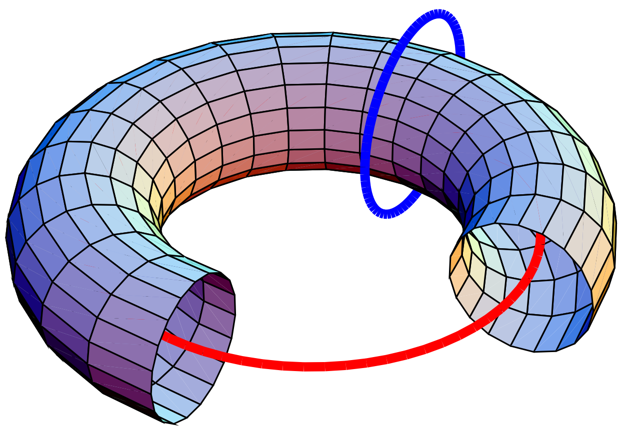
Computational Topology: some snapshots



Gert Vegter

University of Groningen, NL

URL: `http://www.cs.rug.nl/~gert`

Topics in Computational Topology

- Cutting and pasting of triangulated surfaces

  [Vegter& Yap (1989), Dey (1994)]

- Homotopy of closed curves on triangulated surfaces

  [Dey& Schipper (1995)]

- Regular homotopy of closed curves in the plane

  [Whitney & Graustein (1936), Mehlhorn & Yap (1988), Vegter (1989)]

- Realizability of orientable triangulated surfaces in $\mathbb{R}^3$

  [Bokowski&Sturmfels, Richter-Gebert, Brehm, . . .]

- Knot theory: trivial knot problem

  [Haken (1961): decidable problem]

- Computing homology groups

  [Donald& Chang (1991), Delfinado & Edelsbrunner (1995), Friedman (1996)]

Contents

1. Curves
2. Combinatorial surfaces, or:

$$\chi(S) = \sum_{\sigma \text{ simplex of } S} (-1)^{\dim \sigma}$$

   but also

$$\chi(S) = \frac{1}{2\pi} \sum_{v \in V} K(v)$$

3. Complexity of triangulations
4. Homology Theory, or:

$$\chi(S) = \sum_{i=0}^{d} (-1)^i \beta_i(S)$$

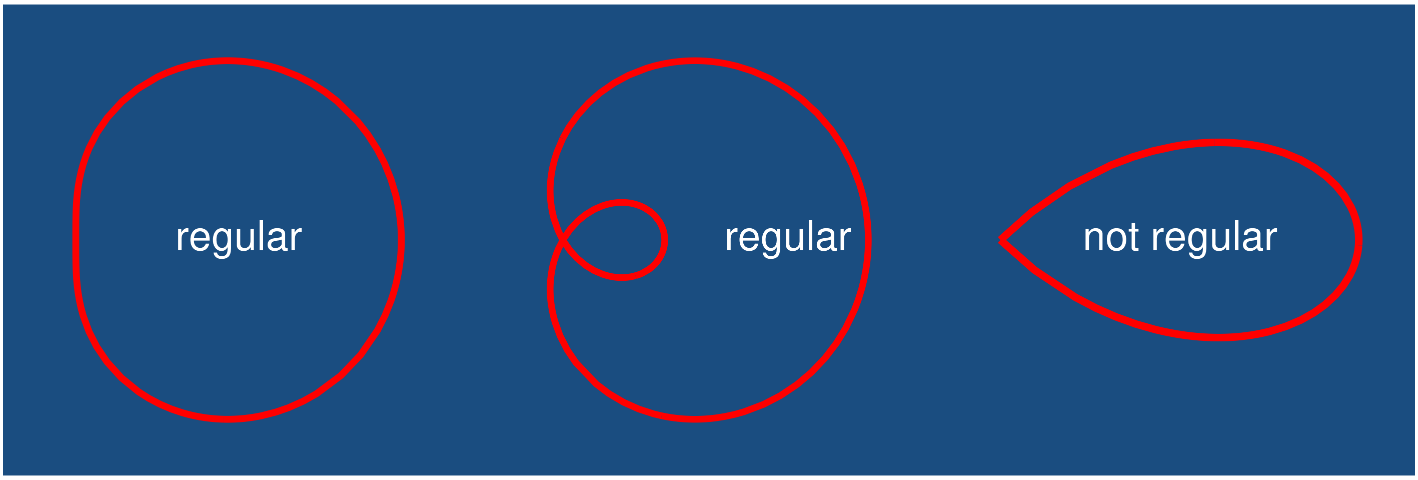
5. Morse Theory, or:

$$\chi(S) = \sum_{i=0}^{d} (-1)^i \nu_i(S)$$

Regular Closed Curves

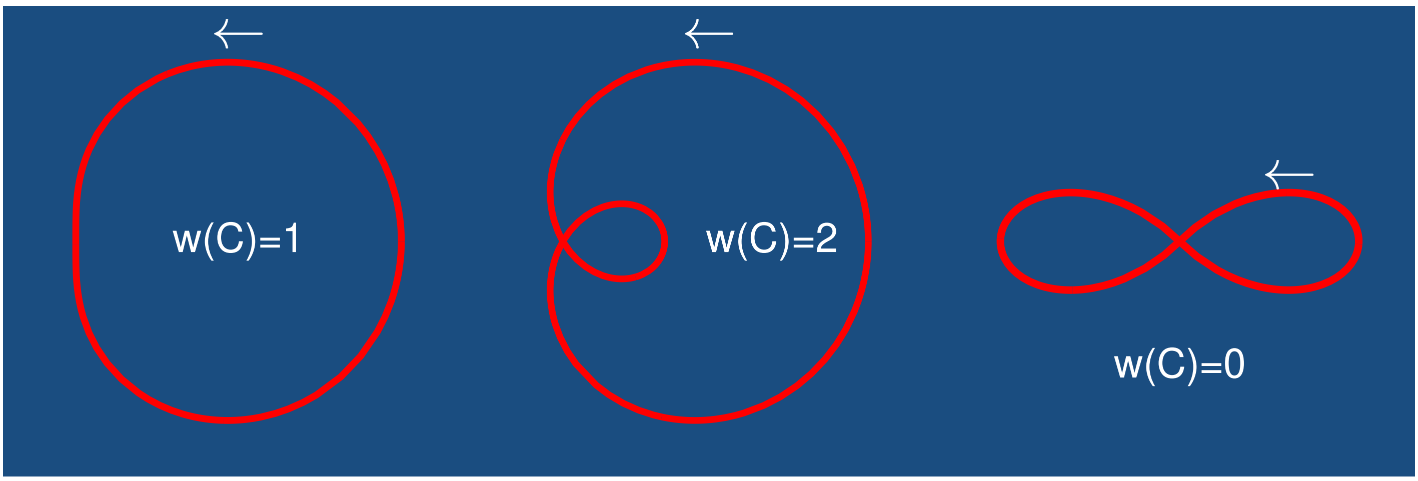
Regular closed curve:

$$f : \mathbb{S}^1 \to \mathbb{R}^2, \text{ with } f'(t) \neq 0.$$



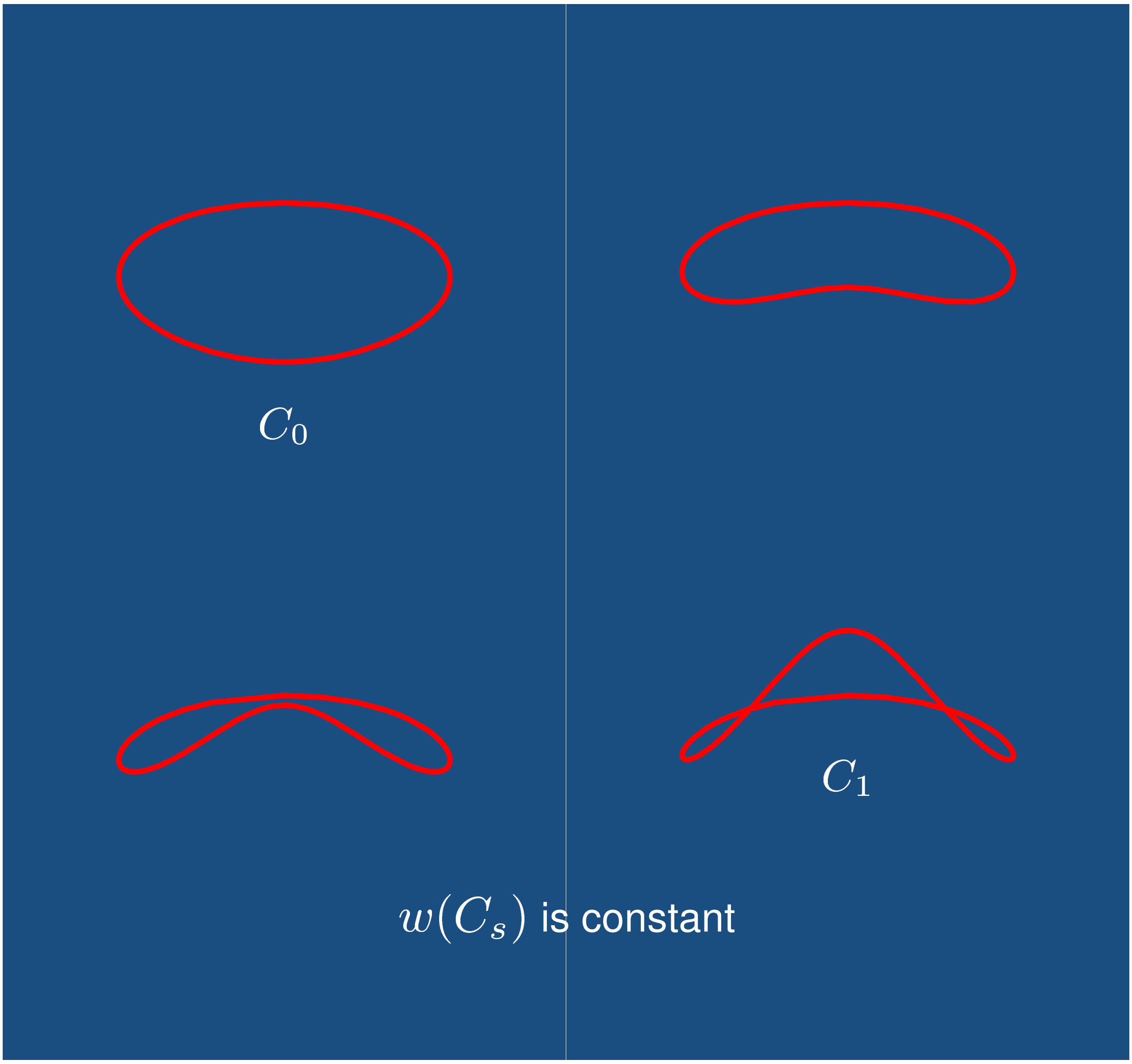
Winding Number w(C) of regular closed curve C:

