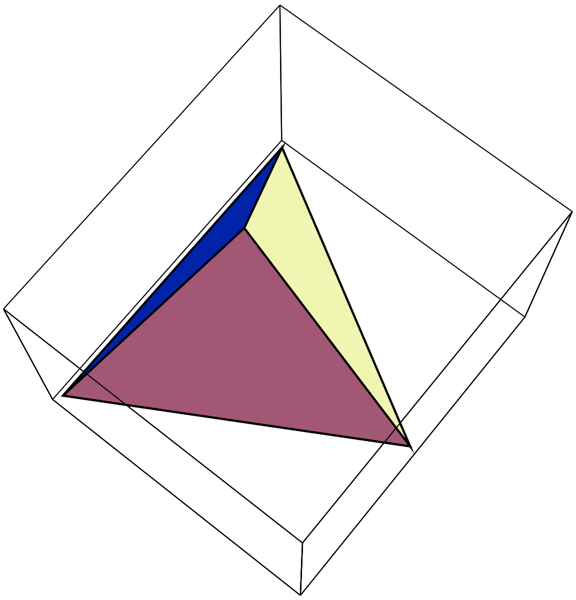
Homology Theory

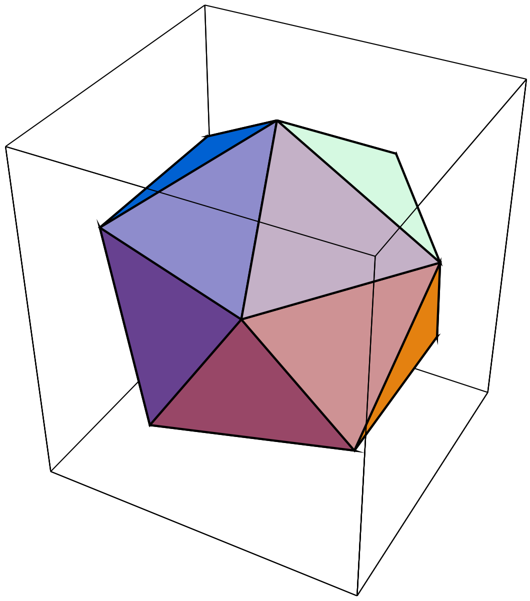
or

$$\chi(K) = \sum_{i=0}^{d} (-1)^i \beta_i(K)$$

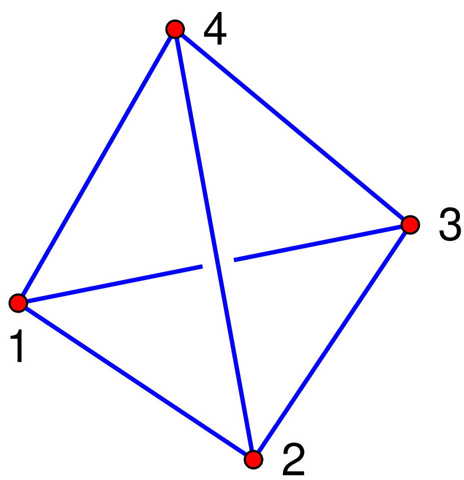
Topological features

Betti-numbers measure number of *connected components* (β_0), *tunnels* (β_1) and *voids* (β_2)





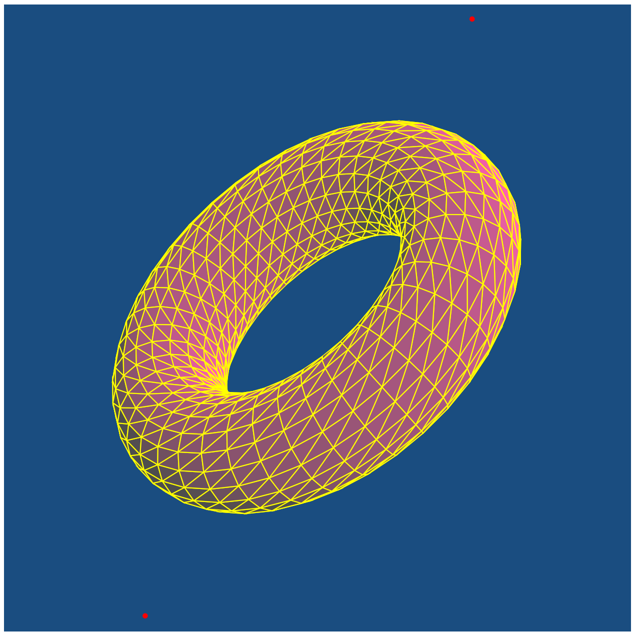
Connected: $\beta_0 = 1$. No tunnels: $\beta_1 = 0$. One void: $\beta_2 = 1$.

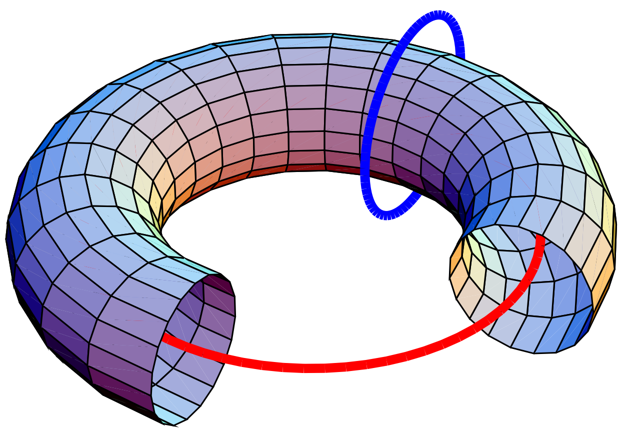


1-2-3-1 $=$ 1-2-4-1 $\oplus$ 2-3-4-2 $\oplus$ 3-1-4-3.

4 tunnels, 3 independent: $\beta_1 = 3$

Topological features





Connected: $\beta_0 = 1$. Two tunnels: $\beta_1 = 2$. One void: $\beta_2 = 1$.

Homology of Graphs

- K: graph with vertices $v_1, \ldots, v_n$ and (oriented) edges $e_1, \ldots, e_m$.
- $C_0(K)$: vector space of 0-chains, i.e. formal sums of the form

$$a_1\overline{v_1} + \cdots + a_n\overline{v_n}, \quad a_i \in \mathbb{R}$$

- $C_1(K)$: vector space of 1-chains, i.e. formal sums of the form

$$b_1\overline{e_1} + \cdots + b_m\overline{e_m}, \quad b_i \in \mathbb{R}.$$

- Boundary operator $\partial_1 : C_1(K) \to C_0(K)$ defined by

$$\partial_1(\overline{v_i v_j}) = \overline{v_j} - \overline{v_i}, \quad \text{if } \overline{v_i v_j} \in E.$$

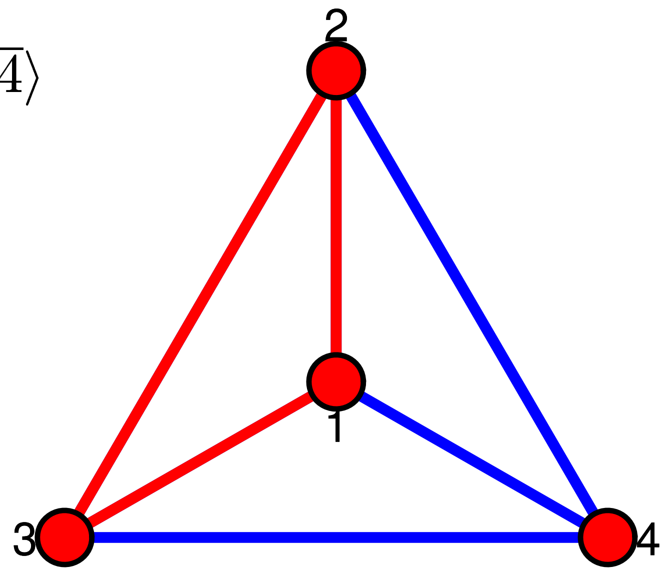
Example: K_4

$C_0(K) = \langle \overline{1}, \overline{2}, \overline{3}, \overline{4} \rangle$

$C_1(K) = \langle \overline{12}, \overline{13}, \overline{14}, \overline{23}, \overline{24}, \overline{34} \rangle$

$$\partial_1(\overline{12}) = \overline{2} - \overline{1}$$

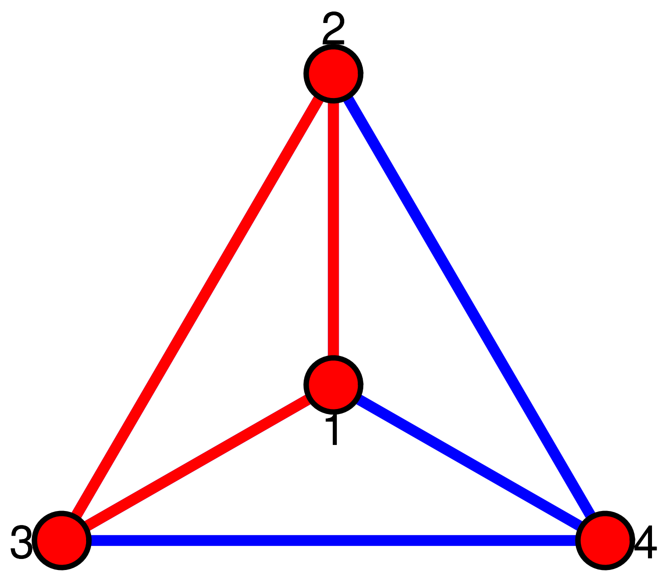
$$\partial_1(\overline{12} + \overline{23} - \overline{13}) = 0$$



Example (continued)

$\partial_1(\overline{12}) = \overline{2} - \overline{1}$

$\partial_1(\overline{12} + \overline{23} - \overline{13}) = 0$



∂_1	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{23}$	$\overline{24}$	$\overline{34}$
$\overline{1}$	-1	-1	-1	0	0	0
$\overline{2}$	1	0	0	-1	-1	0
$\overline{3}$	0	1	0	1	0	-1
$\overline{4}$	0	0	1	0	1	1

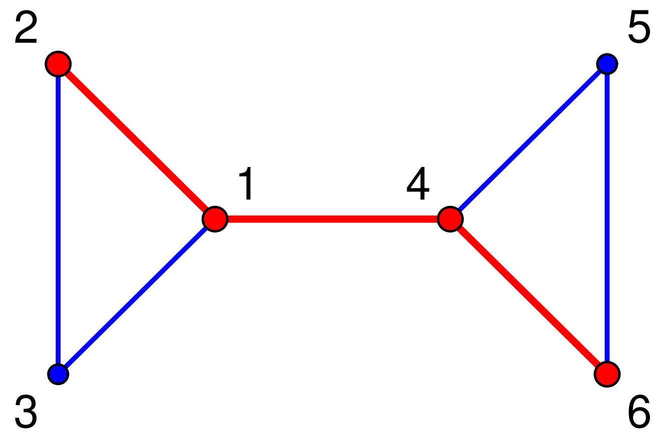
0-Homology

Two 0-chains $\overline{v}, \overline{w} \in C_0(K)$ are 0-homologous if there is a 1-chain $\gamma \in C_1(K)$ with $\partial_1(\gamma) = \overline{v} - \overline{w}$.

