Medial Axis Homotopy
(Rapport de recherche LMC/IMAG)

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Abstract

Medial Axis Transform is sometimes used as an intermediate representation in algorithms for meshing or recognition of shapes from digitized data. This raises the question whether the Medial Axis captures fundamental topological invariants of the object. The (positive) answer has been known already in the case of smooth objects. The main result presented here is *the homotopy equivalence of any bounded open subset of* $\mathbb{R}^n$ *with its Medial Axis*. The largest part of the paper is the proof of this statement.

Résumé

Ce travail présente la preuve de *l'équivalence d'homotopie entre tout ouvert borné de* $\mathbb{R}^n$ *et son axe médian*. L'axe médian joue un rôle important dans plusieurs domaines de recherches comme la modélisation géométrique des solides, la géométrie algorithmique et la morphologie mathematique. Il peut etre utile de pouvoir s'appuyer sur des resultats generaux concernant sa topologie, même lorsque sa frontière n'est pas lisse.

1 Introduction

We present here the proof that *any bounded open subset of* $\mathbb{R}^n$ *has the same hompotopy type than its Medial Axis*. The openness condition is optimal as the medial axis is generally defined for open sets (cf. [7]).

The topology of the Medial Axis transform without smoothness assumtion is addressed in [7] , where the author proves the connectivity of the closure of the medial axis of a connected open set. However, he is unable to prove the connectivity of the medial axis itself even if it "seems rather plausible" ([7],p. 217). This connectivity is a corollary of theorem 4.19 below.

The homotopy equivalence has been proved in the case of an open set with a piecewise C^2 boundary in the plane and in the case of an open set with a C^2 boundary in $\mathbb{R}^n$ [8]. Of course, this result is also known in the case of the complement of a finite number of points in $\mathbb{R}^n$: in this case, the Medial Axis is the Voronoi diagram. These results are paticular cases of theorem 4.19.

It has been proved that, in the smoother case of a piecewise analytic boundary in the plane, the Medial Axis is made of a finite number of curves [9].

Note that there are in fact several closely related objects, the Medial Axis, the Skeleton and the Cut Locus (cf. [7] and [8]). We call here *Medial Axis* of a bounded open set $\mathcal{O}$, the set of points of $\mathcal{O}$ who have at least two closest point on the boundary $\partial\mathcal{O}$. The *Skeleton* is the set of centers of maximal [1] open balls included in the open set, while the *Cut Locus* is the closure of the Medial Axis. The Medial Axis is a subset of the Skeleton, which is itself a subset of the Cut Locus.

The definition of the *Homotopy Equivalence* is taken from [11, 12] and reminded in section 3 below.

The next two sections, (2) (Definitions and Notations) and (3) (Sketch of the proof), are aimed to give a quick overview of the proof, while the section (4) (proof) consists in the detailed proof of each lemma.

2 Definitions and Notations

We use the following definitions and notations. In the whole paper, $\mathcal{O}$ and $\mathcal{M}$ always denote respectively a bounded open subset of $\mathbb{R}^n$ and its Medial Axis defined below. For any set X, $\overline{X}$, X° and X^c denote respectively the closure, the interior and the complement of X. $\mathbb{B}_{x,r}$ and $\mathbb{B}^\circ_{x,r}$ respectively denote the closed and open ball of center x and radius r in $\mathbb{R}^n$. We denote by $\mathbb{S}_{x,r}$ the corresponding hyper-sphere, that is, $\mathbb{S}_{x,r} = \mathbb{B}_{x,r} - \mathbb{B}^\circ_{x,r}$, where "$-$" denotes the set theoretic difference.

We denote by $X \oplus \mathbb{B}_{0,r}$ (resp. $X \oplus \mathbb{B}^\circ{}_{0,r}$) the Minkowski sum of X and the closed (resp. open) ball of radius r, that is :

$$X \oplus \mathbb{B}_{0,r} = \{y,\ \exists x \in X,\ \exists s \in \mathbb{B}_{0,r},\ y = x + s\}$$

and

$$X \oplus \mathbb{B}^\circ{}_{0,r} = \{y,\ \exists x \in X,\ \exists s \in \mathbb{B}^\circ{}_{0,r},\ y = x + s\} = \bigcup_{x \in X} \mathbb{B}^\circ{}_{x,r}$$

For any point $x \in \mathcal{O}$, we denote by $\because(x)$ the set of closest boundary points, that is :

$$\because(x) = \{y \in \mathcal{O}^c,\ d(x,y) = d(x,\mathcal{O}^c)\}$$

Because $\mathcal{O}^c$ is closed and $\mathcal{O}$ is bounded, $\because(x)$ can not be empty.

Definition 2.1 (Medial Axis) *The Medial Axis $\mathcal{M}$ of the open set $\mathcal{O}$ is the set of points x of $\mathcal{O}$ who have at least 2 closest boundary points :*

$$\mathcal{M} = \{x \in \mathcal{O},\ cardinal(\because(x)) \geq 2\}$$

[1] Maximal for the inclusion order.

The strictly positive, real valued function $\mathcal{R}$ defined on $\mathcal{O}$ is the distance to the boundary.

$$\mathcal{R}(x) = d(x, \mathcal{O}^c)$$

One can check that $\mathcal{R}$ is 1-Lipschitz. The real valued, positive function $\mathcal{F}$ is defined as follow (cf. figure below).

$$\mathcal{F}(x) = \inf\{r, \exists y \in \mathbb{R}^n, \mathbb{B}_{y,r} \supset \because (x)\}$$

Moreover, there exists a unique ball with this minimal radius $\mathcal{F}(x)$ and enclosing $\because (x)$[2]. In other words $\mathcal{F}(x)$ is the radius of the smallest closed ball enclosing $\because (x)$.