number of turns of tangent vector.



Regular Deformations

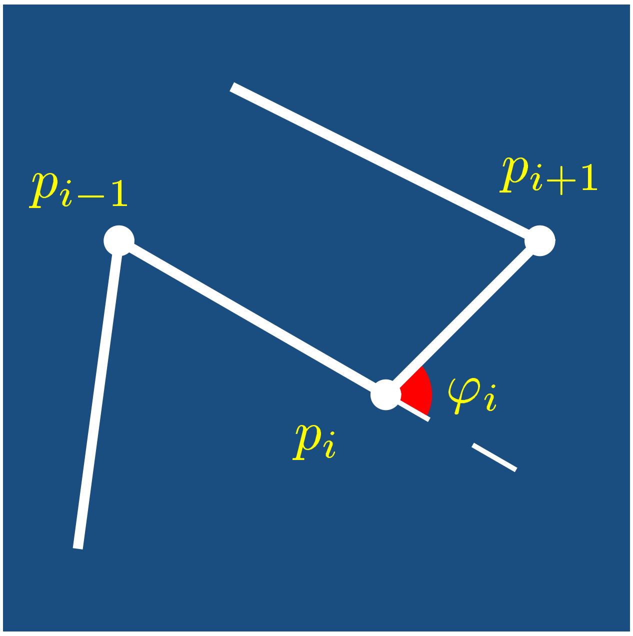
$C \sim C'$ iff there is a continuous family C_s, $0 \leq s \leq 1$, of regular closed curves with $C_0 = C$ and $C_1 = C'$.



Theorem (Whitney & Graustein, 1937)

$$C \sim C' \Leftrightarrow w(C) = w(C').$$

Equivalence of regular polygons

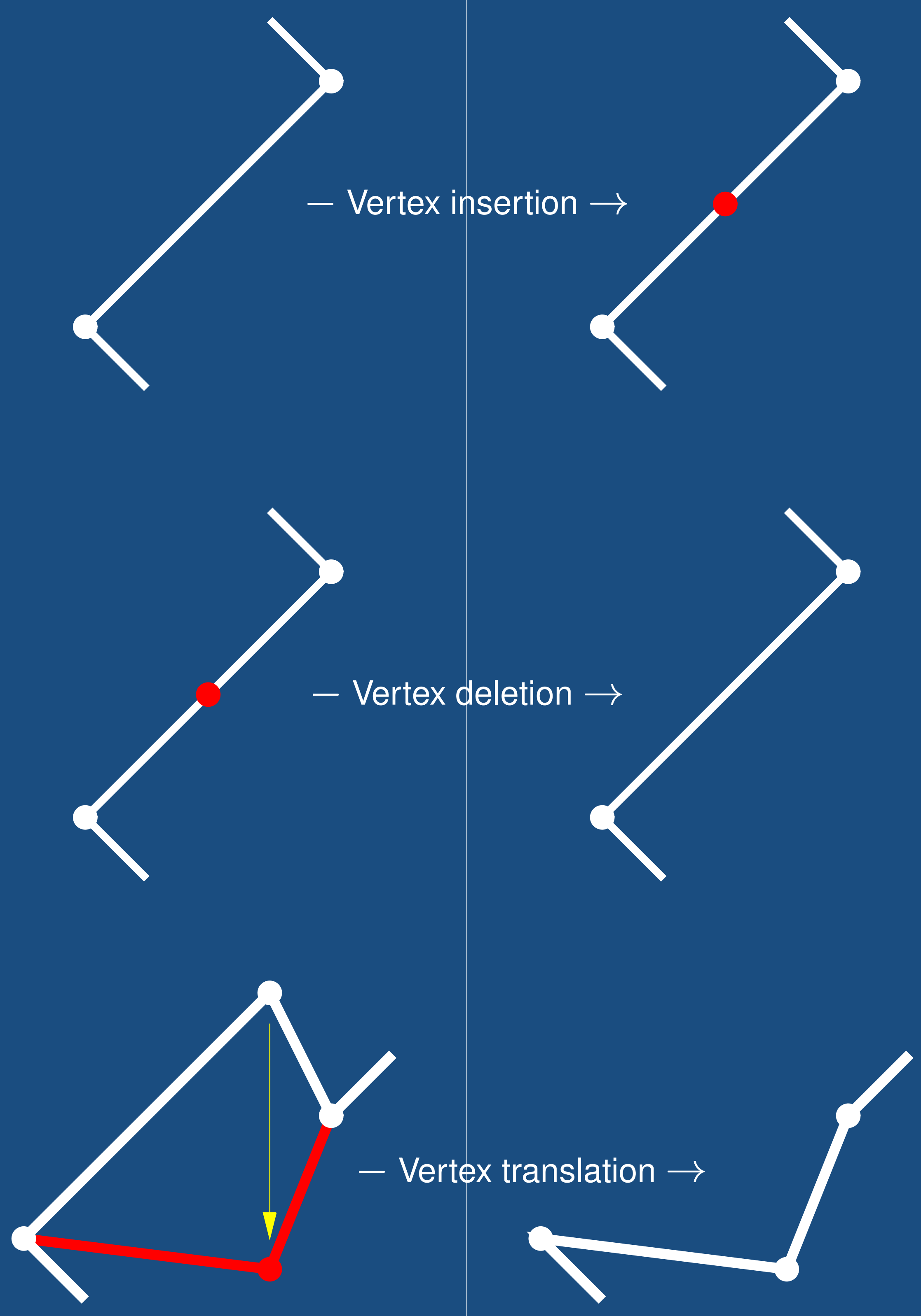


$P = (p_1, \ldots, p_n)$ is a regular polygon if $-\pi < \varphi_i < \pi$, for $1 \leq i \leq n$.

Winding number:

$$w(P) = \frac{1}{2\pi} \sum_{i=1}^{n} \varphi_i$$

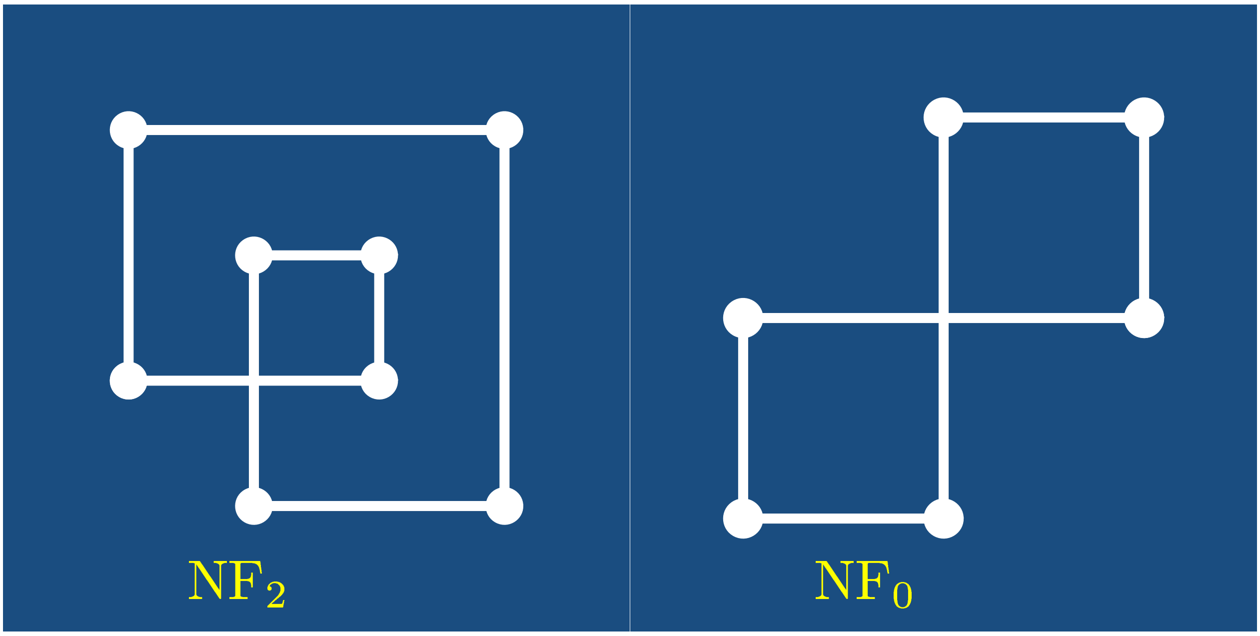
Elementary Transformations



Theorem: If $w(P) = w(P')$, there is a sequence of $O(|P| + |P'|)$ elementary transformations deforming P into P'. This sequence can be determined in $O(|P| + |P'|)$ time.