0-Homology is an equivalence relation; equivalence classes form zero-th homology group:

$$H_0(K) = C_0(K) \big/ \operatorname{Im} \partial_1$$

Example



$\overline{2}$ and $\overline{6}$ are 0-homologous since

$$\partial_1(-\overline{12} + \overline{14} + \overline{46}) = (-\overline{2} + \overline{1}) + (\overline{4} - \overline{1}) + (\overline{6} - \overline{4}) = \overline{6} - \overline{2}.$$

$$H_0(K) = \langle \overline{2} \rangle \cong \mathbb{R}$$

$\dim H_0(K)$: number of connected components of K.

1-Homology

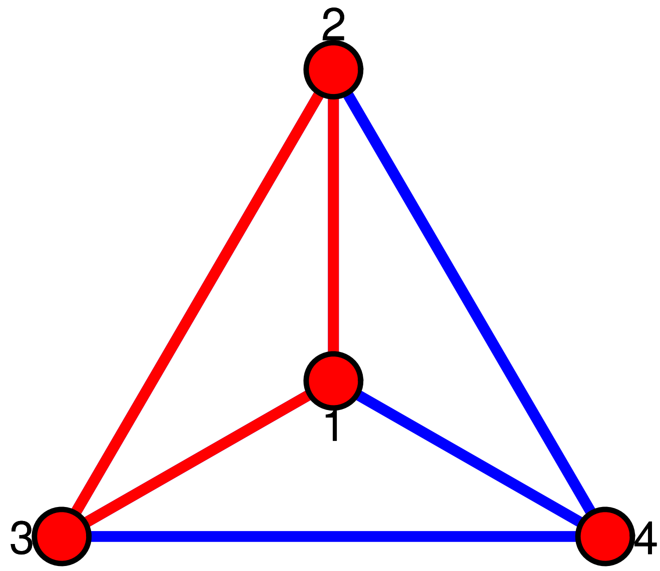
First homology group:

$$H_1(K) = \operatorname{Ker} \partial_1$$

Example: K_4

$$\partial_1(\overline{12}) = \overline{2} - \overline{1}$$

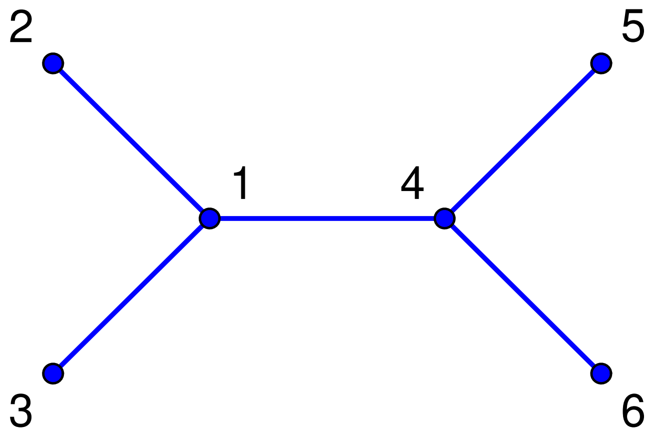
$$\partial_1(\overline{12} + \overline{23} - \overline{13}) = 0$$



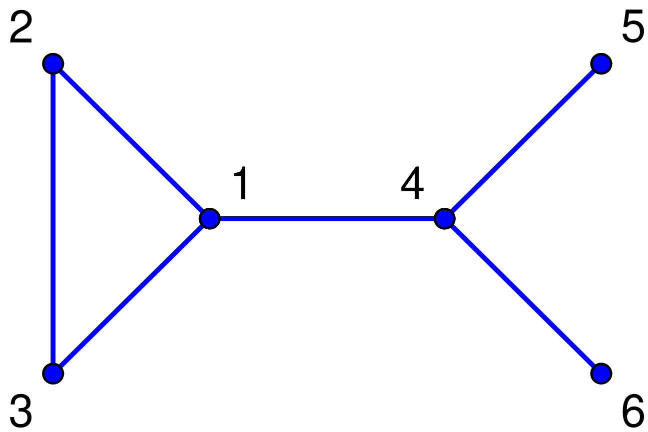
$$\operatorname{Ker} \partial_1 = \langle \overline{12} + \overline{23} - \overline{13},\ \overline{13} + \overline{34} - \overline{14},\ \overline{14} + \overline{42} - \overline{12} \rangle$$

$\dim H_1(K)$ counts number of (independent) loops

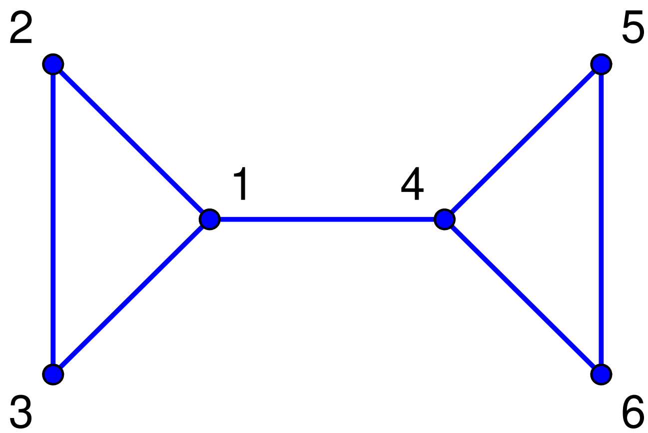
$\dim H_1(K) = 0$



$\dim H_1(K) = 1$



$\dim H_1(K) = 2$

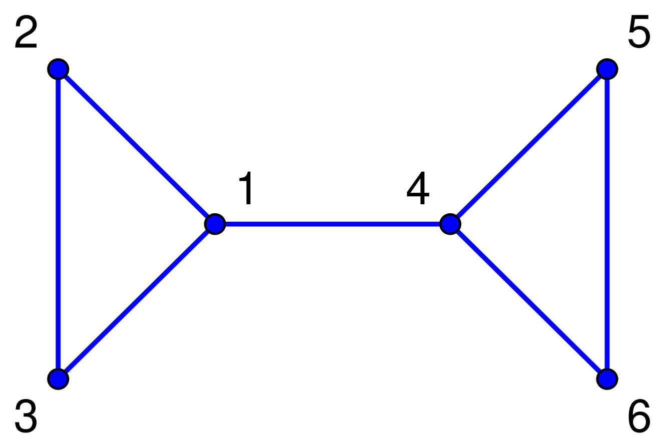


Betti numbers

$$\beta_0 = \dim H_0(K), \quad \beta_1 = \dim H_1(K).$$

Property: $\beta_0 - \beta_1 = \#V - \#E.$

$\dim H_1(K) = 2$



$$\#V = 6, \quad \#E = 7$$

$$\beta_0 = 1, \quad \beta_1 = 2$$

Homology of 2-Complexes

- K: 2-complex with vertices $V = \{v_1, \dots, v_n\}$, (oriented) edges $E = \{e_1, \dots, e_m\}$ and (oriented) faces $F = \{f_1, \dots, f_k\}$.

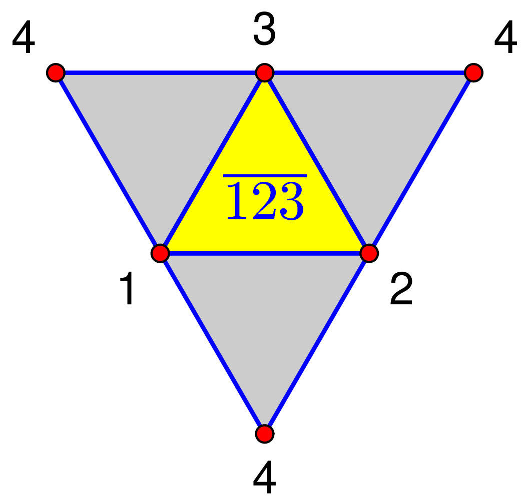
- $C_2(K)$: vector space of 2-chains, i.e. formal sums of the form

$$c_1\overline{f_1} + \cdots + c_k\overline{f_k}, \quad c_i \in \mathbb{R}.$$

- Boundary operator $\partial_2 : C_2(K) \to C_1(K)$ defined by

$$\partial_1(\overline{v_i v_j v_k}) = \overline{v_j v_k} - \overline{v_i v_k} + \overline{v_i v_j}, \quad \text{if } \overline{v_i v_j v_k} \in F.$$

Example: Tetrahedron



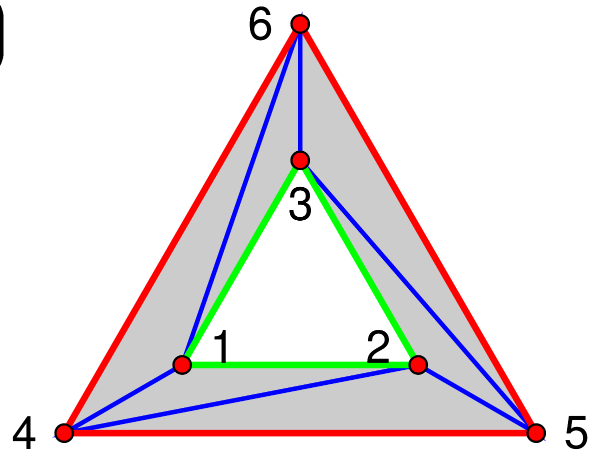
$$\partial_2(\overline{123}) = \overline{23} - \overline{13} + \overline{12}, \;\; \partial_1\partial_2(\overline{123}) = 0.$$

2-Homology

$$C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K)$$

- $\partial_1\partial_2 = 0 \Rightarrow \operatorname{Im}\partial_2 \subset \operatorname{Ker}\partial_1$.
- $\alpha \in C_1(K)$ is a 1-cycle if $\alpha \in \operatorname{Ker}\partial_1$ (i.e., $\partial_1(\alpha) = 0$)
- Two 1-cycles α, β are called 1-homologous if there is a 2-chain $\gamma \in C_2(K)$ with $\partial_2(\gamma) = \alpha - \beta$.

Example: Annulus



$\alpha = \overline{45} + \overline{56} - \overline{46}$

$\beta = \overline{12} + \overline{23} - \overline{13}$

Then $\alpha - \beta = \partial_2(\gamma)$, with

$\gamma = \overline{142} + \overline{245} + \overline{253} + \overline{356} + \overline{136} + \overline{164}$.

$$\boxed{C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K)}$$

- 1-Homology is equivalence relation on set of 1-cycles ($\mathrm{Ker}\,\partial_1$).
  Equivalence classes form 1st homology group:

$$H_1(K) = {\mathrm{Ker}\,\partial_1} \big/ {\mathrm{Im}\,\partial_2}$$

- Other homology groups:
  - $H_2(K) = \mathrm{Ker}\,\partial_2$
  - $H_0(K) = {C_0(K)} \big/ {\mathrm{Im}\,\partial_1}$

- Betti-numbers:

$$\beta_i = \dim H_i(K), \quad i = 0, 1, 2$$

  (number of 'i-dimensional holes')

- Euler characteristic: $\chi(K) = \#V - \#E + \#F$.

