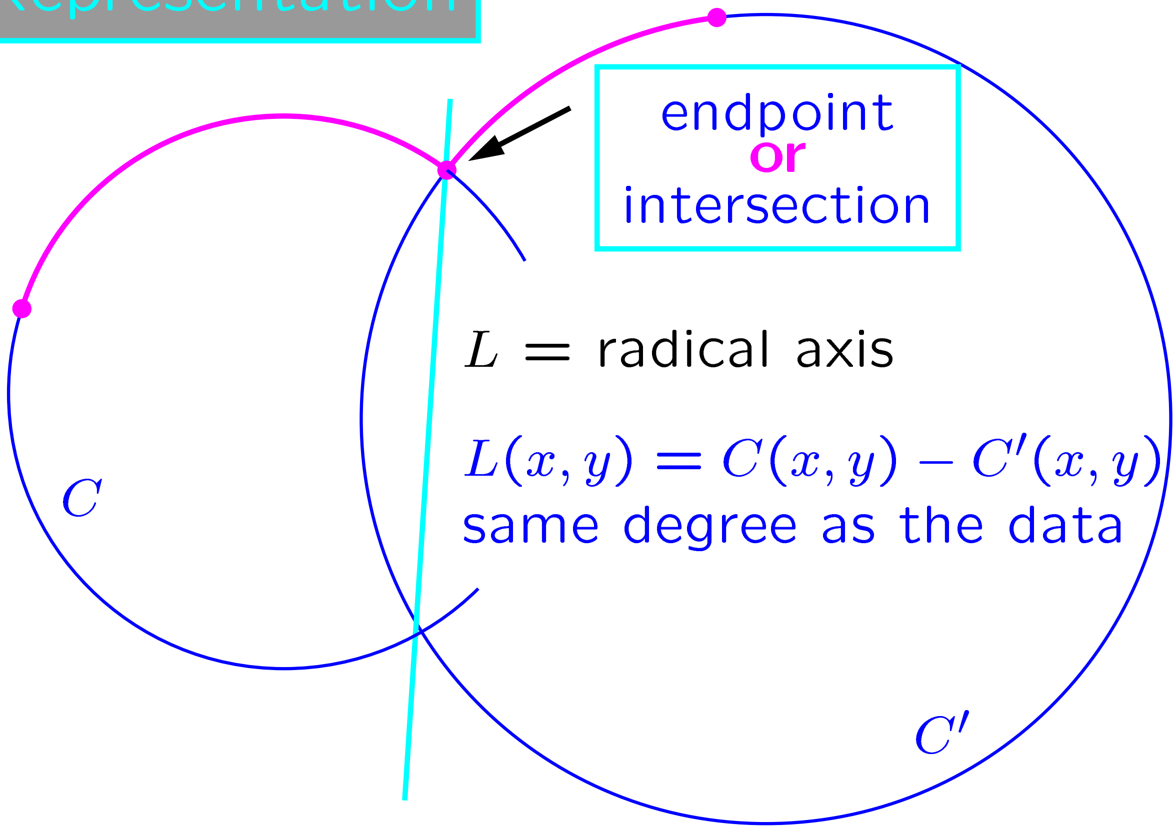
$px + qy + s = 0$

data:

α, β, p, q degree 1

r, s degree 2

Representation



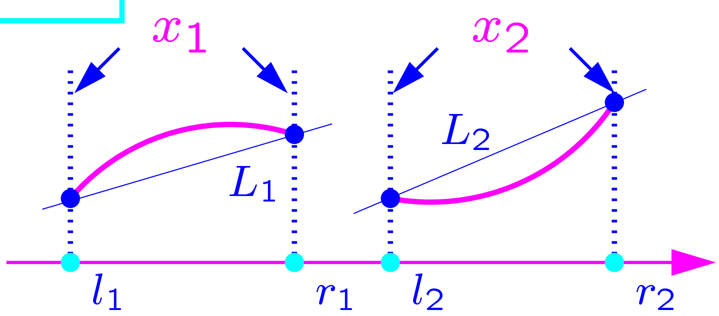
L = radical axis

$L(x,y) = C(x,y) - C'(x,y)$

same degree as the data

x-comparison

$$\boxed{x_i = L_i \cap C_i \in \{l_i, r_i\}}$$



Naive method

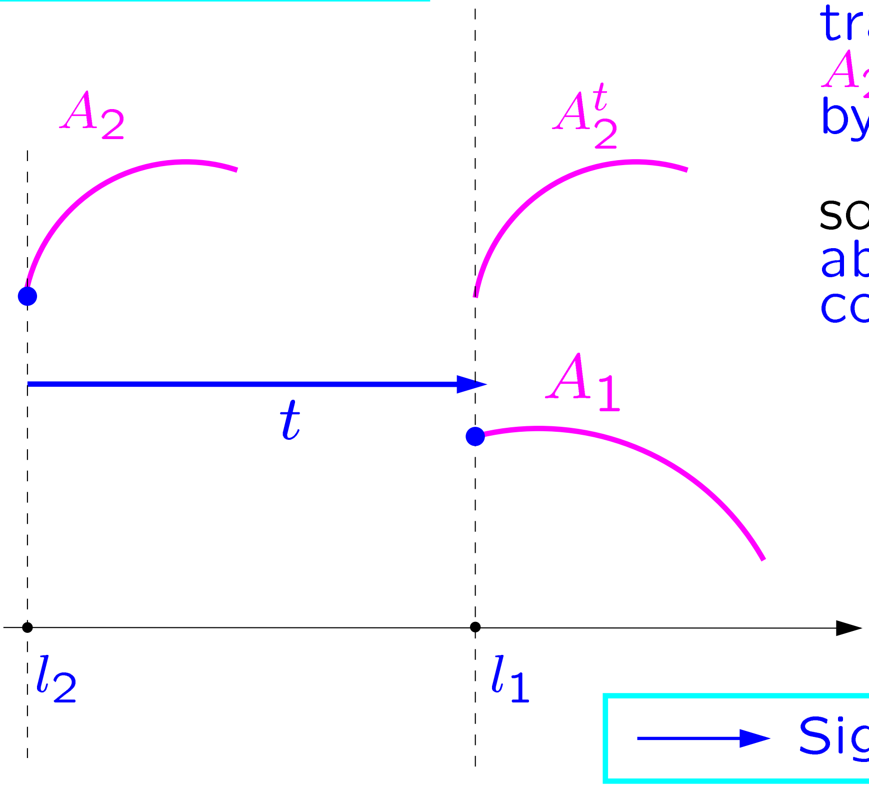
- compute x_1 and x_2

  involves $\sqrt{\cdots}$

- compare exactly

  use expensive arithmetic

x-comparison



translate
A_2
by t

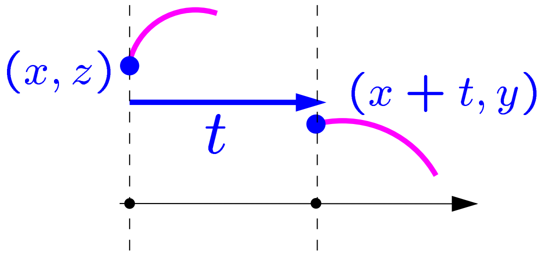
so that
abscissae
coincide

$\longrightarrow$ Sign of t ?

x-comparison

sign of t such that

$$\begin{cases} C_1(x+t,y) = 0 \\ L_1(x+t,y) = 0 \\ C_2(x,z) = 0 \\ L_2(x,z) = 0 \end{cases}$$



has a solution x, y, z

Bezoutian resultant

$\longrightarrow \quad P(t)$

Simplification

$$P(t) = P_0 t^4 + P_1 t^3 + P_2 t^2 + P_3 t + P_4$$

P_i polynomials in

$$\alpha_1, \beta_1, r_1, p_1, q_1, s_1$$
$$\alpha_2, \beta_2, r_2, p_2, q_2, s_2$$

degrees 8 to 12

659 monomials. . .

$\longrightarrow$ sign of a root t ?