Isothetic Polgon:

1. All edges are parallel to coordinate axes
2. Successive edges are perpendicular.

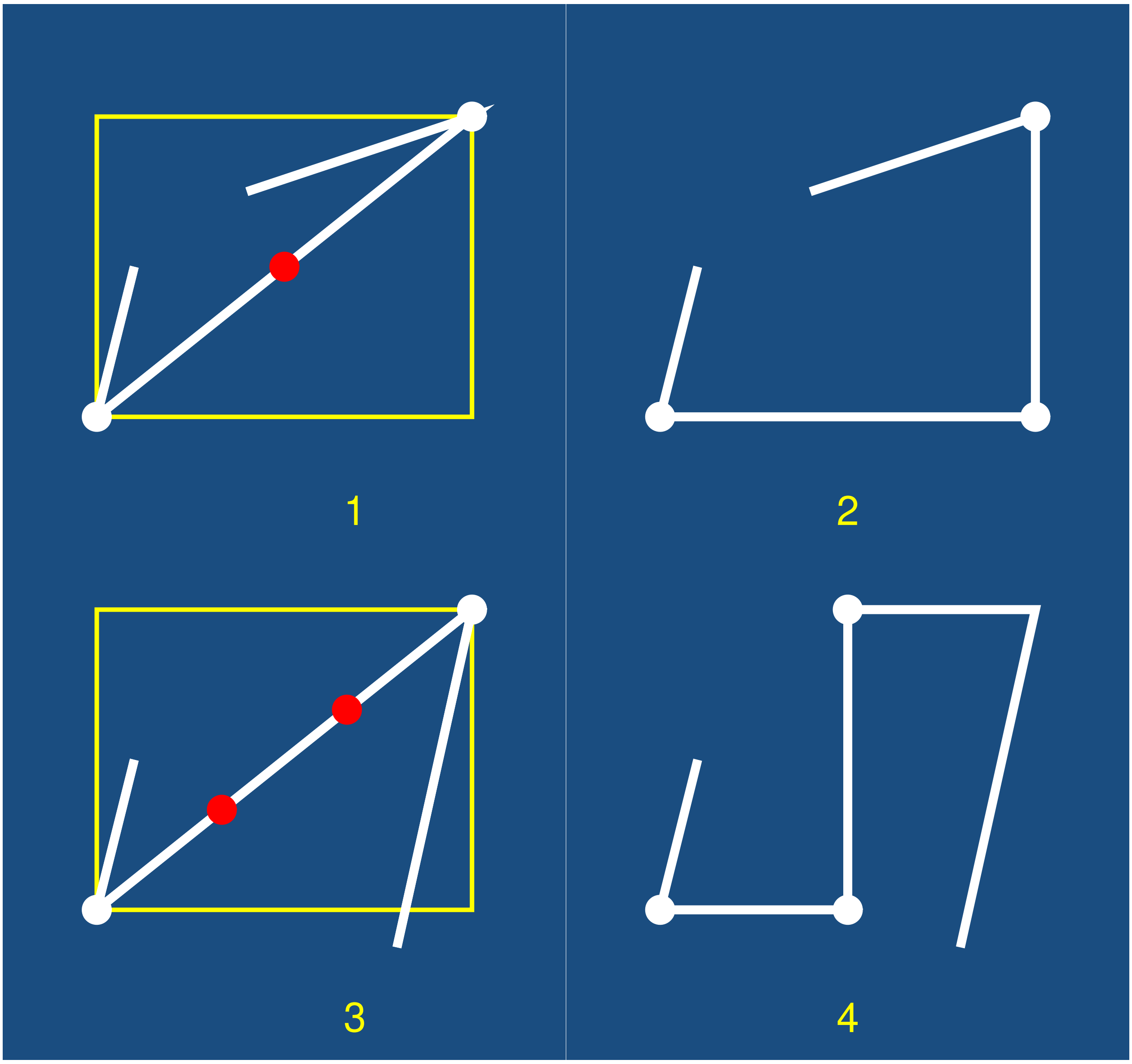


Normal Form: $P \in \mathrm{NF}_m$ if:

- all exterior angles are $\pi/2$, if $m > 0$
- all exterior angles are - $\pi/2$, if $m < 0$
- 3 exterior angles are - $\pi/2$, remaining 3 are $-\pi/2$, if $m = 0$

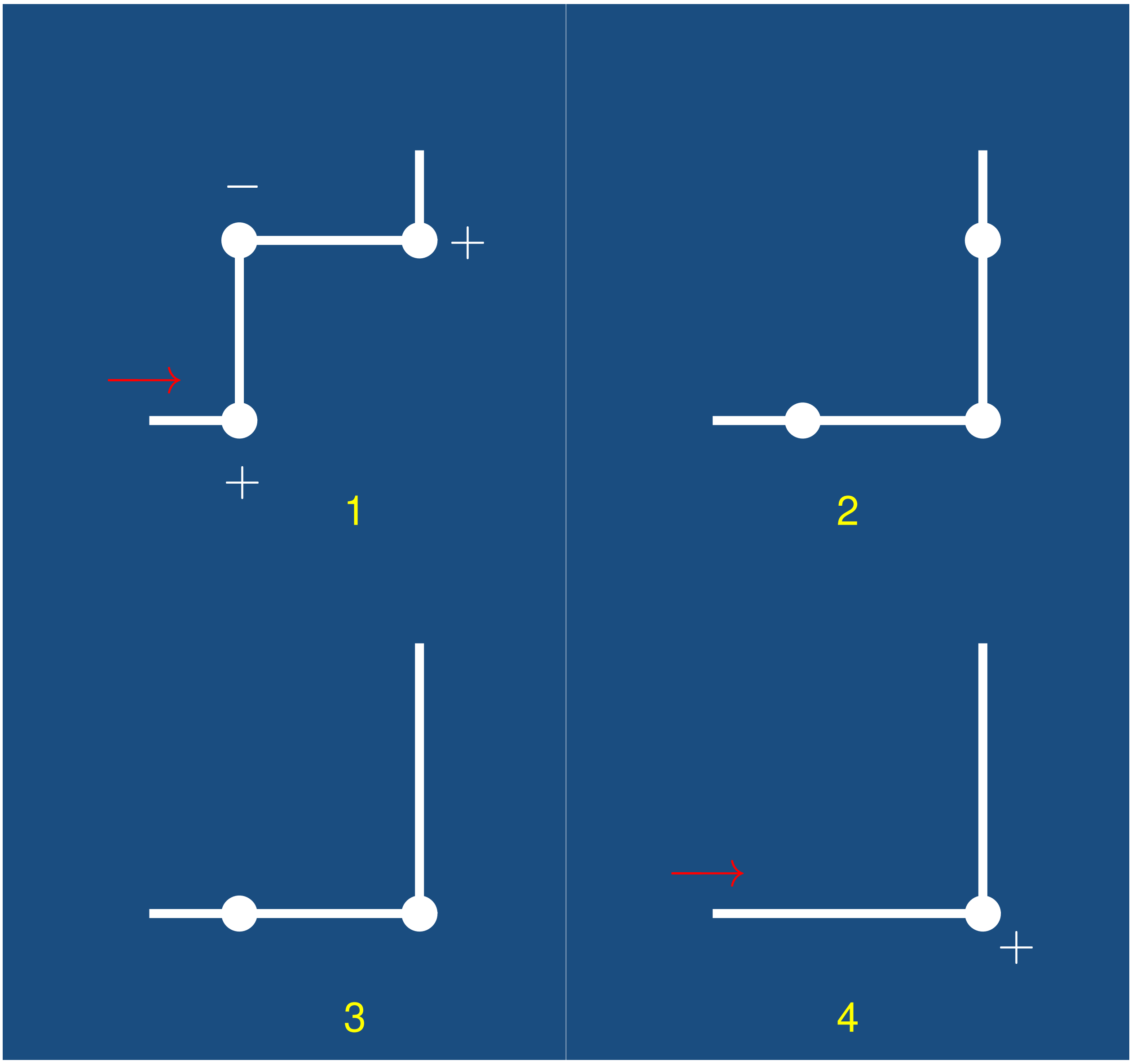
$$w(\mathrm{NF}_m) = m$$

Lemma: A regular polygon P can be brought into isothetic normal form in $O(|P|)$ time, using $O(|P|)$ elementary transformations.