Properties of Betti numbers

- Betti numbers are topological invariants (independent of triangulation)

- $\chi(K) = \beta_0 - \beta_1 + \beta_2$

Theorem

$$\chi(K) = \sum_{i=0}^{d} (-1)^i \beta_i(K)$$

Lemma. If $\eta : V \to W$ is a linear map between (finite dimensional) vector spaces, then

$$\dim \operatorname{Ker} \eta = \dim V - \dim \operatorname{Im} \eta$$

Proof of Theorem

$$\ldots \xrightarrow{\partial_{i+1}} C_i(K) \xrightarrow{\partial_i} C_{i-1}(K) \xrightarrow{\partial_{i-1}} \ldots$$

$$H_i(K) = {\operatorname{Ker} \partial_i} \Big/ {\operatorname{Im} \partial_{i+1}}$$

so

$$\begin{aligned} \beta_i(K) &= \dim H_i(K) \\ &= \dim \operatorname{Ker} \partial_i - \dim \operatorname{Im} \partial_{i+1} \\ &= \dim C_i(K) - \dim \operatorname{Im} \partial_i - \dim \operatorname{Im} \partial_{i+1} \end{aligned}$$

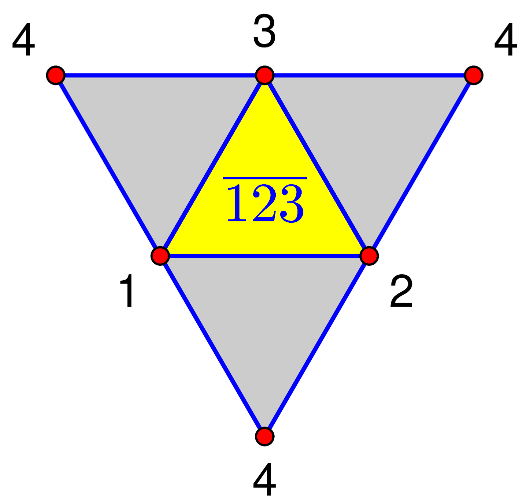
Now:

$$\sum_{i=0}^{d}(-1)^i\left(\dim\operatorname{Im}\partial_i + \dim\operatorname{Im}\partial_{i+1}\right) = 0$$

Hence:

$$\begin{aligned}\sum_{i=0}^{d}(-1)^i\beta_i(K) &= \sum_{i=0}^{d}(-1)^i\dim C_i(K)\\ &= \sum_{i=0}^{d}(-1)^i|S_i|\\ &= \chi(S)\end{aligned}$$

Example: Tetrahedron



∂_1	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{23}$	$\overline{24}$	$\overline{34}$
$\overline{1}$	-1	-1	-1	0	0	0
$\overline{2}$	1	0	0	-1	-1	0
$\overline{3}$	0	1	0	1	0	-1
$\overline{4}$	0	0	1	0	1	1

$\dim C_0(K) = 4$

$\dim \mathrm{Im}\, \partial_1 = 3$

$\dim \mathrm{Ker}\, \partial_1 = 3$

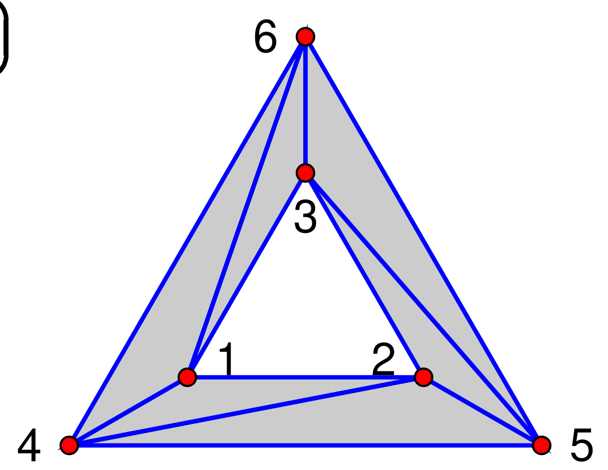
∂_2	$\overline{123}$	$\overline{134}$	$\overline{142}$	$\overline{243}$
$\overline{12}$	1	0	-1	0
$\overline{13}$	-1	1	0	0
$\overline{14}$	0	-1	1	0
$\overline{23}$	1	0	0	-1
$\overline{24}$	0	0	-1	1
$\overline{34}$	0	1	0	-1

$\dim \mathrm{Im}\, \partial_2 = 3$

$\dim \mathrm{Ker}\, \partial_2 = 1$

Tetrahedron: $\beta_0 = 1$, $\beta_1 = 0$, $\beta_2 = 1$

Example: Annulus



$\dim C_0(K) = 6, \quad \dim \operatorname{Im} \partial_1 = 5, \quad \dim \operatorname{Ker} \partial_1 = 7$

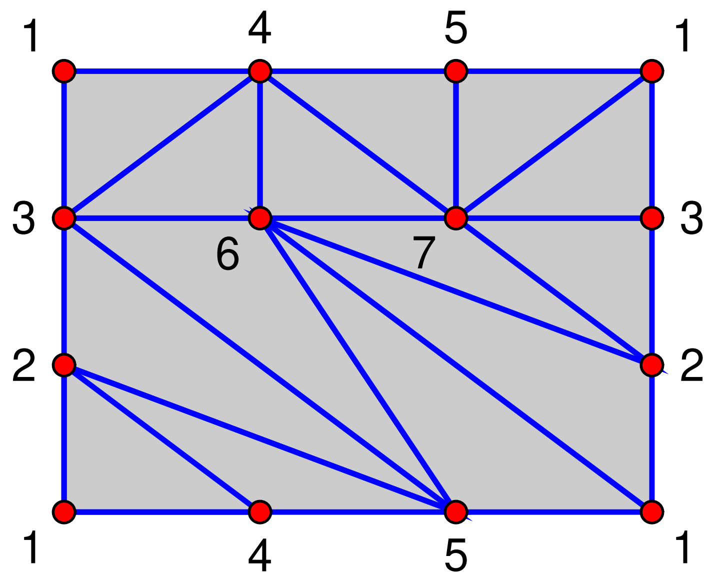
∂_1	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{16}$	$\overline{23}$	$\overline{24}$	$\overline{25}$	$\overline{35}$	$\overline{36}$	$\overline{45}$	$\overline{46}$	$\overline{56}$
$\overline{1}$	-1	-1	-1	-1	0	0	0	0	0	0	0	0
$\overline{2}$	1	0	0	0	-1	-1	-1	0	0	0	0	0
$\overline{3}$	0	1	0	0	1	0	0	-1	-1	0	0	0
$\overline{4}$	0	0	1	0	0	1	0	0	0	-1	-1	0
$\overline{5}$	0	0	0	0	0	0	1	1	0	1	0	-1
$\overline{6}$	0	0	0	1	0	0	0	0	1	0	1	1

$\dim \operatorname{Im} \partial_2 = 6, \quad \dim \operatorname{Ker} \partial_2 = 0$

∂_2	$\overline{142}$	$\overline{245}$	$\overline{253}$	$\overline{356}$	$\overline{136}$	$\overline{164}$
$\overline{12}$	1	0	0	0	0	0
$\overline{13}$	0	0	0	0	1	0
$\overline{14}$	1	0	0	0	0	-1
$\overline{16}$	0	0	0	0	-1	1
$\overline{23}$	0	0	-1	0	0	0
$\overline{24}$	-1	1	0	0	0	0
$\overline{25}$	0	-1	1	0	0	0
$\overline{35}$	0	0	-1	1	0	0
$\overline{36}$	0	0	0	-1	1	0
$\overline{45}$	0	1	0	0	0	0
$\overline{46}$	0	0	0	0	0	-1
$\overline{56}$	0	0	0	1	0	0

Annulus: $\beta_0 = 1, \quad \beta_1 = 1, \quad \beta_2 = 0$

Example: Torus



14 Faces:

$\overline{142}, \overline{245}, \overline{253}, \overline{356}, \overline{165}, \overline{126}, \overline{276},$
$\overline{237}, \overline{173}, \overline{157}, \overline{475}, \overline{467}, \overline{134}, \overline{364}$

21 Edges:

$\overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}, \overline{23},$
$\overline{24}, \overline{25}, \overline{26}, \overline{27}, \overline{34}, \overline{35}, \overline{36},$
$\overline{37}, \overline{45}, \overline{46}, \overline{47}, \overline{56}, \overline{57}, \overline{67}$

7 Vertices:

$\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}$

Torus: $\beta_0 = 1, \quad \beta_1 = 2, \quad \beta_2 = 1$

∂_2	$\overline{142}$	$\overline{245}$	$\overline{253}$	$\overline{356}$	$\overline{165}$	$\overline{126}$	$\overline{276}$	$\overline{237}$	$\overline{173}$	$\overline{157}$	$\overline{475}$	$\overline{467}$	$\overline{134}$	$\overline{364}$
$\overline{12}$	1	0	0	0	0	1	0	0	0	0	0	0	0	0
$\overline{13}$	0	0	0	0	0	0	0	0	-1	0	0	0	1	0
$\overline{14}$	1	0	0	0	0	0	0	0	0	0	0	0	-1	0
$\overline{15}$	0	0	0	0	-1	0	0	0	0	1	0	0	0	0
$\overline{16}$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$\overline{17}$	0	0	0	0	0	0	0	0	1	-1	0	0	0	0
$\overline{23}$	0	0	-1	0	0	0	0	1	0	0	0	0	0	0
$\overline{24}$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
$\overline{25}$	0	-1	1	0	0	0	0	0	0	0	0	0	0	0
$\overline{26}$	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
$\overline{27}$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$\overline{34}$	0	0	0	0	0	0	0	0	0	0	0	0	1	-1
$\overline{35}$	0	0	-1	1	0	0	0	0	0	0	0	0	0	0
$\overline{36}$	0	0	0	-1	0	0	0	0	0	0	0	0	0	1
$\overline{37}$	0	0	0	0	0	0	0	1	-1	0	0	0	0	0
$\overline{45}$	0	1	0	0	0	0	0	0	0	0	-1	0	0	0
$\overline{46}$	0	0	0	0	0	0	0	0	0	0	0	1	0	-1
$\overline{47}$	0	0	0	0	0	0	0	0	0	0	1	-1	0	0
$\overline{56}$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$\overline{57}$	0	0	0	0	0	0	0	0	0	1	-1	0	0	0
$\overline{67}$	0	0	0	0	0	0	-1	0	0	0	0	1	0	0

Computing Betti Numbers for complexes with n cells

- *Donald & Chang, 1991*: probabilistic algorithms for sparse matrix methods.

  $O(n^2)$ expected time (under certain restrictions).

- *Delfinado & Edelsbrunner, 1995*: Computation of Betti numbers for complexes embedded in $\mathbb{R}^3$.

  $O(n\alpha(n))$ time.

  Incremental: maintain Betti numbers, add one cell at a time (uses Mayer-Vietoris property from Homology theory).

- *Friedman, 1996*: Betti numbers via combinatorial Laplacian (based on Hodge theory of harmonic forms).

  Complexity? (seems to work well in general).

Incremental computation of Betti numbers

Situation: Attach k-dimensional simplex σ to simplicial complex K.