Similarly, $\Theta(x)$ denotes the center of this smallest enclosing ball. Of course, when $x \notin \mathcal{M}$, we have $\because (x) = \{\Theta(x)\}$ and $\mathcal{F}(x) = 0$.

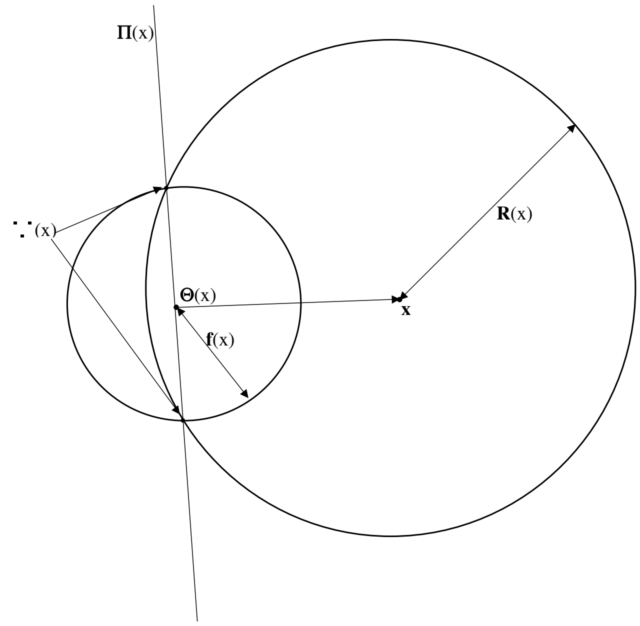

Figure 1: A 2-dimensional example, with 2 closest points

The vector function ∇, that plays a central role in the proof below, is an extension of the gradient of $\mathcal{R}$ outside $\mathcal{O} - \overline{\mathcal{M}}$ (cf. [7]).

[2]The existence follows from the compactness of the set of balls whose radius is bounded by a given value B and containing $\because (x)$ and the unicity from the fact that if two distinct balls contain $\because (x)$ there exist another ball of strictly smaller radius enclosing it.

It is defined as follows:

$$\nabla(x) = \frac{x - \Theta(x)}{\mathcal{R}(x)}$$

If $x \neq \Theta(x)$, the (hyper-) sphere $\mathbb{S}_{x,\mathcal{R}(x)}$ is cut by the (hyper-) sphere $\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$ along a circle in the tridimensional case ($n = 3$, cf. figure below) and an instance of S^{n-2} in general.

We denote by $\Pi(x)$ the (hyper-)plane supporting the intersection circle, that is the radical plane of $\mathbb{S}_{x,\mathcal{R}(x)}$ and $\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$. $\Pi(x)$ is clearly orthogonal to $\nabla(x)$.

When $\mathcal{F}(x) = 0$, $\Pi(x)$ is the (hyper-)plane orthogonal to $\nabla(x)$ and going through $\Theta(x)$ (in this case $\because(x) = \{\Theta(x)\}$).

The minimality condition in the definition of $\mathbb{B}_{\Theta(x),\mathcal{F}(x)}$ entails that $\Pi(x)$ cuts $\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$ along a great circle (a maximal $n-2$ sphere), that is :

$$\Theta(x) \in \Pi(x)$$

There is at least one (in fact two) points of $\because(x)$ in $\Pi(x) \cap \mathbb{S}_{\Theta(x),\mathcal{F}(x)}$. If y is such a point, one can apply Pythagora theorem to the triangle $x, \Theta(x), y$ an we get:

$$\|x - \Theta(x)\|^2 + \mathcal{F}(x)^2 = \mathcal{R}(x)^2$$

and then,

$$\|\nabla(x)\|^2 + \frac{\mathcal{F}(x)^2}{\mathcal{R}(x)^2} = 1$$

in particular: $0 \leq \|\nabla(x)\| \leq 1$ and $\|\nabla(x)\| < 1 \Leftrightarrow x \in \mathcal{M}$.

Along the proof, we often denote by e the unit vector defined when $\nabla(x) \neq 0$,

$$e = \frac{\nabla(x)}{\|\nabla(x)\|}$$

We denote by $\mathcal{O}_\rho$ the set of points of $\mathcal{O}$ whose distance to $\mathcal{O}^c$ is greater than ρ :

$$\mathcal{O}_\rho = \{x \in \mathcal{O}, \mathcal{R}(x) \geq \rho\}$$

We have, because $\mathcal{O}$ is open, $\mathcal{O} = \bigcup_{\rho > 0} \mathcal{O}_\rho$.

We say that a map f, defined on $X \subset \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$ is *upper semicontinuous* if:

$$\forall x \in X, \quad \forall \epsilon > 0, \quad \exists \alpha > 0, \quad \forall y \in X, \quad \|y - x\| < \alpha \Rightarrow f(y) < f(x) + \epsilon$$

or, equivalently :

$$\forall \lambda \in \mathbb{R}, \quad \{x \in X, f(x) < \lambda\} \quad \text{is open}$$

It results from the last definition that a bounded upper semicontinuous function is Lebesgues integrable. Note as well that an upper semicontinuous function reaches its supremum (its lowest upper bound) on any compact set. The *lower semicontinuity* is defined similarly.