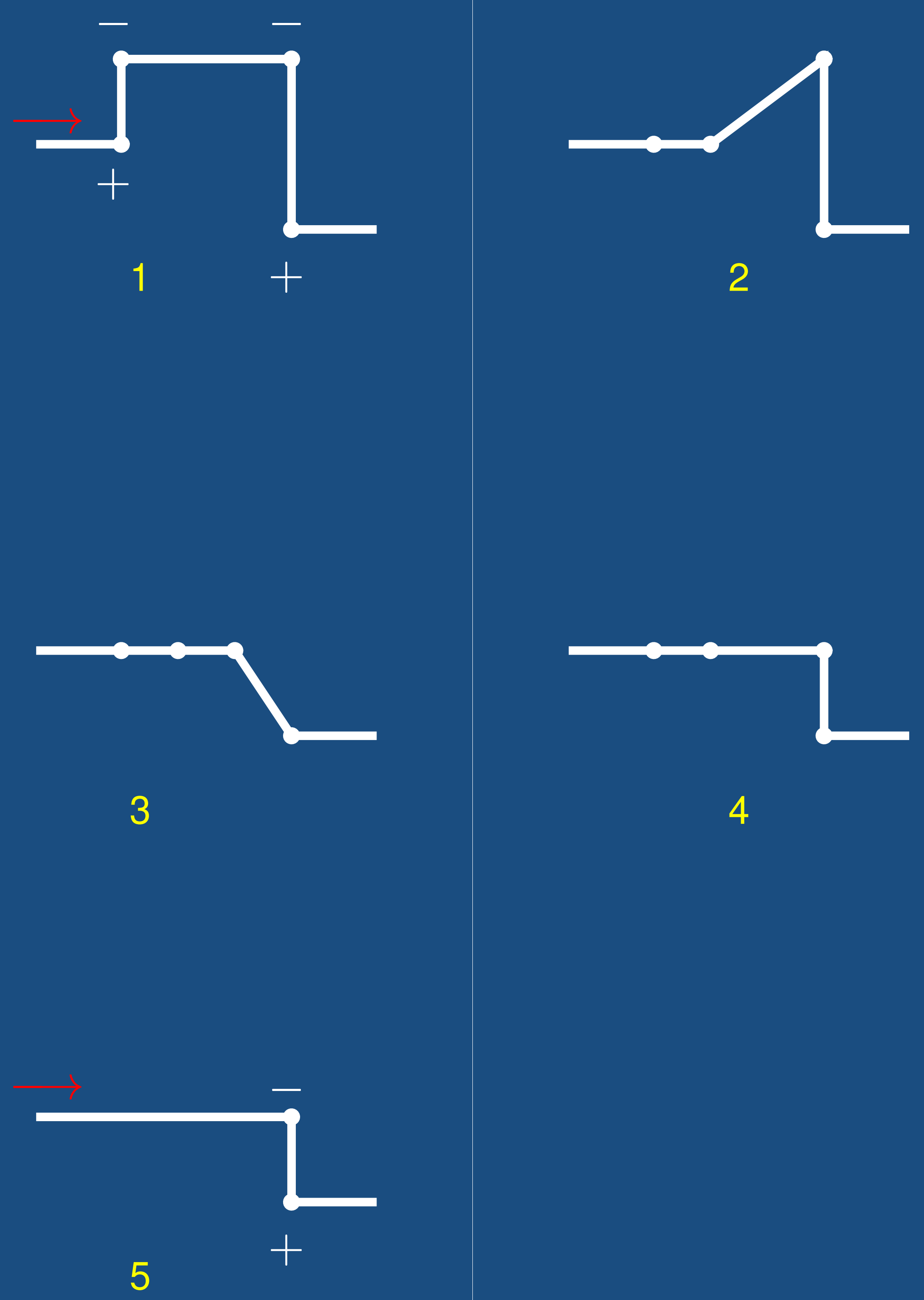


Lemma: An isothetic polygon P with n vertices can be brought into normal form $N_{w(P)}$ in $O(n)$ time and $O(n)$ elementary transformations.

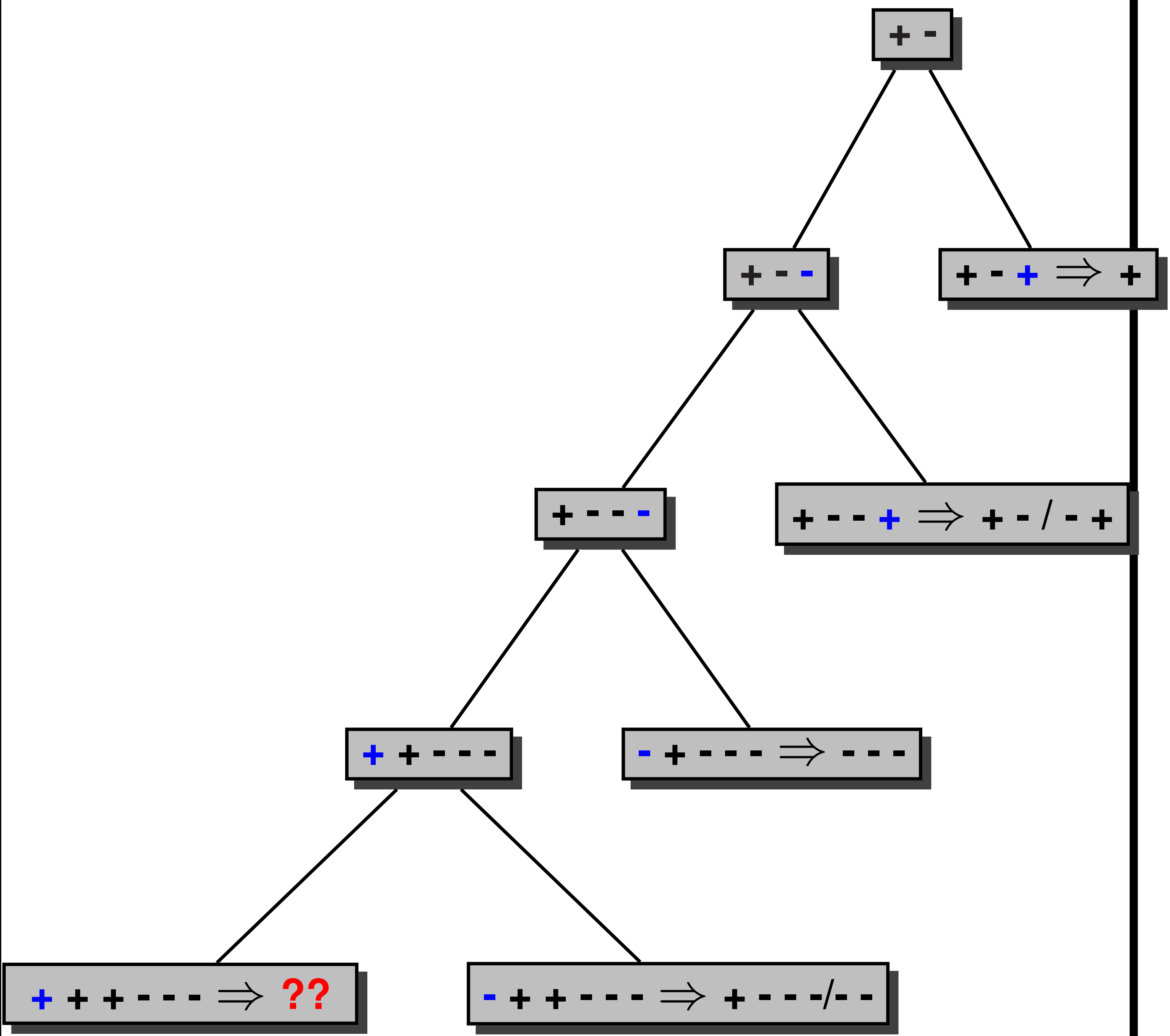
Case 1: $+, -, + \longrightarrow +$



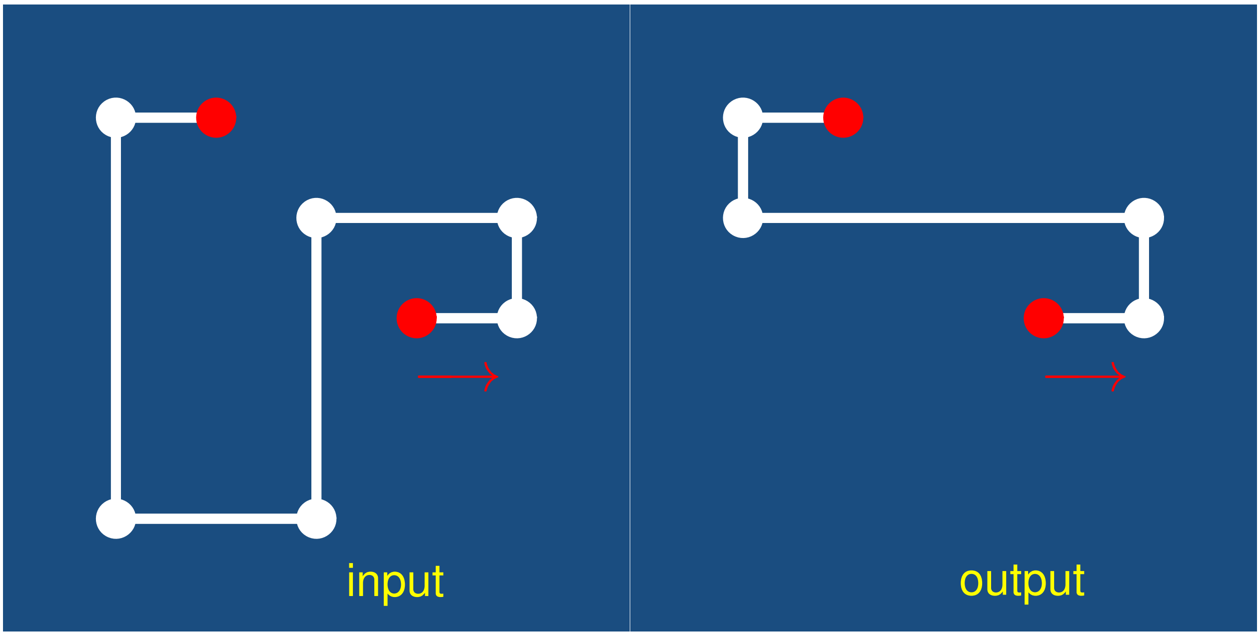
Case 2: $+, -, -, + \longrightarrow -, +$