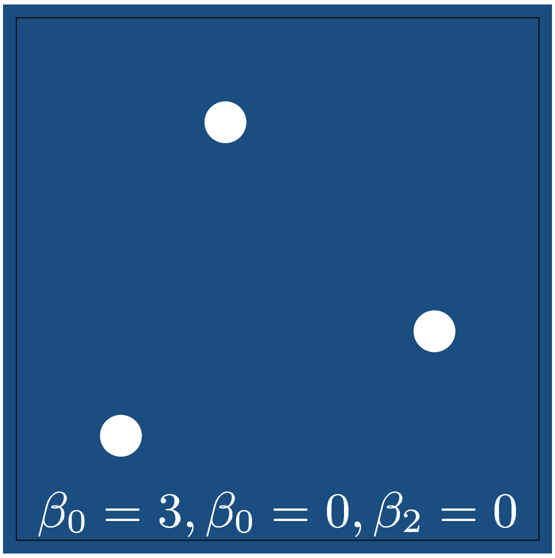
- $\gamma = \partial_k \sigma$
- $K' = K \cup \sigma$

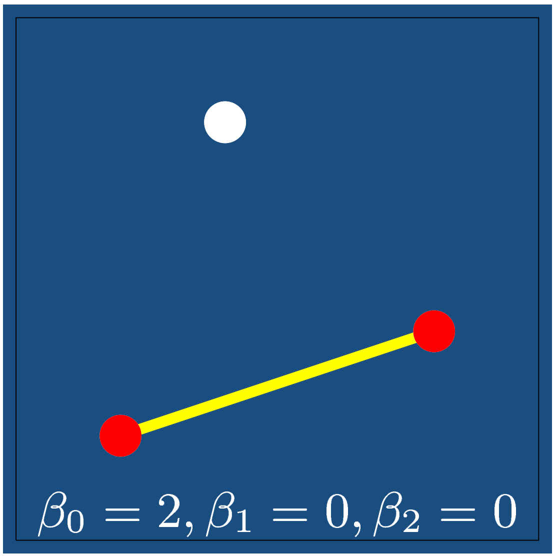
Case 1: γ bounds in K: $\partial'_k \sigma \in \operatorname{Im} \partial_k$

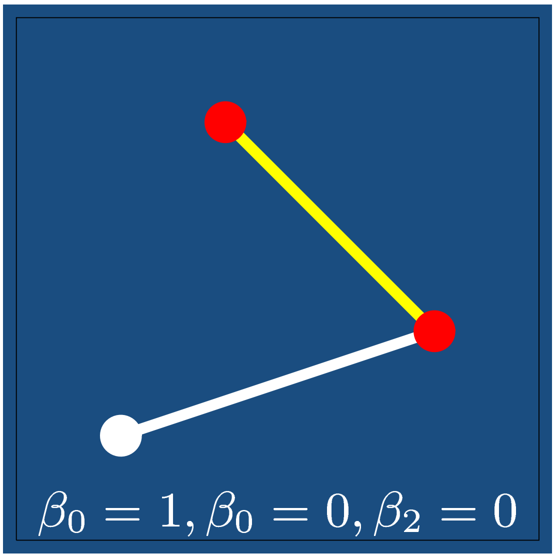
$$\beta_i(K') = \begin{cases} \beta_i(K) & \text{if } i \neq k \\ \beta_k(K) + 1 & \text{if } i = k \end{cases}$$

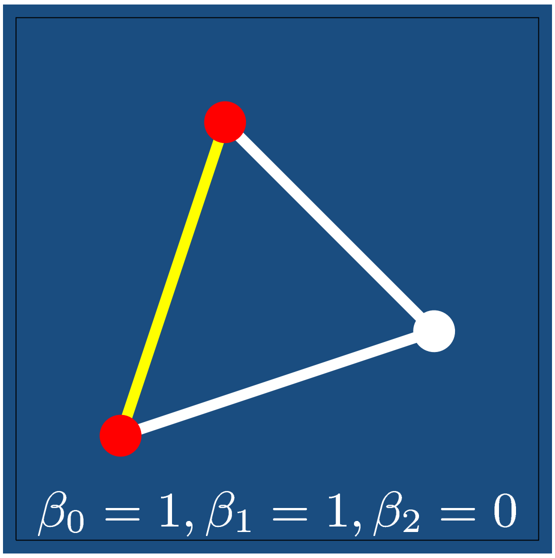
Case 2: γ does not bound in K: $\partial'_k \sigma \notin \operatorname{Im} \partial_k$

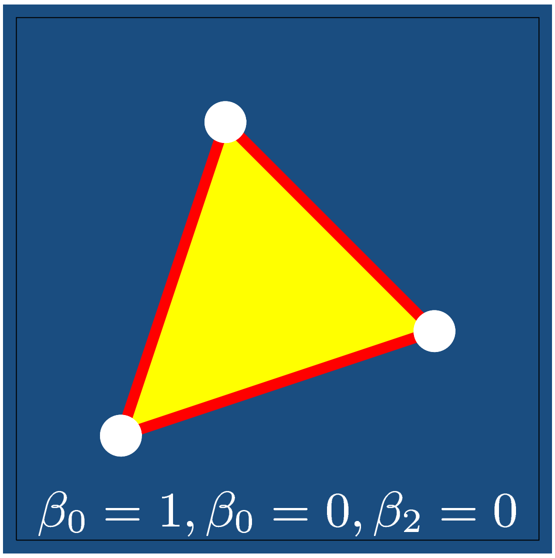
$$\beta_i(K') = \begin{cases} \beta_i(K) & \text{if } i \neq k-1 \\ \beta_{k-1}(K) - 1 & \text{if } i = k-1 \end{cases}$$


$\beta_0 = 3, \beta_0 = 0, \beta_2 = 0$


$\beta_0 = 2, \beta_1 = 0, \beta_2 = 0$


$\beta_0 = 1, \beta_0 = 0, \beta_2 = 0$


$\beta_0 = 1, \beta_1 = 1, \beta_2 = 0$


$\beta_0 = 1, \beta_0 = 0, \beta_2 = 0$

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial'_{k+1}} & C_k(K') & \xrightarrow{\partial'_k} & C_{k-1}(K') & \xrightarrow{\partial'_{k-1}} & \cdots \\ & & \| & & \| & & \\ & & C_k(K) \oplus \mathbb{R}[\sigma] & & C_{k-1}(K) & & \end{array}$$

Case 1: $\partial'_k \sigma \in \operatorname{Im} \partial_k$

Then $\operatorname{Im} \partial'_i = \operatorname{Im} \partial_i$, for all i, so:

$$\dim \operatorname{Im} \partial'_i = \dim \operatorname{Im} \partial_i, \text{ for all } i.$$

Therefore:

$$\begin{aligned} \dim \operatorname{Ker} \partial'_k &= \dim C_k(K') - \dim \operatorname{Im} \partial'_k \\ &= 1 + \dim C_k(K) - \dim \operatorname{Im} \partial_k \\ &= 1 + \dim \operatorname{Ker} \partial_k \end{aligned}$$

Furthermore, for $i \neq k$:

$$\dim \operatorname{Ker} \partial'_i = \dim \operatorname{Ker} \partial_i$$

Hence (recall $\dim H_i(K') = \dim \operatorname{Ker} \partial'_i - \dim \operatorname{Im} \partial'_{i+1}$):

$$\dim H_i(K') = \begin{cases} \dim H_i(K) & \text{if } i \neq k \\ 1 + \dim H_k(K) & \text{if } i = k \end{cases}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial'_{k+1}} & C_k(K') & \xrightarrow{\partial'_k} & C_{k-1}(K') & \xrightarrow{\partial'_{k-1}} & \cdots \\
 & & \| & & \| & & \\
 & & C_k(K) \oplus \mathbb{R}[\sigma] & & C_{k-1}(K) & &
\end{array}$$

Case 2: $\partial'_k \sigma \notin \operatorname{Im} \partial_k$

Then

$$\operatorname{Im} \partial'_i = \begin{cases} \operatorname{Im} \partial_i & \text{if } i \neq k \\ \operatorname{Im} \partial_k \oplus \mathbb{R}[\partial'_k \sigma] & \text{if } i = k. \end{cases}$$

$$\begin{aligned}
& \dim \operatorname{Ker} \partial'_i \\
= \; & \dim C_i(K') - \dim \operatorname{Im} \partial'_i \\
= \; & \begin{cases} \dim C_i(K) - \dim \operatorname{Im} \partial_i & \text{if } i \neq k \\ 1 + \dim C_k(K) - (1 + \dim \operatorname{Im} \partial_k) & \text{if } i = k \end{cases} \\
= \; & \dim \operatorname{Ker} \partial_i
\end{aligned}$$

Hence (recall $\dim H_i(K') = \dim \operatorname{Ker} \partial'_i - \dim \operatorname{Im} \partial'_{i+1}$):

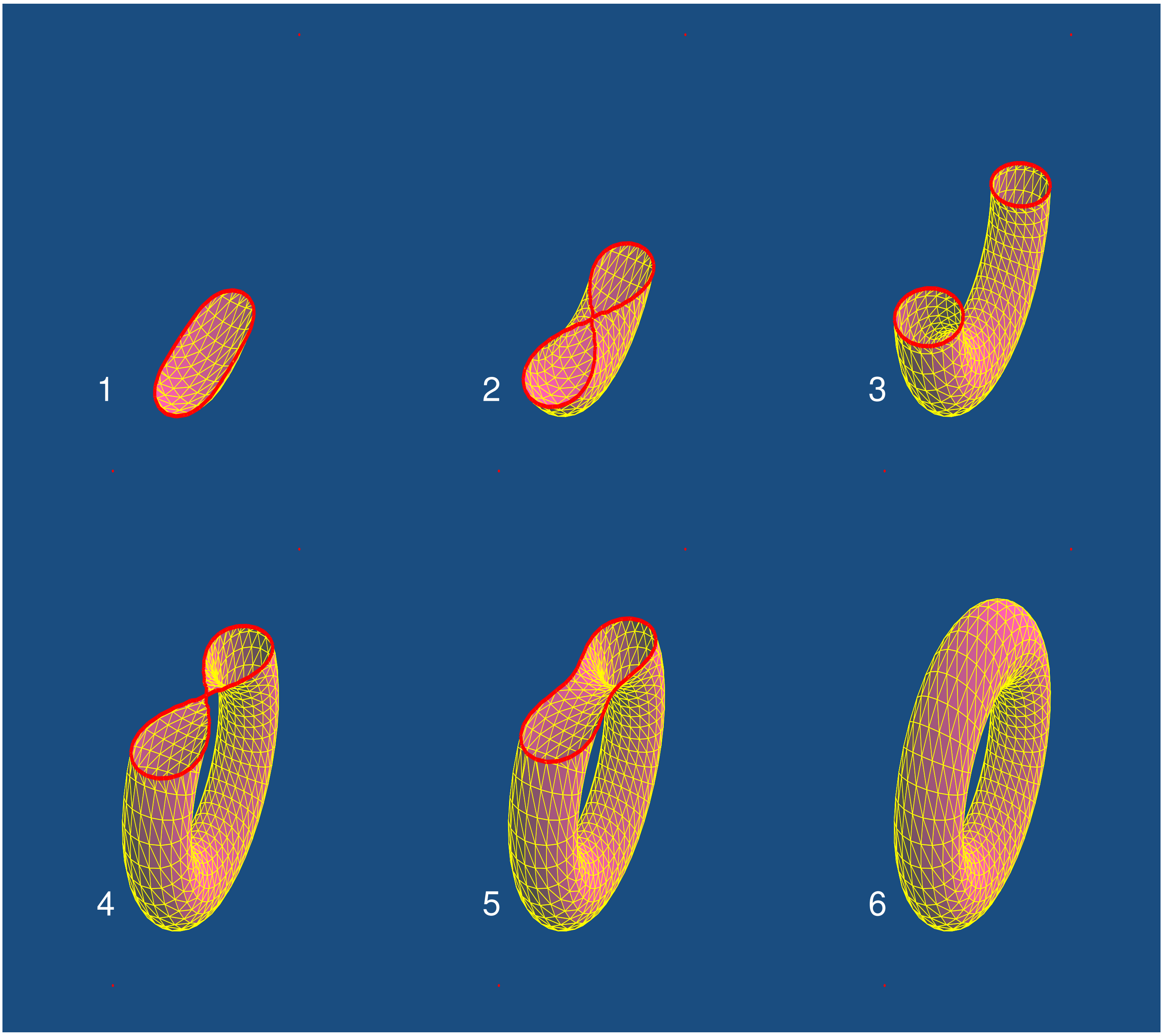
$$\dim H_i(K') = \begin{cases} \dim H_i(K) & \text{if } i \neq k-1 \\ \dim H_{k-1}(K) - 1 & \text{if } i = k-1 \end{cases}$$

Morse Theory

or

$$\chi(S) = \sum_{i=0}^{d} (-1)^i \nu_i(S)$$

Morse Theory: passing critical levels and attaching cells



Level sets for increasing values of the height function on the torus.