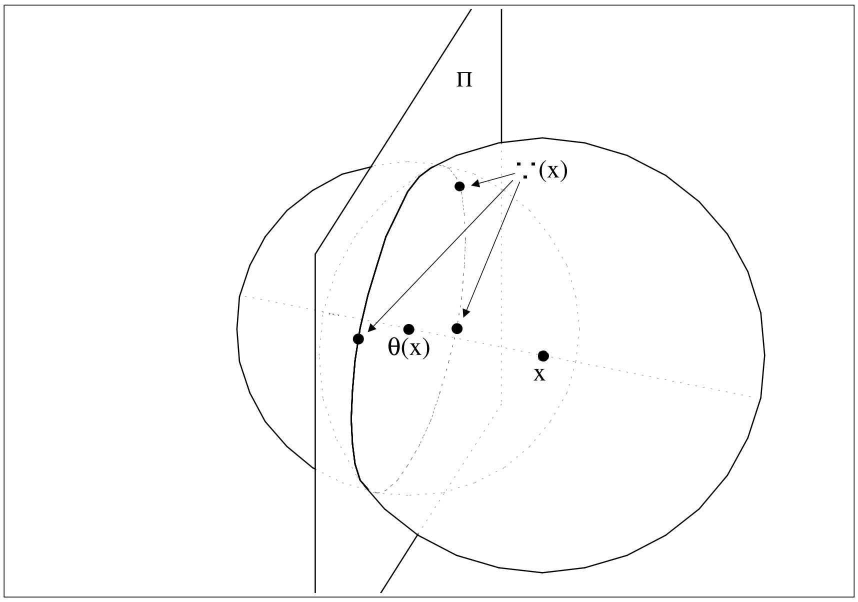

Figure 2: A 3-dimensional example, with 3 closest points

3 Sketch of the proof

A common technique used to prove the homotopy equivalence between a set and one of its subsets is to build a deformation retract [3] ([10]). We use instead the general characterization of homotopy equivalence (cf. [11] pages 171-172 and [12] pages 108):

Two spaces X and Y are said to have the same homotopy type if there are continuous maps $f : X \mapsto Y$ and $g : Y \mapsto X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y.

Having the same homotopy type entails the one to one correspondence of arcwise connected components and the isomorphism of their respective fundamental groups.

We use the characterization above to state that $\mathcal{O}$ and $\mathcal{M}$ have the same homotopy type. We take the canonical inclusion ($g : \mathcal{M} \mapsto \mathcal{O}$ is the identity in $\mathcal{M} \subset \mathcal{O}$) for the map g. The proof consists in building the map f and the associated homotopy.

For that, we build a continuous map

$$\mathfrak{C} : \mathbb{R}^+ \times \mathcal{O} \mapsto \mathcal{O}$$

We will see that in some sense, this map integrates the vector field ∇.

For that we first need to prove a generalization of the classical Cauchy-Lipschitz-Picard theorem [1, 2] that states the existence and uniqueness of the solution of a differential equation under the assumption of a Lipschitz condition. This generalization uses a condition significantly weaker than the Lipschitz condition, as it holds

[3]It is the technique used for the smooth case

for the not continuous vector field ∇. It allows us to define locally the continuous map $\mathfrak{C}$ as developed in section 4.1, lemmas 4.1, 4.4, 4.5.

Then, in section 4.2, one state the right differentiability of the map $t \mapsto \mathfrak{C}(t,x)$ (lemmas 4.6, 4.8,4.9 and 4.10), and $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ (lemma 4.11) as well as some integral equations (lemma 4.12). One consequence is the fact that the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ is increasing and we get from that the global existence of $\mathfrak{C}$ on $\mathbb{R}^+ \times \mathcal{O}$ (lemma 4.13).

Then in section 4.3, we prove that the map $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing (lemmas 4.14, 4.15, 4.16 and 4.17), which immediately entails that for $x \in \mathcal{M}$, we have $\forall t \in \mathbb{R}^+, \;\; \mathfrak{C}(t,x) \in \mathcal{M}$.

Now, let D be an upper bound of the diameter of $\mathcal{O}$. We prove in proposition 4.18 in section 4.4 that, $\forall x \in \mathcal{O}$, $\mathfrak{C}(D,x) \in \mathcal{M}$ and we can define the map $f : \mathcal{O} \mapsto \mathcal{M}$ as $f(x) = \mathfrak{C}(D,x)$

$f : \mathcal{O} \mapsto \mathcal{M}$ is homotopic to the identity in $\mathcal{O}$ and the map $f_{\mathcal{M}} : \mathcal{M} \mapsto \mathcal{M}$, restriction of f to $\mathcal{M}$ is homotopic to the identity in $\mathcal{M}$.

These homotopies are given by the map H,

$$H : [0,1] \times \mathcal{O} \mapsto \mathcal{O}$$

given by :

$$H(t,x) = \mathfrak{C}(Dt,x)$$

and the map $H_{\mathcal{M}} : [0,1] \times \mathcal{M} \mapsto \mathcal{M}$, restriction of H to $\mathcal{M}$. It results that $\mathcal{O}$ and $\mathcal{M}$ have same homotopy type (theorem 4.19 in section 4.4).

4 proof

4.1 Local existence of $\mathfrak{C}$

We start with a geometric lemma.