Local reduction of isothetic polygons

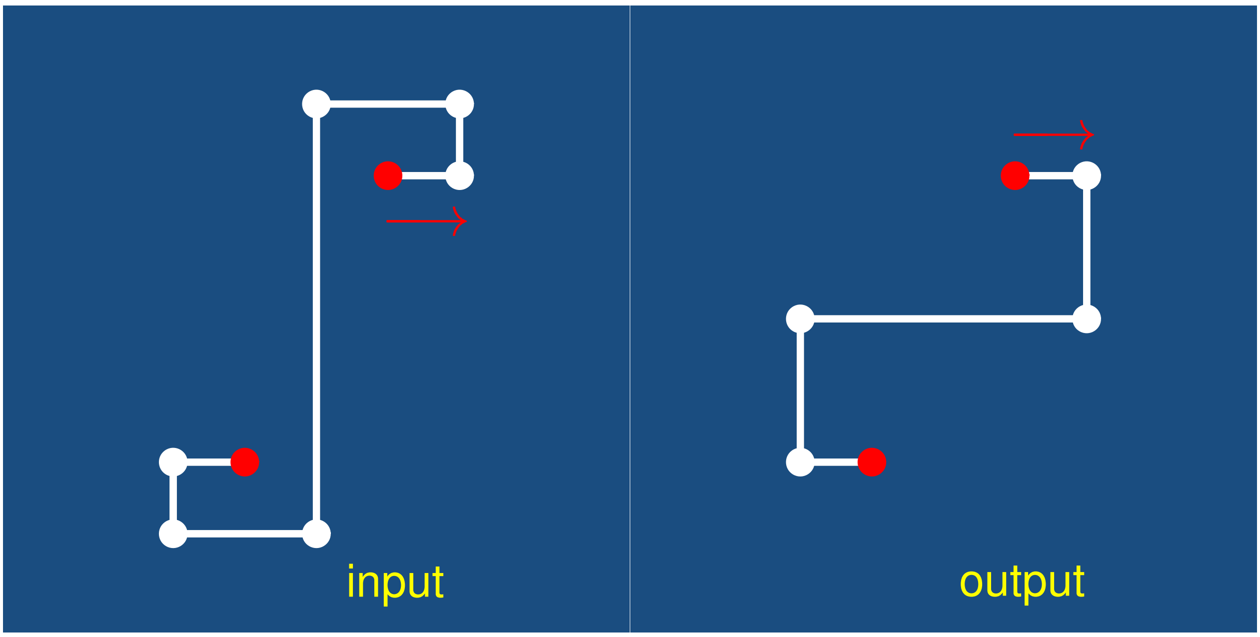


Case 3a: $+, +, +, -, -, - \longrightarrow +, +, -, -$

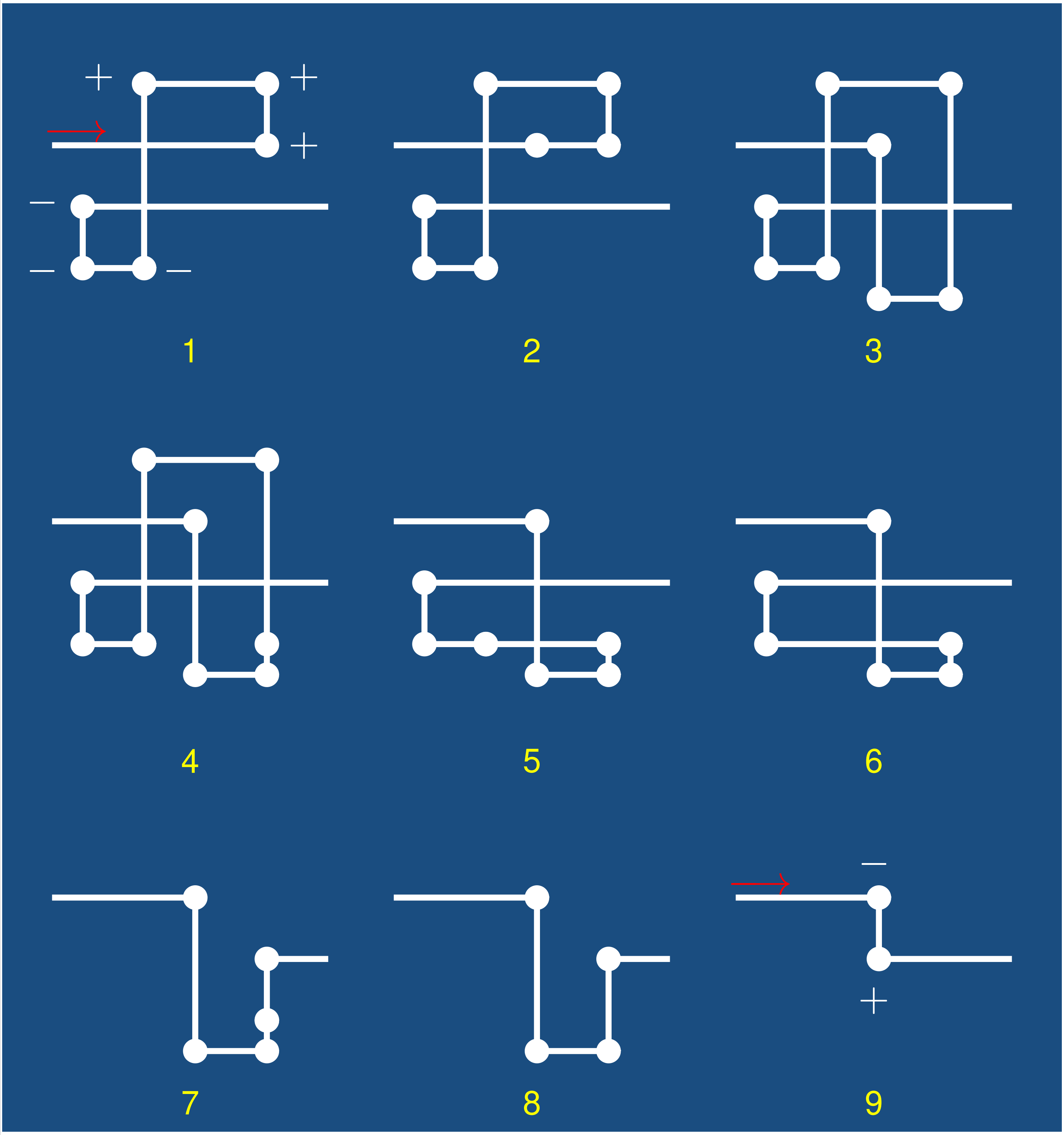


Exercise:

Case 3b: $+, +, +, -, -, - \longrightarrow -, -, +, +$

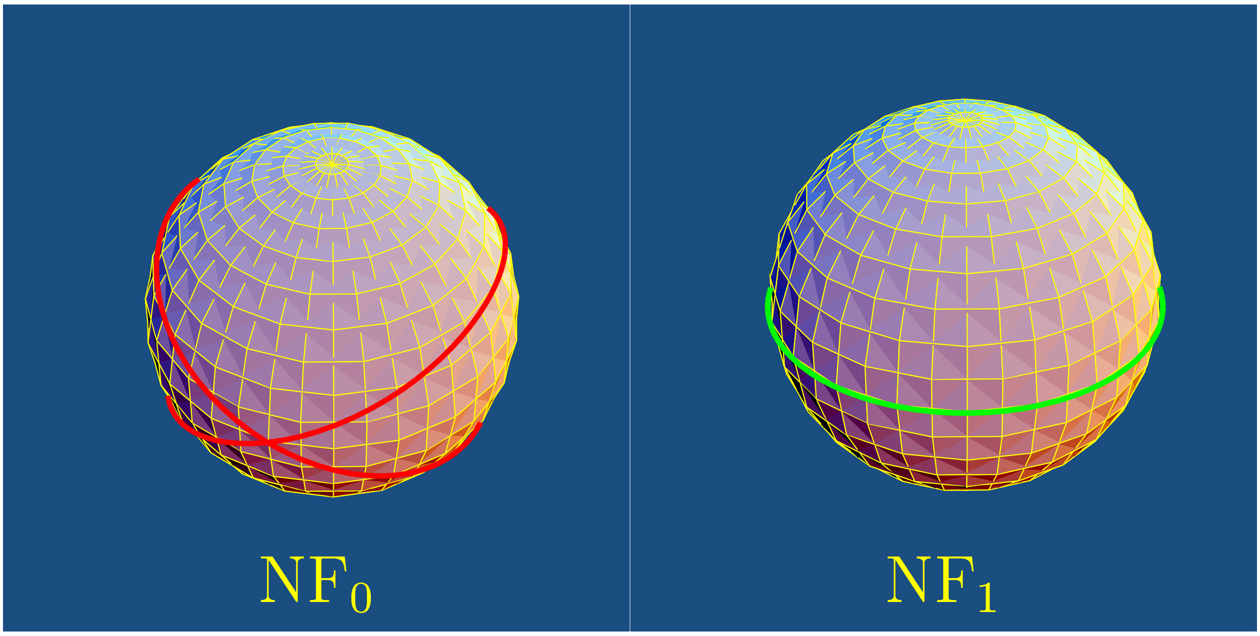


Example/Case 3c: $+,+,+,-,-,- \longrightarrow +,-$



Regular closed curves on the two-sphere

Two normal forms under regular deformations



Regular closed curves on surfaces (Advanced)

If $\gamma : \mathbb{S}^1 \to S$ is a regular closed curve on surface S, then

$$\tilde{\gamma} : t \mapsto \frac{\gamma'(t)}{|\gamma'(t)|} : \mathbb{S}^1 \to T_1(S)$$

is a closed curve in the space of unit tangent vectors of S

Theorem (Smale)

$\tilde{\gamma}$ and $\tilde{\gamma'}$ are regularly equivalent iff $[\tilde{\gamma}] = [\tilde{\gamma'}] \in \pi_1(T_1(S))$.