Questions:

1. Local form near singular points / singular level sets?

2. Changes of topology: explanation?

Morse Theory: Smooth surfaces

$S \subset \mathbb{R}^3$: smooth surface in $\mathbb{R}^3$.

Regular parametrization of S:

$$\Phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

such that

- $\Phi(u,v) \in S$, for all $(u,v) \in U$
- $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, such that

$$\begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 1: Sperical coordinates

S: sphere with radius R and center $(0,0,0) \in \mathbb{R}^3$;

Parameter domain

$U = \{(u,v) \mid 0 \leq u < 2\pi, \pi/2 < v < \pi/2\}$ Parametrization of $S \setminus \{(0,0,\pm 1)\}$:

$$\Phi(u,v) = (R\cos u \cos v, R\sin u \cos v, R\sin v),$$

Example 1a: Lower Hemisphere

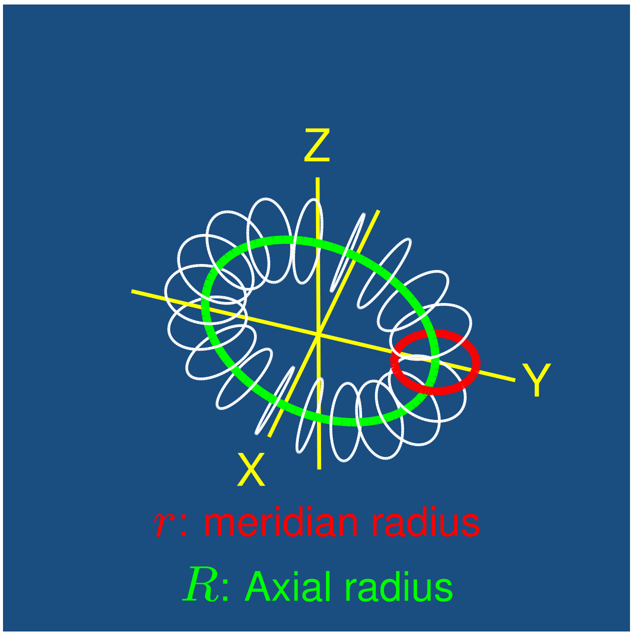
Parameter domain $U = \{(x,y) \mid x^2 + y^2 < 1\}$;

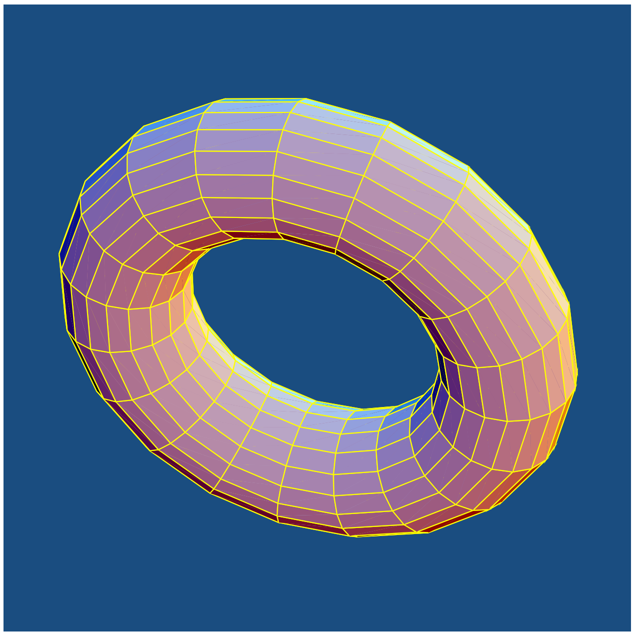
Parametrization $\Phi : U \to \mathbb{R}^3$ defined by:

$$\Phi(x,y) = (x, y, -\sqrt{1 - x^2 - y^2})$$

Example 2: Torus of revolution

Torus S obtained by rotating circle in x, y-plane with center $(0, R, 0)$ and radius r around x-axis.





Parameter domain $U = \{(u, v) \mid 0 \leq u, v < 2\pi\}$;
Parametrization $\Phi : U \to \mathbb{R}^3$ defined by:

$$\Phi(u, v) = (r \sin u, (R - r\cos u) \sin v, (R - r\cos u) \cos v).$$

Example 2a: Local form of torus of revolution

For (x, y, z) near $(0, 0, \pm(R - r))$:

$$z = \pm\sqrt{R^2 + r^2 - x^2 - y^2 - 2R\sqrt{r^2 - x^2}}.$$

Functions on surfaces

Height functions on surface S.

$h : S \to \mathbb{R}$ defined by: $h(x, y, z) = z,$ for $(x, y, z) \in S$.

C^k functions on surface S.

$f : S \to \mathbb{R}$ is C^k on S if for any regular parametrization $\Phi : U \to \mathbb{R}^3$ of S the function $f \circ \Phi : U \to \mathbb{R}$ is C^k.
(C^0: continuous; C^1: differentiable, continuous derivatives, . . .)

Height functions are C^k.

Example 1: Height function on the standard sphere.

1. Away from the poles $(0, 0, \pm 1)$:

$$h(\Phi(u, v)) = R \sin v$$

2. Near the poles $(0, 0, \pm 1)$:

$$h(\Phi_1(x, y)) = \pm\sqrt{1 - x^2 - y^2}$$

Example 2: Height function on the torus of revolution.

$$h(\Phi(u, v)) = (R - r \cos u) \cos v$$

Singularities of functions on surfaces

Point p on S is a singularity of $f : S \to \mathbb{R}$ if there is a regular parametrization $\Phi : U \to \mathbb{R}^3$ of S with

- $\Phi(u_0, v_0) = p$, for some $(u_0, v_0) \in U$
- $\tilde{f}(u, v) = f(\Phi(u, v))$ has a singularity at (u_0, v_0):

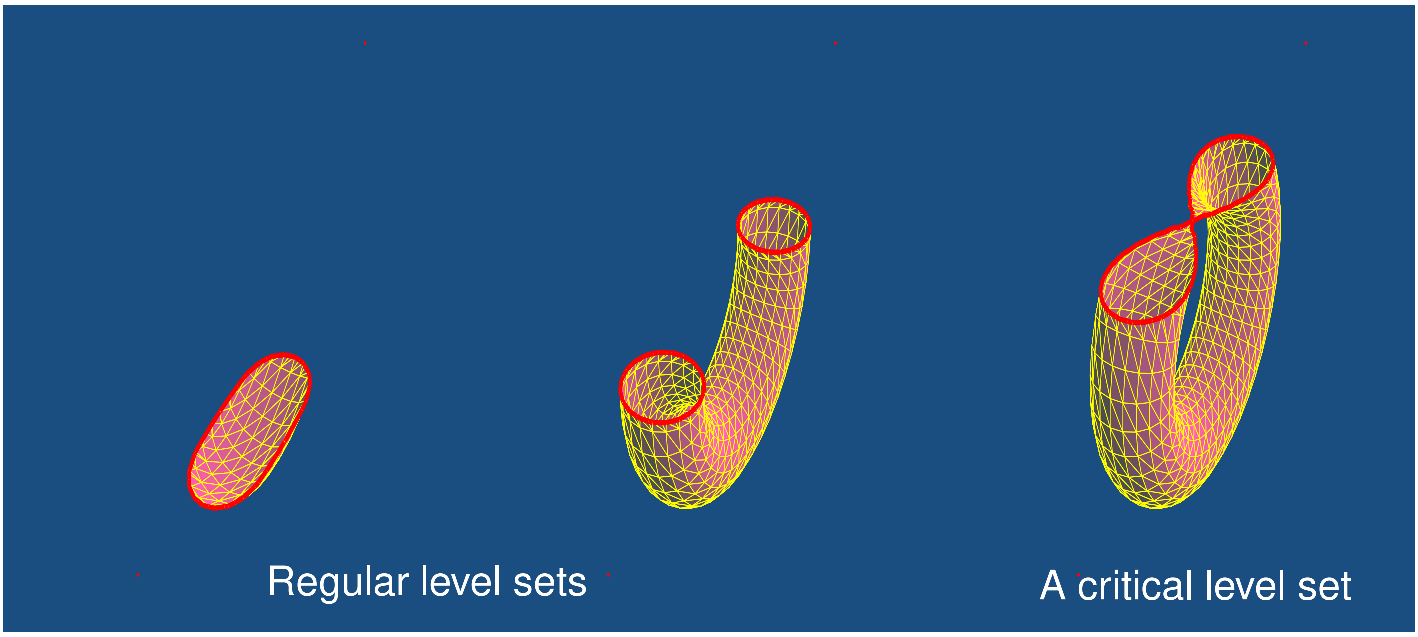
$$\frac{\partial \tilde{f}}{\partial u}(0,0) = \frac{\partial \tilde{f}}{\partial v}(0,0) = 0.$$

$c \in \mathbb{R}$ is a regular value of f if $f(p) \neq c$ for all singular points p.

Regular level sets.

If c is a regular value of $f : S \to \mathbb{R}$, then the *level set* $f^{-1}(c)$ is a regular curve on S.

(Proof: use Implicit Function Theorem.)