Lemma 4.1 *$\forall x_1, x_2 \in \mathcal{O}_\rho$, we have the inequality*

$$(\nabla(x_1) - \nabla(x_2)) \cdot (x_1 - x_2) \leq \frac{1}{\rho}(x_1 - x_2)^2$$

Proof We consider x_1 and x_2 in $\mathcal{O}_\rho$. If $x_1 = x_2$, the inequality is trivially satisfied. Let us assume that $x_1 \neq x_2$.

We denote by π_{12} the radical hyper-plane of the spheres $\mathbb{S}_{x_1,\mathcal{R}(x_1)}$ and $\mathbb{S}_{x_2,\mathcal{R}(x_2)}$.

Let p_0 be a point in π_{12}: we denote by π_{12}^+ (resp. π_{12}^-) the closed half-space $\pi_{12}^+ = \{y \in \mathbb{R}^n, (y-p_0)\cdot(x_1-x_2) \geq 0\}$ (resp. $\pi_{12}^- = \{y \in \mathbb{R}^n, (y-p_0)\cdot(x_1-x_2) \leq 0\}$).

(Recall that if $\mathbb{B}_{x_1,\mathcal{R}(x_1)}$ and $\mathbb{B}_{x_2,\mathcal{R}(x_2)}$ intersect, π_{12} contains the "circle" $\mathbb{S}_{x_1,\mathcal{R}(x_1)} \cap \mathbb{S}_{x_2,\mathcal{R}(x_2)}$.)

We know that :

$$\because (x_1) \subset \mathbb{S}_{x_1,\mathcal{R}(x_1)} \cap [\mathbb{B}^{\circ}_{x_2,\mathcal{R}(x_2)}]^c$$

And we can deduce that $\because (x_1) \subset \pi_{12}^{+}$ which implies :

$$\Theta(x_1) \in \pi_{12}^{+}$$

In the same manner, one has:

$$\Theta(x_2) \in \pi_{12}^{-}$$

which gives us :

$$(\Theta(x_2) - \Theta(x_1)) \cdot (x_1 - x_2) \leq 0 \tag{1}$$

We study the map λ, defined by :

$$\lambda(x_1, x_2) = (\nabla(x_1) - \nabla(x_2)) \cdot (x_1 - x_2)$$

And we want to prove that :

$$\lambda(x_1, x_2) \leq \frac{1}{\rho}(x_1 - x_2)^2$$

Now, because λ is symmetric, we can assume, without loss of generality, that $\mathcal{R}(x_1) \leq \mathcal{R}(x_2)$.

In this case, one of the two following statement is true: either x_1 and x_2 are on opposite side of the radical plane, that is $x_1 \in \pi_{12}^{+}$ and $x_2 \in \pi_{12}^{-}$, either they both lay on the side of the sphere $\mathbb{B}_{x_2,\mathcal{R}(x_2)}$, that is $x_1 \in \pi_{12}^{-}$ and $x_2 \in \pi_{12}^{-}$. From which we get that :

$$(x_2 - \Theta(x_1)) \cdot (x_1 - x_2) \leq 0 \tag{2}$$

We have :

$$\lambda(x_1, x_2) = (\nabla(x_1) - \nabla(x_2)) \cdot (x_1 - x_2)$$

$$= (\frac{x_1 - \Theta(x_1)}{\mathcal{R}(x_1)} - \frac{x_2 - \Theta(x_2)}{\mathcal{R}(x_2)}) \cdot (x_1 - x_2)$$

$$= [\frac{x_1 - x_2}{\mathcal{R}(x_1)} + \frac{\Theta(x_2) - \Theta(x_1)}{\mathcal{R}(x_2)} + (\frac{1}{\mathcal{R}(x_1)} - \frac{1}{\mathcal{R}(x_2)})(x_2 - \Theta(x_1))] \cdot (x_1 - x_2)$$

Note that $\frac{1}{\mathcal{R}(x_1)} - \frac{1}{\mathcal{R}(x_2)} \geq 0$ and, from inequations (1) and (2) we get that:

$$\lambda(x_1, x_2) \leq (\frac{x_1 - x_2}{\mathcal{R}(x_1)}) \cdot (x_1 - x_2) \leq \frac{1}{\rho}(x_1 - x_2)^2$$

which completes the proof of the lemma. $\square$

We want to define the flow induced by the vector field ∇. The vector field ∇ is not Lipschitz and not even continuous. Because the classical Lipschitz condition does not apply on ∇ it seems not possible, at first glance, to solve the differential equation :

$$C(0,x) = x, \quad \frac{d}{dt}C(t,x) = \nabla(C(t,x))$$

In order to build a "kind of solution" of this equation, we have to state the Lemmas 4.4 and 4.5 below, that are very similar to the classical theorem [1, 2] for differential equations in the Lipschitz case but with a weaker condition.

Definition 4.2 (h-sampling) *For a real number $T > 0$ we says that a strictly increasing finite sequence of real numbers $(t_i)_{i=0,n} = t_0, t_1, .., t_n$ is a h-sampling of the real interval $[0,T]$ iff :*

$$t_0 = 0, t_n = T$$

$$\forall i = 1, n \quad 0 < t_i - t_{i-1} < h$$

Definition 4.3 (Euler scheme) *Let X be a subset of $\mathbb{R}^n$.*

Given a vector map $V : X \mapsto \mathbb{R}^n$, a point $x \in X$, a real number $T > 0$ and a h-sampling $(t_i)_{i=0,n}$ of $[0,T]$, if the sequence defined by induction :

$$Y_0 = x$$

$$\forall i = 0, n-1, \quad Y_{i+1} = Y_i + (t_{i+1} - t_i)V(Y_i)$$

is well defined, that is if $Y_i \in X$ at each iteration, the Euler scheme $E^V_{t_0 \cdots t_n}(x)$ is the continuous, piecewise linear function defined on $[0,T]$ interpolating the points Y_i.