Simplification

$$
\begin{aligned}
P_0 &= a_1^2\, a_2^2 \\
P_1 &= 4\, a_1\, a_2\, J \\
P_2 &= 4\, J^2 + 2\, a_1\, a_2\, K \\
P_3 &= 4\, J\, K \\
P_4 &= K^2 - 4\, I_1\, I_2 \quad = \quad -4 J J' + J''^2
\end{aligned}
$$

$i = 1, 2$

$$
\begin{aligned}
a_i &= p_i^2 + q_i^2 \\
b_i &= q_i^2\, \alpha_i - p_i\, s_i - p_i\, q_i\, \beta_i \\
c_i &= s_i^2 + 2\, q_i\, s_i\, \beta_i + q_i^2\, \alpha_i^2 + q_i^2\, \beta_i^2 - q_i^2\, \gamma_i
\end{aligned}
$$

Simplification

I_1, I_2, J, J', K invariants of
$Q_i(v,w) = a_i v^2 - 2b_i vw + c_i w^2$

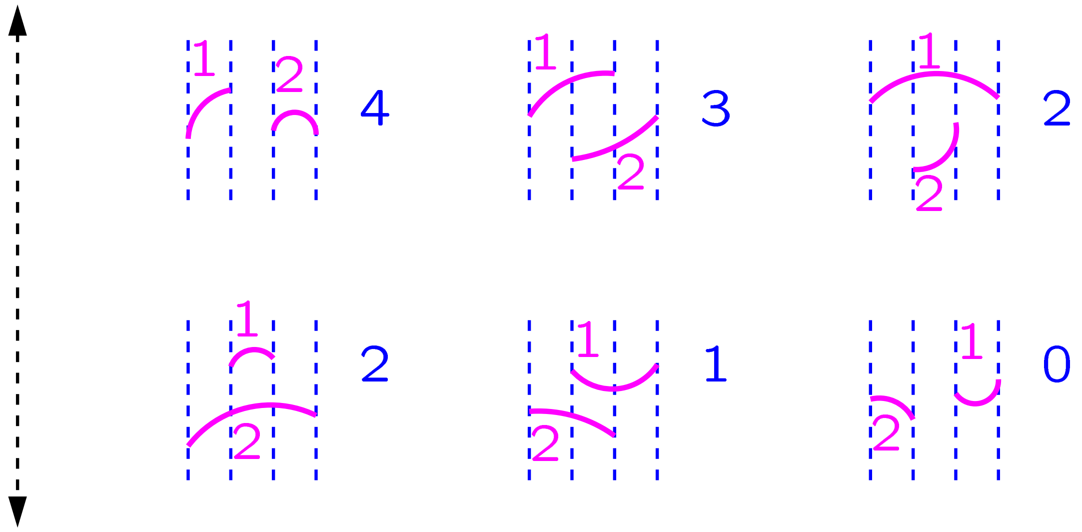
roots of $Q_i(x,1)$
$=$ abscissae of A_i's endpoints

$$
\begin{aligned}
I_1 &= b_1^2 - a_1\,c_1\\
I_2 &= b_2^2 - a_2\,c_2\\
J &= a_1\,b_2 - a_2\,b_1\\
J' &= b_1\,c_2 - b_2\,c_1\\
J'' &= a_1\,c_2 - a_2\,c_1\\
K &= c_1\,a_2 + a_1\,c_2 - 2\,b_1\,b_2
\end{aligned}
$$