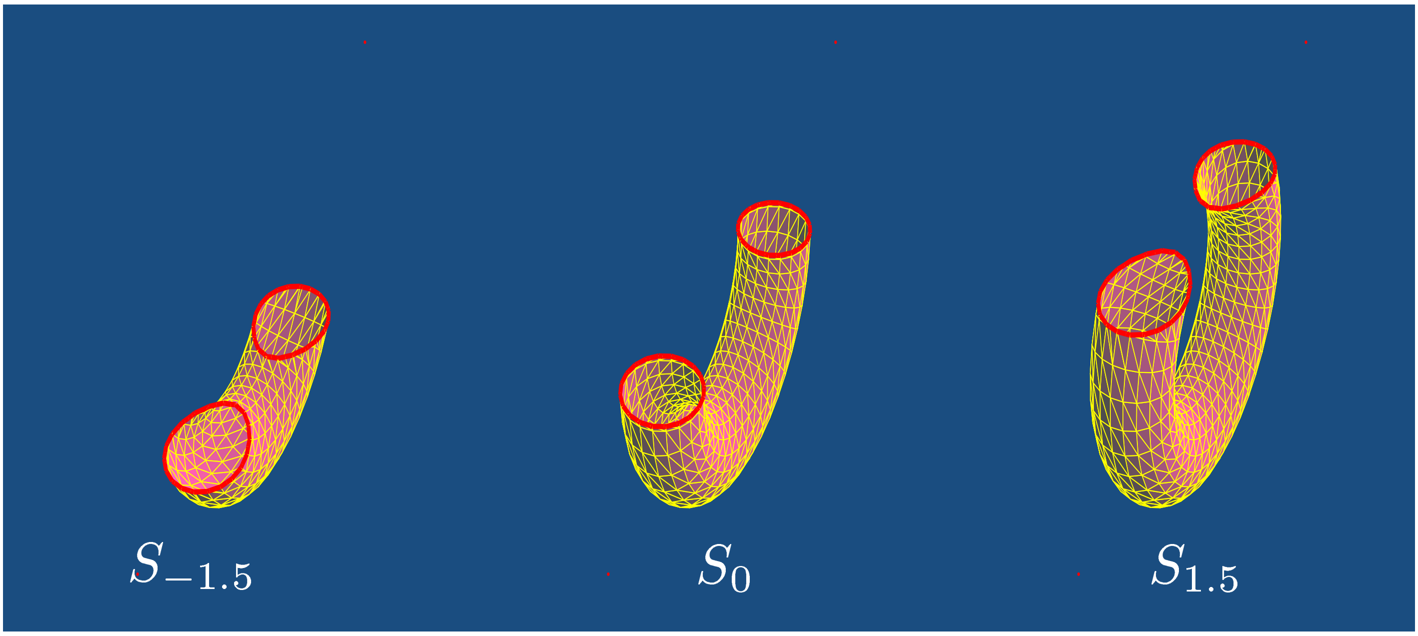
Regular level sets

For a smooth function $f : S \to \mathbb{R}$ and $c \in \mathbb{R}$

$$S_c = \{q \in S \mid f(q) \leq c\}$$

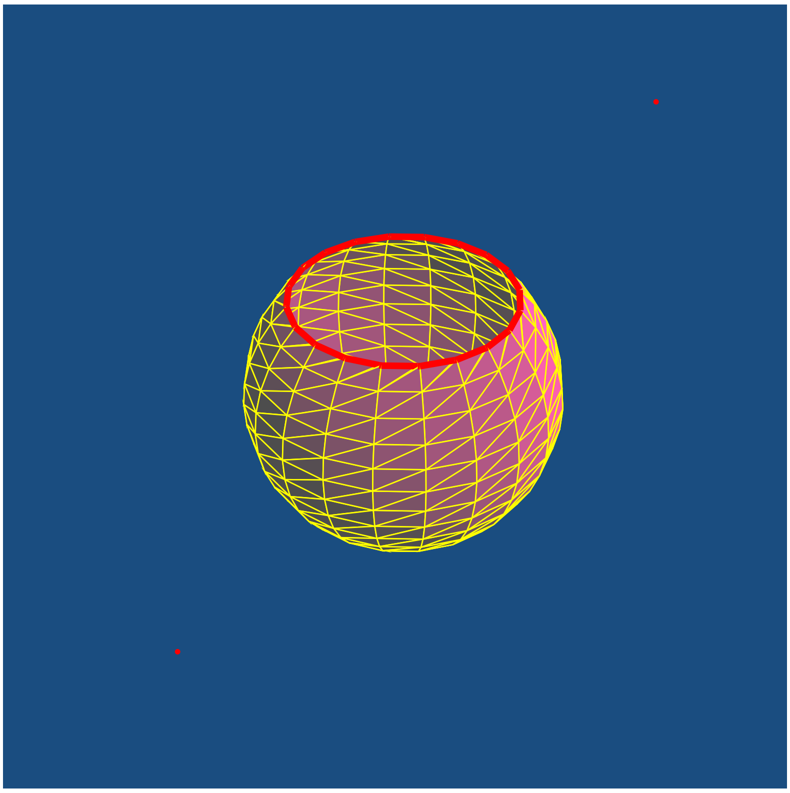
No change of topology near regular level sets.

If $a < b$ and f has no critical levels in $[a, b]$, then $S_a \sim S_b$ (topological equivalence).



Question: What happens upon passage of a critical level?

Singularities of height functions.



Example 1: Height function on the sphere.

1. Away from the poles $(0, 0, \pm 1)$:

$$\tilde{h}(u, v) = h(\Phi(u, v)) = R \sin v,$$

so no singular point (since $-\pi/2 < v < \pi/2$).

2. Near the poles $(0, 0, \pm 1)$:

$$\tilde{h}(x, y) = h(\Phi_1(x, y)) = \pm\sqrt{1 - x^2 - y^2}$$

so the singular points of h are $(0, 0, -1)$ (local minimum) and $(0, 0, 1)$ (local maximum).

Example 2: Height function on the torus of revolution.

$$\tilde{h}(u,v) = h(\Phi(u,v)) = (R - r\cos u)\cos v,$$

so singular points are

(u,v)	$\Phi(u,v)$	type of singularity
$(0,0)$	$(0,0,R-r)$	saddle point
$(0,\pi)$	$(0,0,-R+r)$	saddle point
$(\pi,0)$	$(0,0,R+r)$	maximum
(π,π)	$(0,0,-R-r)$	minimum

Local form of height function near $(0,0,\pm(R-r))$:

$$\begin{aligned} z &= \pm\sqrt{R^2+r^2-x^2-y^2-2R\sqrt{r^2-x^2}} \\ &= \pm\left(R-r-\frac{1}{2r}x^2+\frac{1}{2(R-r)}y^2\right) + \text{Higher Order Terms} \end{aligned}$$

Local form near $(0,0,\pm(R+r))$:

$$z = \pm\left(R+r-\frac{1}{2r}x^2-\frac{1}{2(R+r)}y^2\right) + \text{Higher Order Terms}$$

Morse functions on surfaces

A singular point p of $f : S \to \mathbb{R}$ is called non-degenerate if, for a regular parametrization $\Phi : U \to \mathbb{R}^3$ of f, with $p = \Phi(u_0, v_0)$, the function

$$\tilde{f}(u,v) = f(\Phi(u,v))$$

has a non-degenerate quadratic part at (u_0, v_0). A function is called a Morse function if all its singularities are non-degenerate.

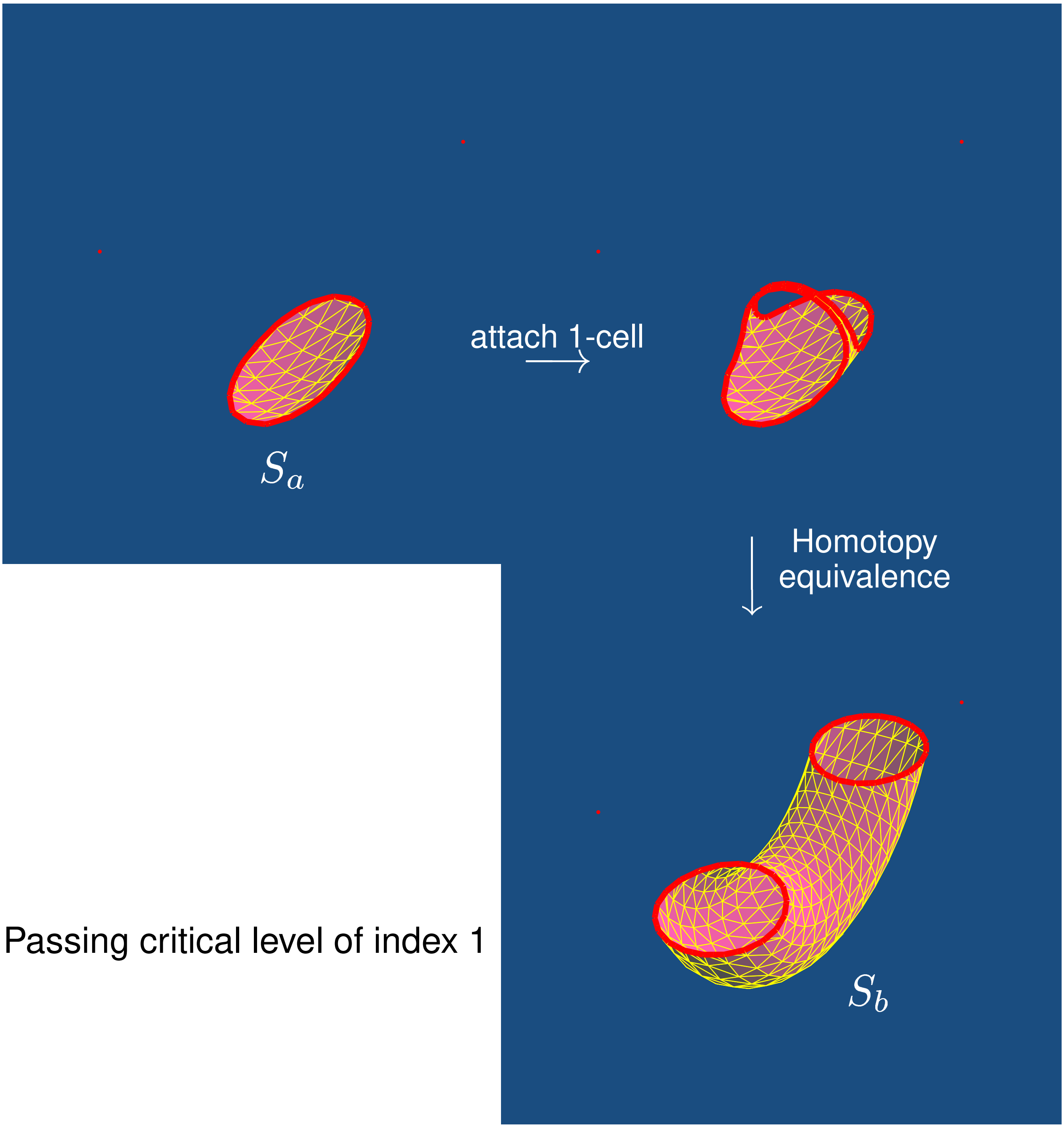
Example: The height functions on the standard 2-sphere and the standard torus of revolution are Morse functions.