Normal forms are parametrized by $\pi_1(T_1(S))$.

Examples

1. $T_1(\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{S}^1$, so $\pi_1(T_1(\mathbb{R}^2)) = \mathbb{Z}$ (winding numbers)

2. $T_1(\mathbb{S}^2) = SO(3)$, so $\pi_1(T_1(\mathbb{S}^2)) = \pi_1(SO(3)) = \mathbb{Z}_2$

Regular deformations of polygonal curves on surfaces

Open problem

Design algorithms for

1. deciding whether two regular closed curves on a closed surface are regularly equivalent
2. constructing a sequence of elementary deformations transforming two equivalent regular closed curves on a surface into each other

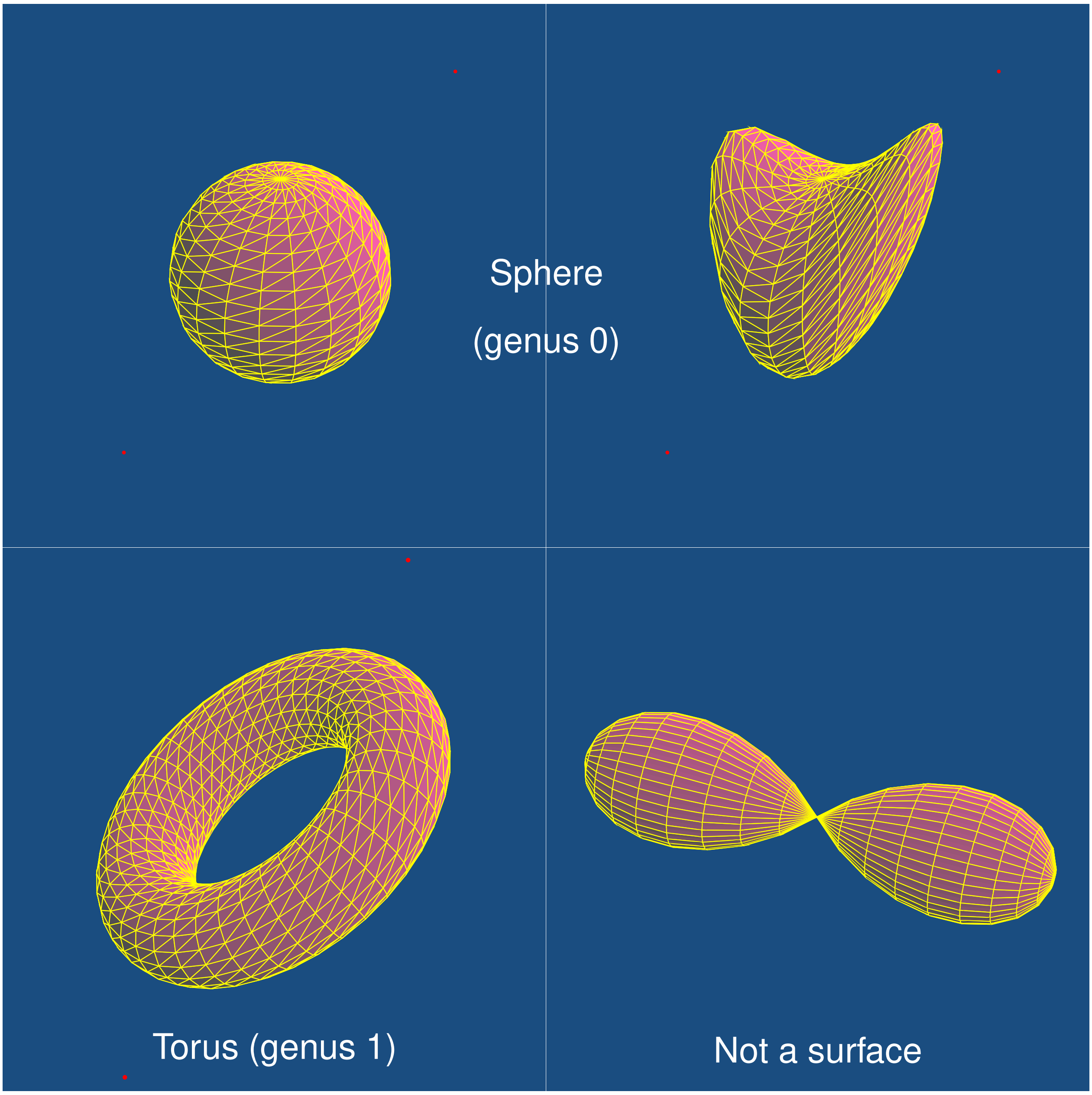
Note: requires solution to homotopy problem for curves on surfaces.

Surfaces

or

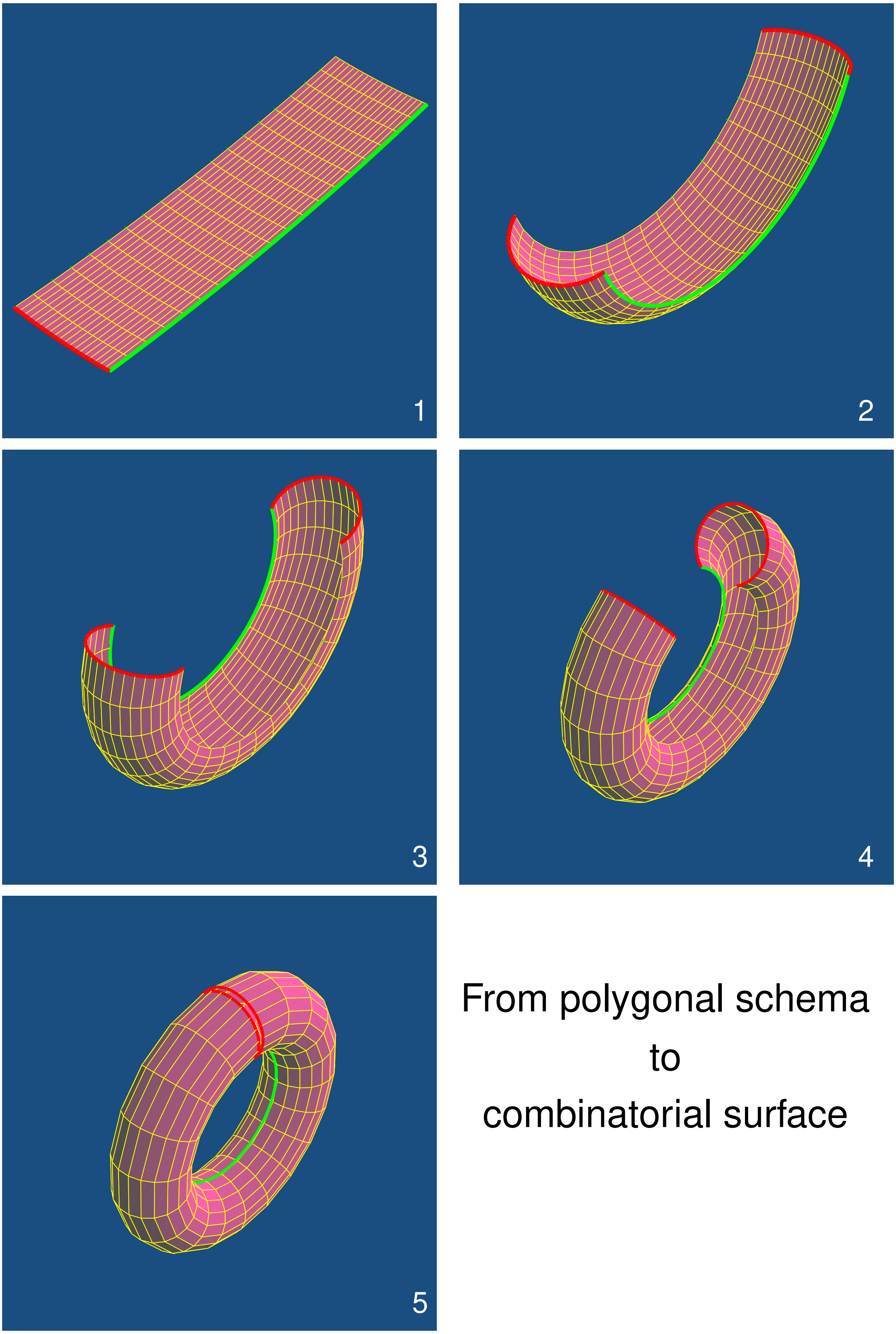
$$\chi(S) = \sum_{\sigma \text{ simplex of } S} (-1)^{\dim \sigma} = \sum_{i=0}^{d} (-1)^i |S_i|$$