In other words, If $t \in [t_i, t_{i+1}]$,

$$E^V_{t_0 \cdots t_n}(x)(t) = Y_i + (t - t_i)V(Y_i)$$

We can now state the two next lemmas allowing the construction of $\mathfrak{C}$ out of ∇.

Lemma 4.4 *Let X be a bounded subset of $\mathbb{R}^n$, $V : X \mapsto \mathbb{R}^n$ a vector map defined on X that satisfies the conditions :*

V is bounded : $\exists B, \forall x \in X, \|V(x)\| < B$

V satisfies the pseudo-Lipschitz condition for the pseudo-Lipschitz constant K :

$$\exists K > 0, \quad \forall x_1, x_2 \in X, \quad (V(x_1) - V(x_2)) \cdot (x_1 - x_2) \le K(x_1 - x_2)^2$$

Then, there is a constant γ such that, for any real number $T > 0$ if $t_0 \cdots t_n$ (resp. $t'_0 \cdots t'_{n'}$) is a h-sampling of $[0,T]$ and x_1 (resp. x_2) a point of X such that the Euler scheme $E^V_{t_0\cdots t_n}(x_1)$ (resp. $E^V_{t'_0\cdots t'_{n'}}(x_2)$) is well defined, then we have

$$\sup_{t\in[0,T]} \|E^V_{t_0\cdots t_n}(x_1)(t) - E^V_{t'_0\cdots t'_{n'}}(x_2)(t)\|^2 \leq \|x_1 - x_2\|^2 \mathbf{e}^{2KT} + h\gamma(\mathbf{e}^{2KT} - 1)$$

Proof Under the assumptions of the lemma, let $u_0 \cdots u_N$ be a H-sampling of $[0,T]$, with $H \leq h$, which is a refinement of both $t_0 \cdots t_n$ and $t'_0 \cdots t'_{n'}$. In other words :

$$\{t_0, \cdots, t_n\} \subset \{u_0, \cdots, u_N\}$$

and

$$\{t'_0, \cdots, t'_{n'}\} \subset \{u_0, \cdots, u_N\}$$

We consider $u_i \in \{u_0, \cdots, u_{N-1}\}$.

There is j, $0 \leq j \leq n-1$ (resp. j', $0 \leq j' \leq n'-1$) such that $t_j \leq u_i < u_{i+1} \leq t_{j+1}$ (resp. $t'_{j'} \leq u_i < u_{i+1} \leq t'_{j'+1}$).

To make shorter, we use the following notation in the computation below, according to the figure below.

$$X_1 = E^V_{t_0\cdots t_n}(x_1)(t_j)$$

$$X_2 = E^V_{t'_0\cdots t'_{n'}}(x_2)(t'_{j'})$$

We use also the notation $E(x_1)$ and $E'(x_2)$ in place of $E^V_{t_0\cdots t_n}(x_1)$ and $E^V_{t'_0\cdots t'_{n'}}(x_2)$.

We have:

$$[E(x_1)(u_{i+1}) - E'(x_2)(u_{i+1})]^2$$

$$= [\ (E(x_1)(u_i) + (u_{i+1} - u_i)V(X_1)) - (E'(x_2)(u_i) + (u_{i+1} - u_i)V(X_2))\ ]^2$$

$$= [\ (E(x_1)(u_i) - E'(x_2)(u_i)) + (u_{i+1} - u_i)(V(X_1) - V(X_2))\ ]^2$$

$$= [E(x_1)(u_i) - E'(x_2)(u_i)]^2$$

$$+2[E(x_1)(u_i) - E'(x_2)(u_i)] \cdot (u_{i+1} - u_i)(V(X_1) - V(X_2))$$

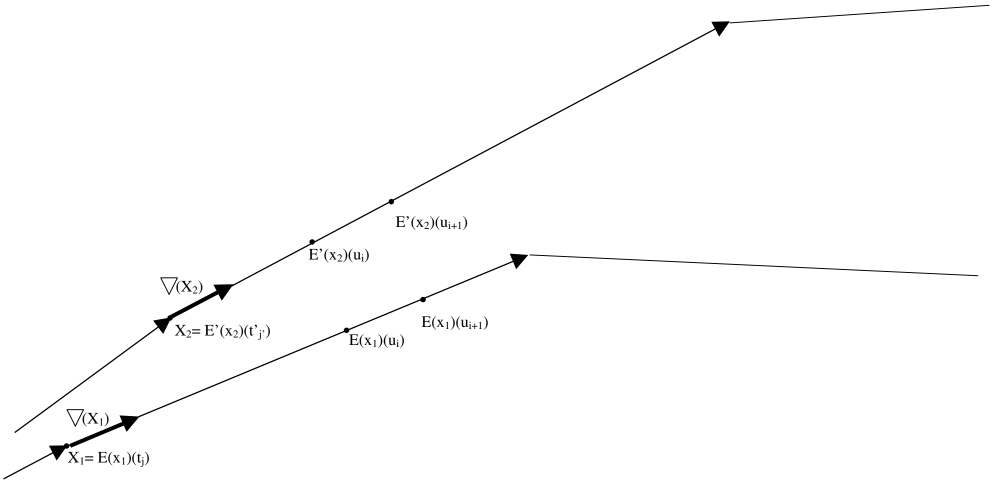

Figure 3: Euler Schemes $E^V_{t_0\cdots t_n}(x_1)$ and $E^V_{t'_0\cdots t'_{n'}}(x_2)$