Morse Lemma. If $f : S \to \mathbb{R}$ has a non-degenerate singularity at $p \in S$, then there is a regular parametrization $\Phi : U \to \mathbb{R}^3$ of S such that $\Phi(0,0) = p$, and $\tilde{f}(u,v) = f(\Phi(u,v))$ is of one of the following forms for (u,v) near $(0,0)$:

Normal form	type	index
$f(p) + u^2 + v^2$	local minimum	0
$f(p) + u^2 - v^2$	saddle point	1
$f(p) - u^2 - v^2$	local maximum	2

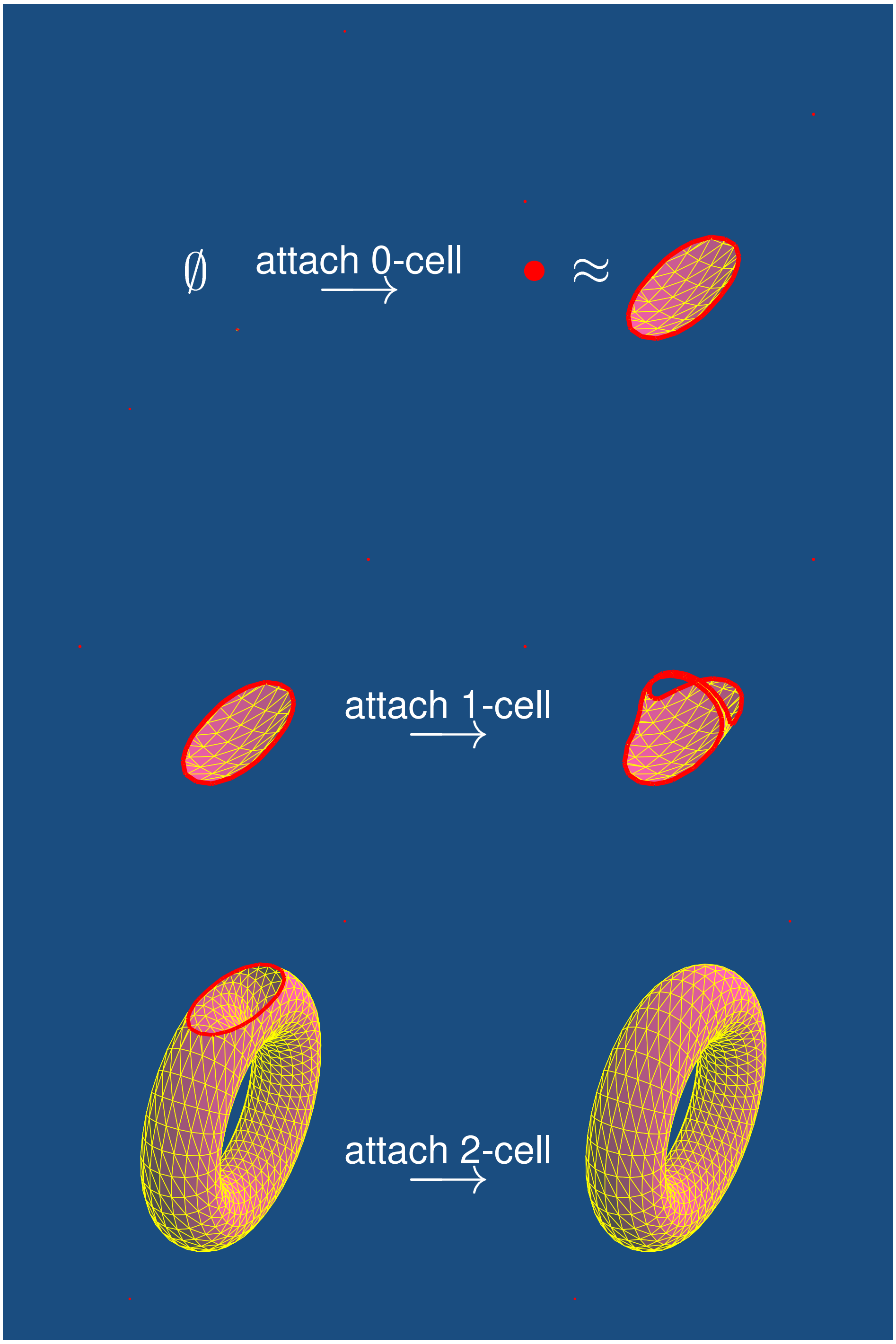
Passing a critical level of index k:

$f : S \to \mathbb{R}$ is a Morse function with one critical point in $f^{-1}[a, b]$ of index k. Then S_b is homotopy-equivalent to S_a with a cell σ of dimension k attached (in particular $\partial\sigma \subset S_a$).