have a geometric interpretation

Strategy

Number of positive roots (translations) of P



Strategy

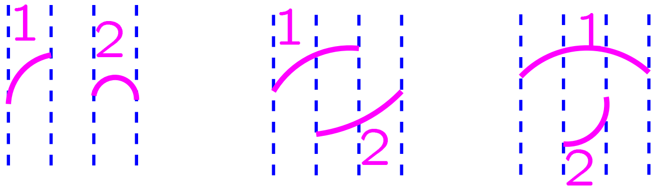
- Descartes rule

  # positive roots
  =
  # sign changes in $P_0, \ldots, P_4$

- Determination of the configuration

  not always necessary
  to compare 2 abscissae

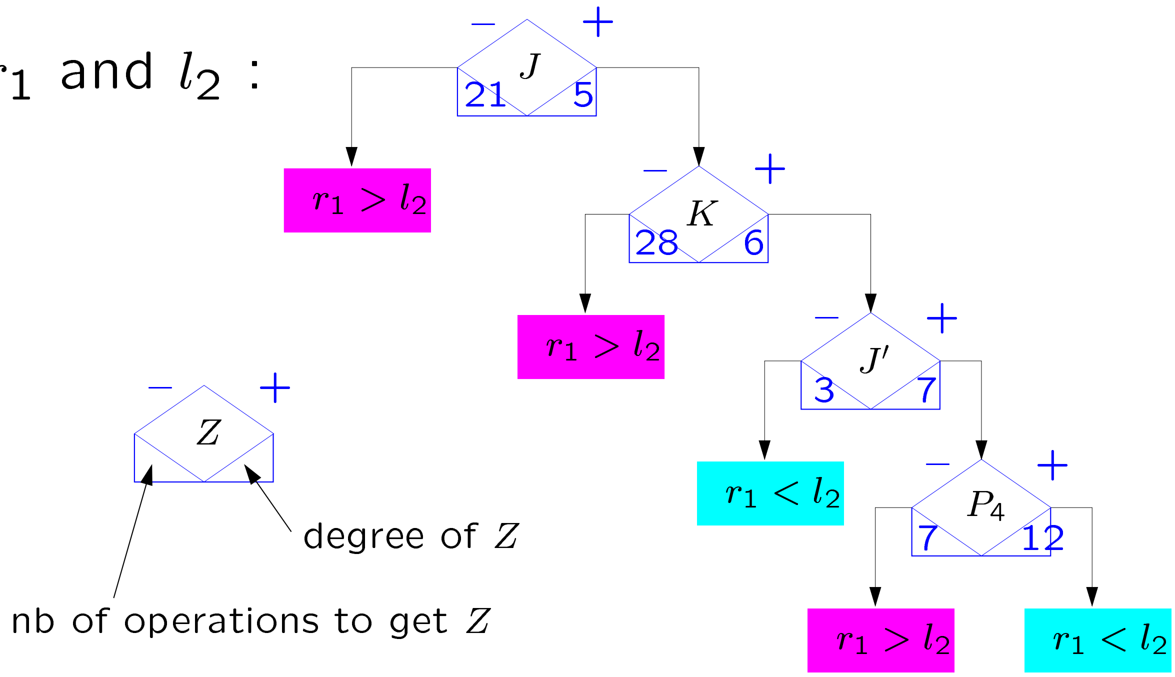
$l_1 < l_2$



Strategy

signs of polynomials
of degrees 5,6,. . . , at most 12

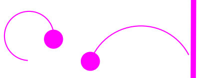
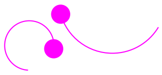
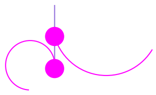
r_1 and l_2 :



Arithmetic

	naive	new
Filters		
static		•
semi-static		•
dynamic (interval arithmetic) [Pion]	•	•
Number types		
double	•	•
LEDA real	•	•
GMPZ		•

Benchmarks

μs		Naive Interval + L_real	New static +semi-static +Interval +GMP	New static +semi-static +Inter. Naive +GMP
left right		2.48	0.36	0.36
		67	24.3	6.8
		2170	129	128

Conclusion

Efficient → to be implemented in CGAL

Practical

General method → study other curves/surfaces

Bézoutian

Bézoutian of $f_0, f_1, ..., f_n$: $\Theta_{f_0,f_1,...,f_n}(u,v) =$

$$\begin{vmatrix} f_0(u) & \Theta_1(f_0)(u,v) & \ldots & \Theta_n(f_0)(u,v) \\ \vdots & \vdots & & \vdots \\ f_n(u) & \Theta_1(f_n)(u,v) & \ldots & \Theta_n(f_n)(u,v) \end{vmatrix}$$

$$\Theta_i(f_j)(u,v) = \frac{f_j(v_0,\ldots,v_{i-1},u_i,\ldots,u_n) - f_j(v_0,\ldots,v_i,u_{i+1},\ldots,u_n)}{u_i - v_i}$$

$\Theta_{f_0,\ldots,f_n}(u,v)$ rewritten as $\sum_{\alpha,\beta} \lambda_{\alpha,\beta}\ u^\alpha v^\beta$.

The determinant $|\lambda_{\alpha,\beta}|_{\{\alpha,\beta\}}$
is a multiple of the resultant.