$$+(u_{i+1}-u_i)^2(V(X_1)-V(X_2))^2$$

$$\leq [E(x_1)(u_i)-E'(x_2)(u_i)]^2$$

$$+2[E(x_1)(u_i)-E'(x_2)(u_i)]\cdot(u_{i+1}-u_i)(V(X_1)-V(X_2))$$

$$+h(u_{i+1}-u_i)(2B)^2$$

We bound now the second term :

$$\mathbf{W}=2[E(x_1)(u_i)-E'(x_2)(u_i)]\cdot(u_{i+1}-u_i)(V(X_1)-V(X_2))$$

$$=2[(X_1+(u_i-t_j)V(X_1))-(X_2+(u_i-t'_{j'})V(X_2)]\cdot(u_{i+1}-u_i)[(V(X_1)-V(X_2))$$

$$\leq 2(u_{i+1}-u_i)[(V(X_1)-V(X_2))(X_1-X2)+(2B)^2h)]$$

$$\leq 2(u_{i+1} - u_i)[K(X_1 - X_2)^2 + (2B)^2 h)]$$

We bound now $(X_1 - X_2)^2$:

$$(X_1 - X_2)^2 = [(E(x_1)(u_i) - E'(x_2)(u_i)) - ((u_i - t_j)V(X_1) - (u_i - t'_{j'})V(X_2))]^2$$

$$\leq [E(x_1)(u_i) - E'(x_2)(u_i)]^2 + 4hB\|E(x_1)(u_i) - E'(x_2)(u_i)\| + 4h^2B^2$$

As X is bounded, let D be a constant such that for $y_1, y_2 \in X, \|y_1 - y_2\| \leq D$. In particular :

$$\|E(x_1)(u_i) - E'(x_2)(u_i)\| \leq D$$

and

$$h \leq D$$

We have

$$(X_1 - X_2)^2 \leq [E(x_1)(u_i) - E'(x_2)(u_i)]^2 + h(4BD + 4DB^2)$$

We can now rewrite the bound on $\mathbf{W}$.

$$\mathbf{W} \leq (u_{i+1} - u_i)[2K[E(x_1)(u_i) - E'(x_2)(u_i)]^2 + 2(K(4BD + 4DB^2) + (2B)^2)h]$$

We define $\beta = 2(K(4BD + 4DB^2) + (2B)^2) + (2B)^2$. Notice that $\beta > 0$.
We can now substitute this upper bound of $\mathbf{W}$ in the initial inequation:

$$[E(x_1)(u_{i+1}) - E'(x_2)(u_{i+1})]^2 \leq [E(x_1)(u_i) - E'(x_2)(u_i)]^2(1 + 2K(u_{i+1} - u_i)) + \beta h(u_{i+1} - u_i)$$

We define now the sequence $(v_i)_{i=0,\cdots,N}$:

$$v_i = [E(x_1)(u_i) - E'(x_2)(u_i)]^2 + \frac{\beta}{2K}h$$

The inequality above can be written :

$$v_{i+1} - \frac{\beta}{2K}h$$

$$\leq v_i(1 + 2K(u_{i+1} - u_i)) - \frac{\beta}{2K}h(1 + 2K(u_{i+1} - u_i)) + \beta h(u_{i+1} - u_i)$$

that is :

$$v_{i+1} \leq v_i(1 + 2K(u_{i+1} - u_i))$$

That gives :

$$v_i \leq v_0 \prod_{k=1,i} (1 + 2K(u_k - u_{k-1}))$$

And, as both members are strictly positive :

$$\ln(v_i) \leq \ln(v_0) + \sum_{k=1,i} \ln(1 + 2K(u_k - u_{k-1}))$$

And because $\forall u \geq 0$, $\ln(1+u) \leq u$, we have :

$$\ln(1 + 2K(u_k - u_{k-1})) \leq 2K(u_k - u_{k-1})$$

We can then write, because $u_0 = 0$:

$$\ln(v_i) \leq \ln(v_0) + 2Ku_i$$

that makes :

$$v_i \leq v_0 \mathbf{e}^{2Ku_i}$$

We have :

$$v_0 = [E(x_1)(0) - E(x_2)(0)]^2 + \frac{\beta}{2K}h = (x_1 - x_2)^2 + \frac{\beta}{2K}h$$

that gives :

$$v_i \leq [(x_1 - x_2)^2 + \frac{\beta}{2K}h]\mathbf{e}^{2Ku_i}$$

and

$$[E(x_1)(u_i) - E'(x_2)(u_i)]^2 \leq [(x_1 - x_2)^2 + \frac{\beta}{2K}h]\mathbf{e}^{2Ku_i} - \frac{\beta}{2K}h$$

$$= (x_1 - x_2)^2\mathbf{e}^{2Ku_i} + h\frac{\beta}{2K}(\mathbf{e}^{2Ku_i} - 1)$$

The inequality above holds for any H-sampling $u_0, \cdots, u_N$. For a given $t \in [0, T]$ we can choose a H-sampling $u_0, \cdots, u_N$ such that, for some i, $t = u_i$. We have then:

$$\forall t \in [0, T], \quad [E(x_1)(t) - E'(x_2)(t)]^2 \leq (x_1 - x_2)^2\mathbf{e}^{2Kt} + h\frac{\beta}{2K}(\mathbf{e}^{2Kt} - 1)$$

and that ends the proof of the lemma.