Surface: ”Connected two-dimensional set which locally looks like the plane”



Representations:

1. Polygonal Schema
2. Combinatorial surface



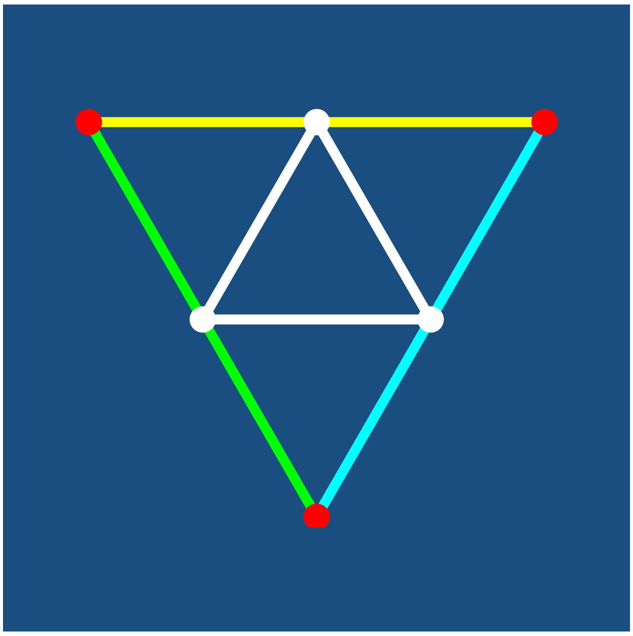
From polygonal schema
to
combinatorial surface

Euler characteristic of a combinatorial surface S:

$$\chi(S) = |V| - |E| + |F|$$

Property: The Euler characteristic is independent of the triangulation.

Example: $\chi(\mathbb{S}^2) = 2$

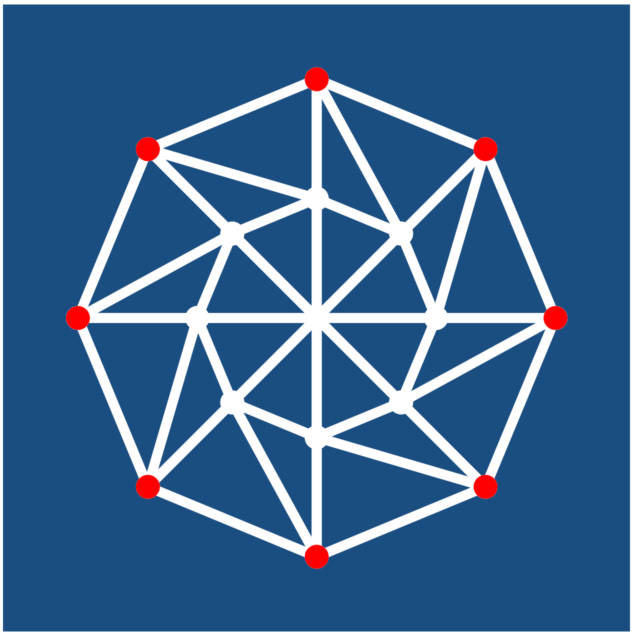


$$\begin{aligned} |V| &= 4 \\ |E| &= 6 \\ |F| &= 4 \end{aligned}$$

Example: Surface with polygonal schema

$$a_1 b_1 \overline{a}_1 \overline{b}_1 \cdots a_g b_g \overline{a}_g \overline{b}_g$$

has Euler characteristic $2 - 2g$:



$$\begin{aligned} |V| &= 1 + 4g + 1 \\ |E| &= 4 \cdot 4g + 2g \\ |F| &= 2 \cdot 4g + 4g \end{aligned}$$

g is called the genus of the surface.

Fundamental Theorem of Surface Topology

Every orientable surface without boundary is uniquely determined up to homeomorphism by its Euler characteristic:

$$S \approx S' \text{ iff } \chi(S) = \chi(S')$$

All the way with Gauss-Bonnet (discrete version)

For a *smooth* closed surface S with Gaussian curvature $K : S \to \mathbb{R}$:

$$\chi(S) = \frac{1}{2\pi} \int_S K \, dA$$

Discrete Gaussian curvature (at vertex v) is angle deficit:

$$K(v) = 2\pi - \sum_{f : v \in \partial f} \alpha(v, f)$$

Discrete Gauss-Bonnet

$$\chi(S) = \frac{1}{2\pi} \sum_{v \in V} K(v)$$

$$\sum_{v \in V} K(v) = 2\pi|V| - \pi|F| = 2\pi(|V| - \frac{1}{2}|F|)$$

$3|F| = 2|E|$, so

$$\chi(S) = |V| - \frac{3}{2}|F| + |F| = |V| - \frac{1}{2}|F|$$

Corollary

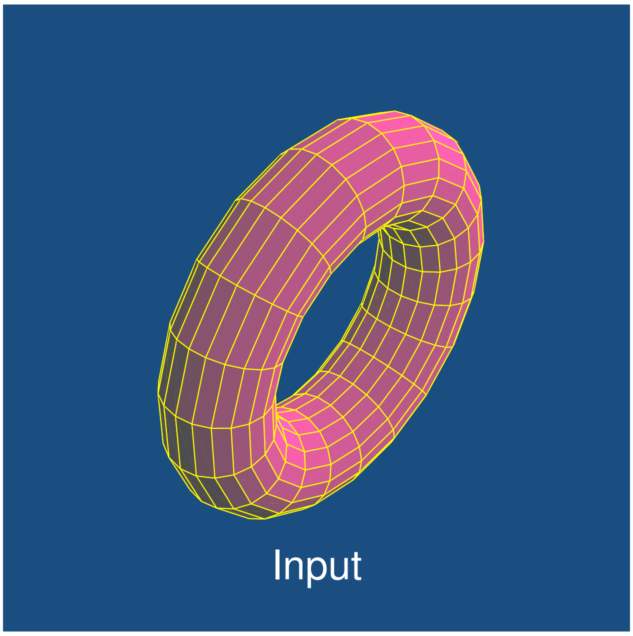
A convex polyhedral surface is isomorphic to the two-sphere.

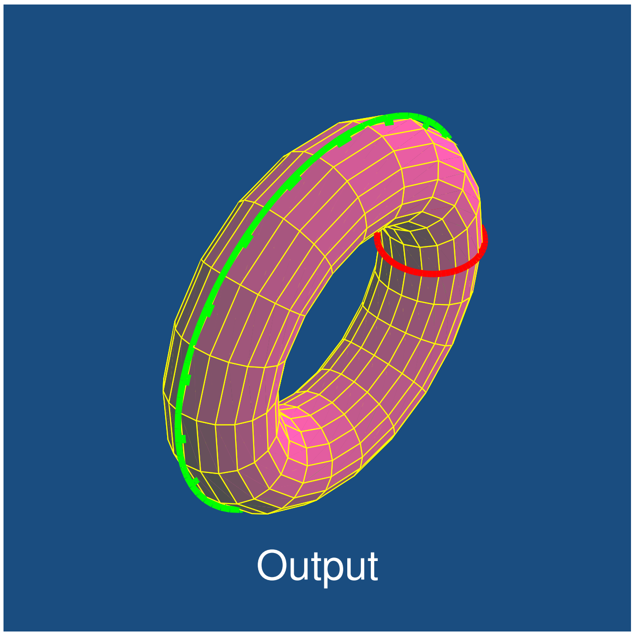
Proof

Convexity implies $K(v) > 0$, for $v \in V$, so

$$\chi(S) = \frac{1}{2\pi} \sum_{v \in V} K(v) > 0,$$

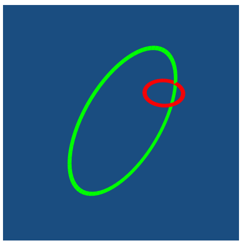
and hence $\chi(S) = 2$. Hence $S \approx \mathbb{S}^2$.





Computing Canonical Generators

Given orientable combinatorial surface S with $\chi(S) = 2 - 2g$. Find $2g$ curves – canonical set of generators — on S



sharing a common base point, whose complement is a disc. The curves should appear on the boundary of this disc in canonical order:

$$a_1 b_1 \overline{a}_1 \overline{b}_1 \cdots a_g b_g \overline{a}_g \overline{b}_g$$