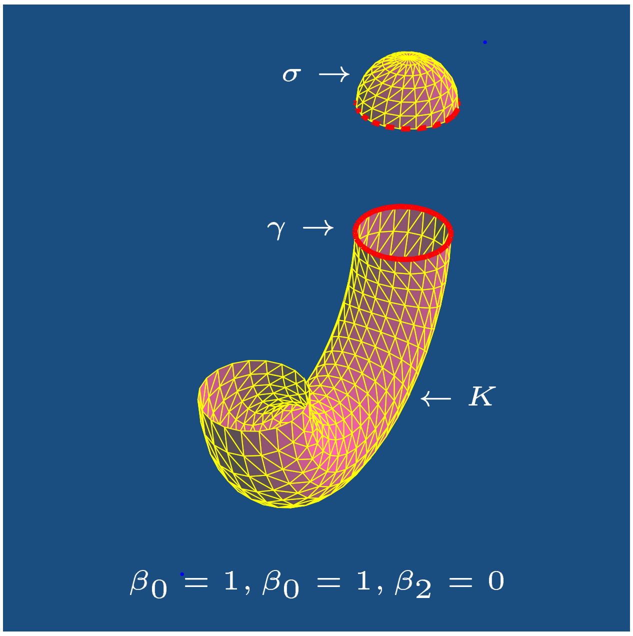


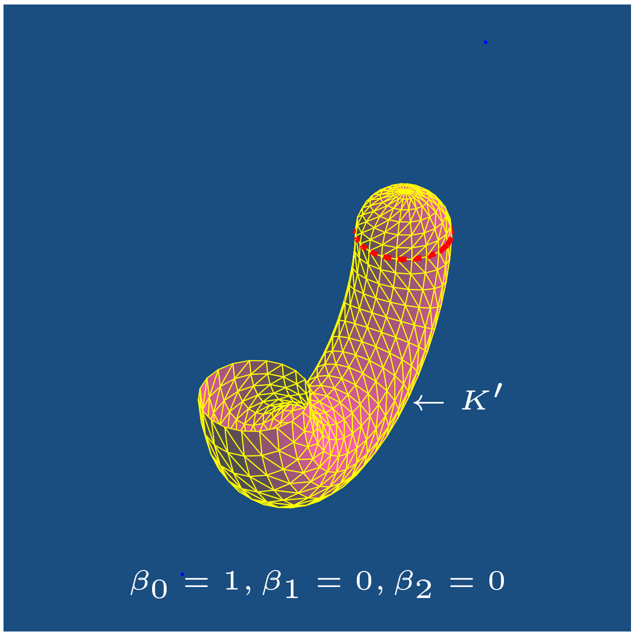
Passing critical level of index 1

Morse Theory: passing critical level



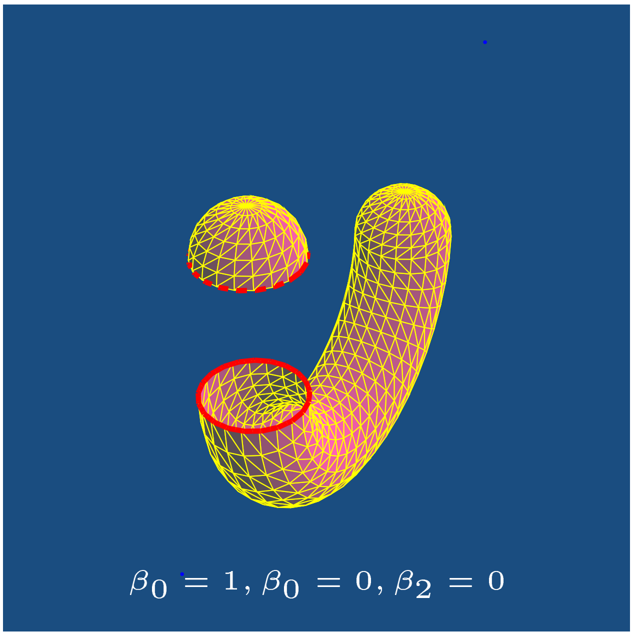
Morse Theory and Homology

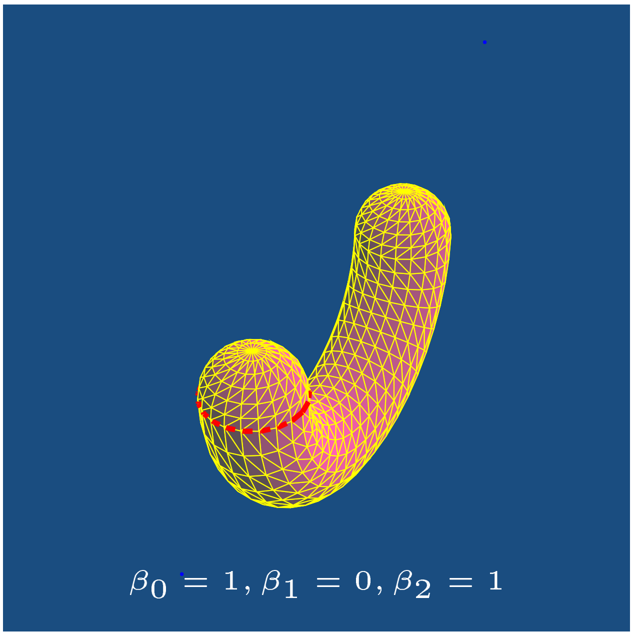




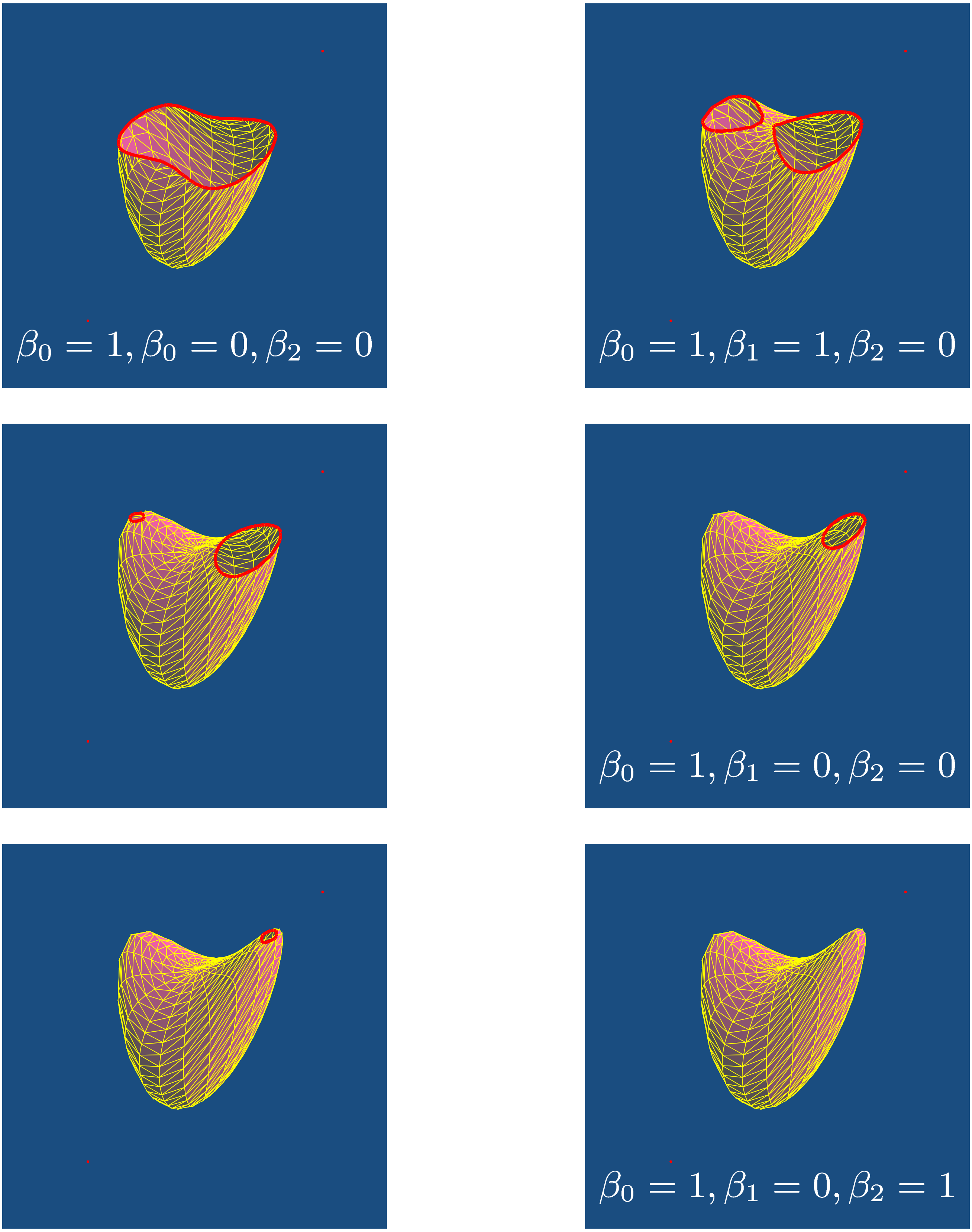
- $K' = K \cup_\gamma \sigma$
- $\gamma = \partial_2 \sigma$ does not bound in K, hence: $\beta_1 := \beta_1 - 1$.

Morse Theory and Homology

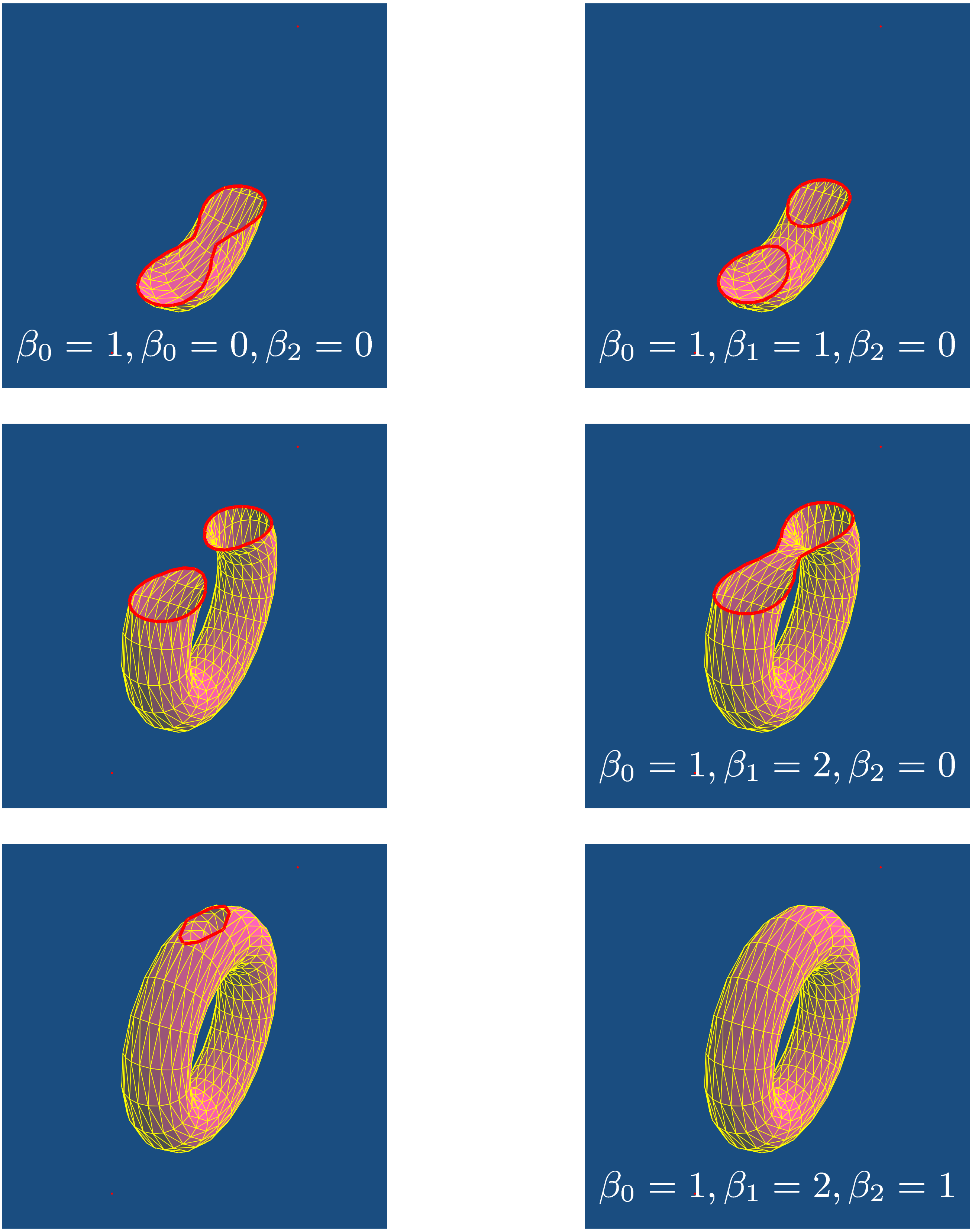




$\partial\sigma$ bounds, hence: $\beta_2 := \beta_2 + 1$.



Homology changes upon passage of a critical level



Homology changes upon passage of a critical level

Morse Inequalities

Observation: Upon passage of a critical level of index k either $\beta_k := \beta_k + 1$, or $\beta_{k-1} := \beta_{k-1} - 1$.

Corollary (Morse Inequalities)

For a Morse function $f : S \to \mathbb{R}$ with ν_k critical points of index k, $0 \leq k \leq d \quad$ (d: dimension, so $d = 2$):

1. $\nu_k \geq \beta_k$
2. $\sum_{i=k}^{d} (-1)^k \nu_k = \sum_{i=k}^{d} (-1)^k \beta_k$

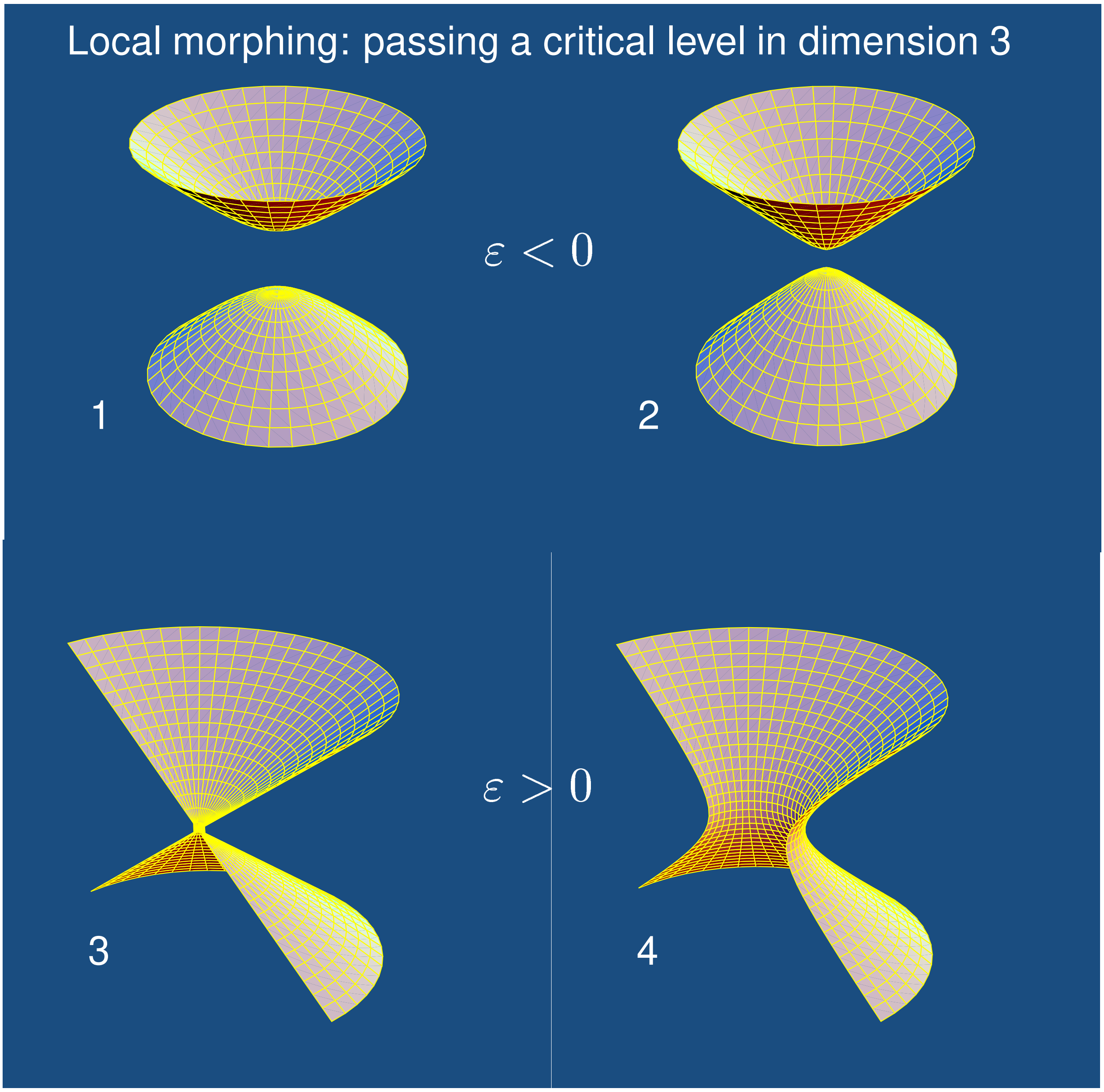
Corollary (Euler characteristic and Morse functions)

For a Morse function $f : S \to \mathbb{R}$ with ν_k critical points of index k, $0 \leq k \leq d$:

$$\chi(S) = \sum_{i=k}^{d} (-1)^k \nu_k$$

Local form of non-degenerate singularities in 3D:

$$f(x, y, z) = f(p) \pm x^2 \pm y^2 \pm z^2$$



Example: $f(x, y, z) = f(p) + x^2 + y^2 - z^2$

Morphing: passing a critical level in dimension 3