□

Remark : the inequality of the Lemma above is very similar to the one used in the Lipschitz case, page 116 in [1] and page 171 in [2].

For an open set X the lemma below states the local existence and uniqueness of a Lipschitz map, uniform limit of the Euler schemes.

Lemma 4.5 *If X is a bounded subset of $\mathbb{R}^n$, and if $V : X \mapsto \mathbb{R}^n$ is a vector map defined on X that satisfies the conditions :*

(1) V is bounded : $\exists B, \forall x \in X, \|V(x)\| < B$

(2) V satisfies the pseudo-Lipschitz condition for the pseudo-Lipschitz constant K:

$$\exists K > 0, \forall x_1, x_2 \in X,\ (V(x_1) - V(x_2)) \cdot (x_1 - x_2) \leq K(x_1 - x_2)^2$$

Let be $x \in X^\circ$ and $\rho > 0$ such that $\mathbb{B}_{x,\rho} \subset X$. Then, there is a unique map

$$C : [0, \frac{\rho}{2B}] \times \mathbb{B}_{x,\frac{\rho}{2}} \mapsto \mathbb{B}_{x,\rho}$$

such that :

(i) For any $y \in \mathbb{B}_{x,\frac{\rho}{2}}$ and $\{h_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} h_i = 0$, if $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of Euler Schemes starting in y such that E_i is based on a h_i sampling of $[0, \frac{\rho}{2B}]$, then the sequence of $\{E_i\}_{i\in\mathbb{N}}$ converges uniformly toward the map $t \mapsto C(t, y)$.

(ii) The Map C is Lipschitz with respect to both variables [4]

(iii) $C(0, y) = y$

Proof Let y be in $\mathbb{B}_{x,\frac{\rho}{2}}$. For any h-sampling of $[0, \frac{\rho}{2B}]$ the Euler scheme based on this h-sampling and starting at y is well defined, because at each iteration Y_i is still in the ball $\mathbb{B}_{x,\rho} \subset X$.

Let $\{h_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} h_i = 0$. Let $\{E_i(y)\}_{i\in\mathbb{N}}$ be a sequence of Euler Schemes starting in y such that each E_i is based on a h_i sampling. Let $\epsilon > 0$ be a real number. Then there is $N \in \mathbb{N}$ such that $i > N \Rightarrow h_i < \epsilon$.

Then, we get from the lemma 4.4 that, for any $i, i' > N$:

$$\sup_{t\in[0,\frac{\rho}{2B}]} \|E_i(y)(t) - E_{i'}(y)(t)\|^2 \leq \epsilon\gamma(\mathbf{e}^{2K\frac{\rho}{2B}} - 1) \quad (3)$$

The sequence of $\{E_i(y)\}_{i\in\mathbb{N}}$ is therefore a Cauchy sequence for the *sup* norm on the complete space of continuous map $\mathbf{C}[0, \frac{\rho}{2B}] \mapsto \mathbb{B}_{x,\rho}$. The sequence of $\{E_i\}_{i\in\mathbb{N}}$ therefore converges uniformly toward a continuous map. We denote this limit : $t \mapsto C(t, y)$.

[4] and therefore uniformly continuous

$t \mapsto C(t,y)$ does not depend on the chosen sequence of $\{E_i(y)\}_{i\in\mathbb{N}}$ as the inequality (3) stands for any sequence of Euler schemes. One checks as well that the bound defined in the lemma 4.4 extends to the limit, that is for $t \in [0, \frac{\rho}{2B}]$ and $y_1, y_2 \in \mathbb{B}_{x,\frac{\rho}{2}}$ we have :

$$\|C(t,y_1) - C(t,y_2)\|^2 \leq \|y_1 - y_2\|^2 \mathbf{e}^{2K\frac{\rho}{2B}}$$

And that gives:

$$\|C(t,y_1) - C(t,y_2)\| \leq \|y_1 - y_2\| \mathbf{e}^{K\frac{\rho}{2B}}$$

Now, for each Euler Scheme $E_i(y)$ from $\{E_i(y)\}_{i\in\mathbb{N}}$, we have the Lipschitz bound with respect to t induced by the bound B on V. For $t_1, t_2 \in [0, \frac{\rho}{2B}]$:

$$\|E_i(y)(t_1) - E_i(y)(t_2)\| \leq B|t_1 - t_2|$$

This extends to the limit :

$$\|C(t_1,y) - C(t_2,y)\| \leq B|t_1 - t_2|$$

and we can write :

$$\forall t_1, t_2 \in [0, \frac{\rho}{2B}], \forall y_1, y_2 \in \mathbb{B}_{x,\frac{\rho}{2}}$$

$$\|C(t_1,y_1) - C(t_2,y_2)\| \leq B|t_1 - t_2| + \|y_1 - y_2\| \mathbf{e}^{K\frac{\rho}{2B}}$$

Both the Lipschitz property and the uniqueness are expressed by this inequality. (iii) is trivial and that completes the proof. □

From lemma 4.1, it is clear that the conditions of lemma 4.5 apply to the vector map ∇ in the set $\mathcal{O}_\rho$ with the pseudo-Lipschitz constant $K = \frac{1}{\rho}$ and the bound $B = 1$ on the map ∇. We have then locally, that is on $\cup_{\rho>0}([0,\rho] \times \mathcal{O}_\rho)$, a continuous map that is the limit of the Euler Schemes built on ∇. We denote this map by $\mathfrak{C}$.