Algorithm for canonical set of generators

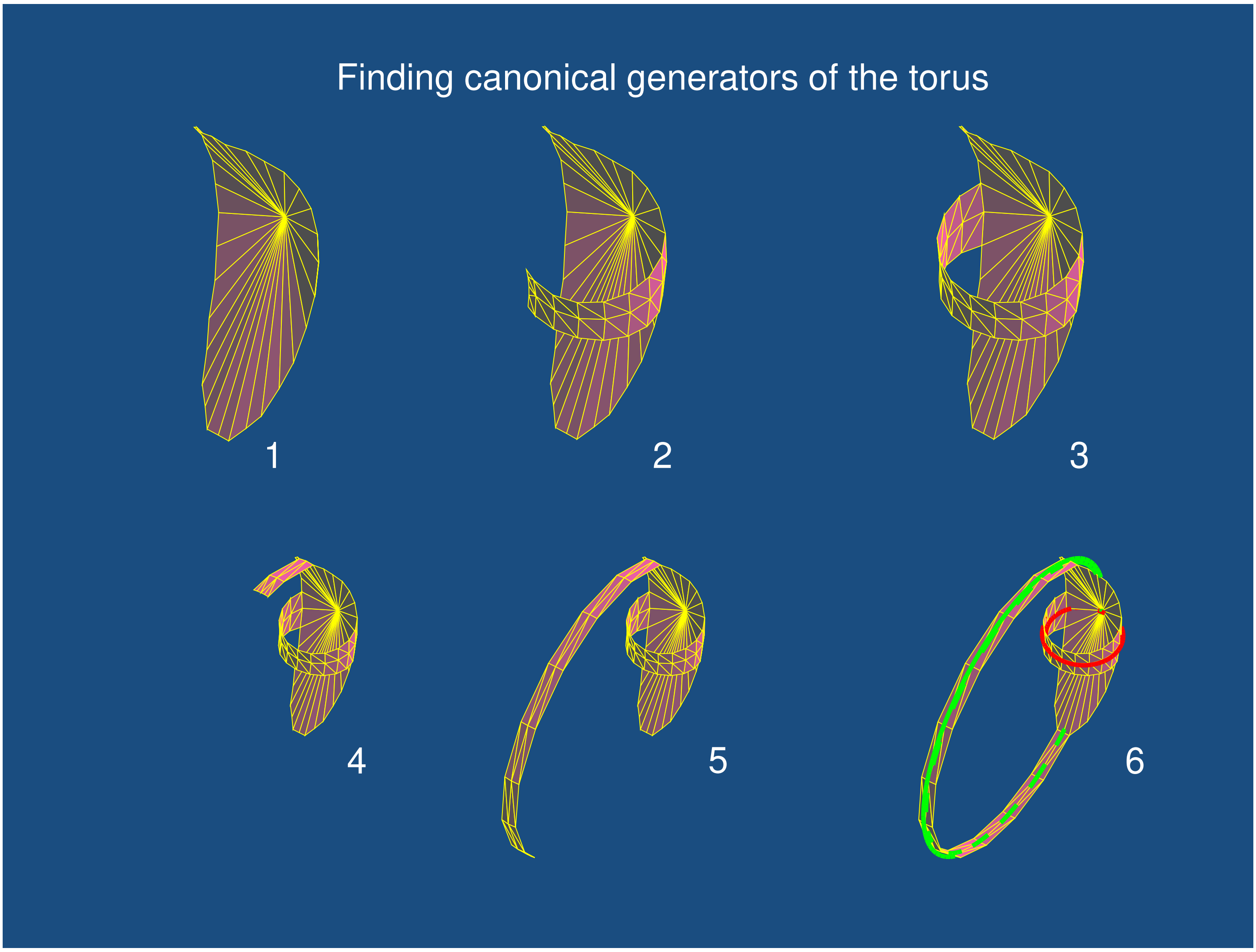
Given an orientable combinatorial surface M of complexity n ($n = |V| + |E| + |F|$), with $\chi(M) = 2 - 2g$.

There is a canonical set of generators, each consisting of n segments, that can be computed in $O(gn)$ time and space.

These bounds are optimal: there is a class of surfaces for which every generator in a canonical set consists of $\Omega(n)$ segments.

C.K. Yap, G.Vegter (1990)

joined by:
F. Lazarus, M. Pocchiola, A. Verroust (2000)



1. Start with the star of a vertex (1)
2. Add triangles incident to current boundary (2), until the boundary splits (3)
3. If the complement is connected, join connected components of boundary (4,5)
4. Construct one pair of generators (6)

Applications of canonical generators algorithm

1. Construction of homeomorphisms between surfaces

   M_1, M_2: oriented combinatorial surfaces with $\chi(M_1) = \chi(M_2)$. Compute subtriangulations M_1' and M_2' with isomorphics face lattices.

   Method: Reduce to construction on a *disk* by first computing a canonical set of generators.

   Time: $O(g^2n^2)$ — n: number of cells, g: genus

2. Construction of homotopies (deformations) of curves

   $\pi : \tilde{M} \to M$ is *universal cover* of M. Then:

   Curve c on M is contractible (to a point) iff. lift $\tilde{c}$ of c to $\tilde{M}$ is a closed curve.

   Method: Construct 'suitable' part of universal cover by glueing copies of canonical polygonal schema of M (first compute canonical generators). Construct lift, check whether it is closed.

   Time: $O(g^2kn)$ — n: number of cells, g: genus, k: number of edges of c [Schipper, 1992].

V-minimal triangulations

Lemma (lower bound on f_0)
For an orientable surface of genus g the number f_0 of vertices satisfies

$$f_0 \geq \left\lceil \frac{1}{2}(7 + \sqrt{48g+1}) \right\rceil = \Theta(\sqrt{g}). \tag{1}$$

Theorem (Jungerman & Ringel, 1980)
An orientable 2-manifold of genus $g \neq 2$ has a triangulation for which equality holds in (**??**). For $g = 2$, the minimal number of vertices is 10.

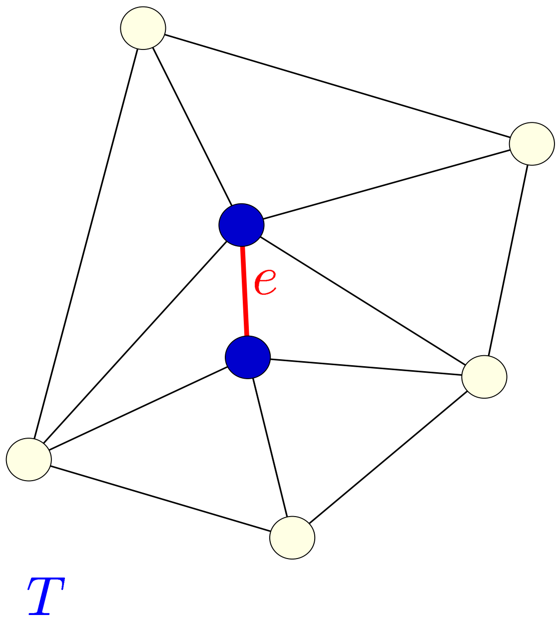
Conjecture
V-minimal triangulations of orientable surfaces are simplexwise linearly embeddable in $\mathbb{R}^3$.

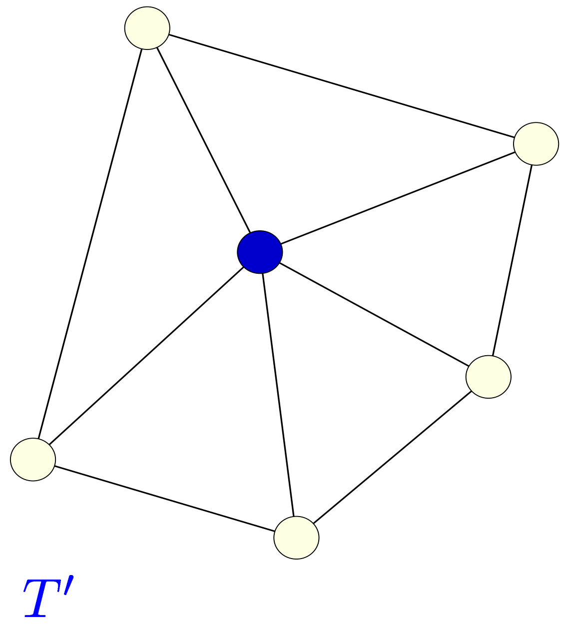
True for

- $g = 0$ (trivial)
- $g = 1$ (Császár, 1949)
- $g = 2, 3$ (Brehm, 1981,1987)
- $g = 4$ (Bokowski & Brehm, 1989)

Open for $g > 4$.

E-minimal triangulations

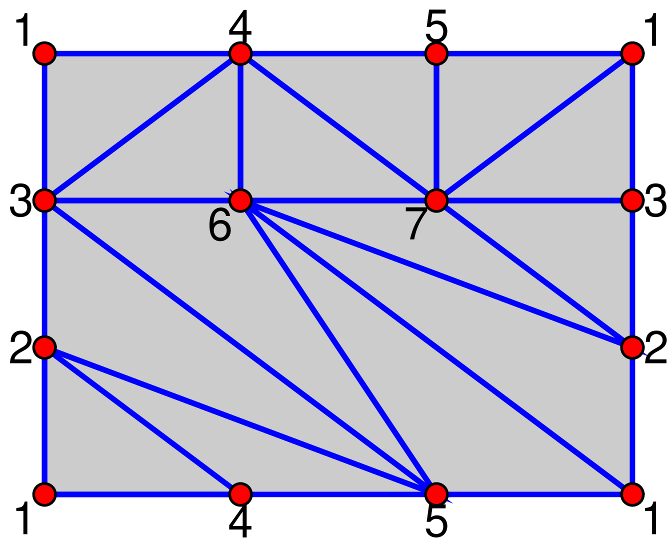




If T' is a triangulation, then edge e is shrinkable

A triangulation is called E-minimal if it contains no shrinkable edges.

An E-minimal triangulation
of the torus
(Császár's Möbius torus)



Theorem (Barnette & Edelson, 1988)

A 2-manifold has finitely many combinatorially distinct E-minimal triangulations.

Theorem (Schipper, 1991)

For every triangulation of a 2-manifold of genus g with $O(n)$ edges a sequence of $O(n)$ edge shrinkings leading to an E-minimal triangulation can be determined in $O(n \log n)$ time.

Conjectures and open problems

- An E-minimal triangulation of a 2-manifold of genus g has $\Theta(g)$ edges and faces, and $O(g)$ vertices.
- Determine bounds for the number of E-minimal triangulations of a surface of genus g.
- Design an algorithm to compute all E-minimal triangulations of a surface of genus g.