4.2 Global existence of $\mathfrak{C}$

In order to prove a global existence of the map $\mathfrak{C}$, that is that $\mathfrak{C}$ can be extended to a single map in $\mathbb{R}^+ \times \mathcal{O}$, we need to state a few propositions.

The lemma below states a kind of semi-continuity on the map $x \mapsto \because(x)$.

Lemma 4.6 *If y is not too far from x, $\because(y)$ is not too far from $\because(x)$, in other words:*

$$\forall x \in \mathcal{O}, \quad \forall \epsilon > 0, \quad \exists \alpha > 0$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \because(y) \subset \because(x) \oplus \mathbb{B}^\circ{}_{0,\epsilon}$$

Proof For any $\epsilon > 0$, the set $\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon}$, as union of open balls, is open. The set $\mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c$ is then closed. If this set is empty, the statement of the lemma is trivially satisfied for any $\alpha > 0$. Let us assume now that it is not empty.

Because $\mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c$ is compact, there is $z \in \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c$ such that:

$$d(x, z) = d(x, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c)$$

Because $z \notin \because (x)$, we have:

$$d(x, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c) = d(x, z) > d(x, \mathcal{O}^c)$$

which entails :

$$d(x, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c) - d(x, \mathcal{O}^c) > 0$$

Let us choose $\alpha > 0$ such that :

$$\alpha < \frac{1}{2}(d(x, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c) - d(x, \mathcal{O}^c))$$

If $y \in \mathbb{B}_{x,\alpha}$, we have :

$$d(y, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c) \geq d(x, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c) - \alpha > d(x, \mathcal{O}^c) + \alpha$$

$$d(y, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})) \leq d(y, \because (x)) \leq d(x, \because (x)) + \alpha = d(x, \mathcal{O}^c) + \alpha$$

We have then

$$d(y, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})) < d(y, \mathcal{O}^c \cap (\because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon})^c)$$

which entails

$$\because (y) \subset \because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon}$$

□

Corollary 4.7 *The map $\mathcal{F}$ is upper semicontinuous.*

Proof It is a trivial consequence of the lemma and the definition of $\mathcal{F}$. □

We consider now a given point x such that $\nabla(x) \neq 0$. The next lemma states some inequalities on $\Theta(y)$ and $\nabla(y)$, when y is sufficiently near x. Recall that, when $\nabla(x) \neq 0$, we denote by $\mathbf{e}$ the unit vector colinear to $\nabla(x)$: $\mathbf{e} = \frac{\nabla(x)}{\|\nabla(x)\|}$.

Lemma 4.8 *Let be x such that $\nabla(x) \neq 0$.*

(i)

$$\forall \epsilon > 0,\ \exists \alpha > 0$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \forall z \in \because (y), z \cdot \mathbf{e} < \Theta(x) \cdot \mathbf{e} + \epsilon$$

(ii)

$$\forall \epsilon > 0,\ \exists \alpha > 0$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \Theta(y) \cdot \mathbf{e} < \Theta(x) \cdot \mathbf{e} + \epsilon$$

(iii)

$$\forall \epsilon > 0,\ \exists \alpha > 0$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \nabla(y) \cdot \mathbf{e} > \|\nabla(x)\| - \epsilon$$

Proof Recall that, if $x \neq \Theta(x)$, we denote by $\Pi(x)$ the radical plane of $\mathbb{S}_{x,\mathcal{R}(x)}$ and $\mathbb{S}_{\Theta(x),\mathcal{F}(x)}$and that $\Pi(x)$ is the plane orthogonal to $\nabla(x)$ and going through $\Theta(x)$.

$\Pi(x)$ splits the space into two half-spaces. We denote by $\Pi^+(x)$ the closed half-space that contains x. The other one is denoted $\Pi^-(x)$.

Note that, because $\because (x) \subset \mathbb{S}_{x,\mathcal{R}(x)} \cap \mathbb{B}_{\Theta(x),\mathcal{F}(x)}$, we have:

$$\because (x) \subset \Pi^-(x)$$

that is, $\forall z \in \because (x),\ z \cdot \mathbf{e} \leq \Theta(x) \cdot \mathbf{e}$. If we call $\Pi^{-\epsilon}(x)$ the open half-space of equation

$$\Pi^{-\epsilon}(x) = \{z \in \mathbb{R}^n,\ z \cdot \mathbf{e} < \Theta(x) \cdot \mathbf{e} + \epsilon\}$$

we get from lemma 4.6 :

$$\forall \epsilon > 0,\ \exists \alpha > 0,$$

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \because (y) \subset \because (x) \oplus \mathbb{B}^\circ{}_{0,\epsilon} \subset \Pi^-(x) \oplus \mathbb{B}^\circ{}_{0,\epsilon} = \Pi^{-\epsilon}(x)$$

This proves (i) and entails that the center $\Theta(y)$ of the minimal ball enclosing $\because (y)$ is in this half-space : $\Theta(y) \in \Pi^{-\epsilon}(x)$, that is (ii).

Now we know that $\mathcal{R}$ is 1-Lipschitz. Let $\epsilon > 0$ be a positive number. There is α_1 such that :

$$y \in \mathbb{B}_{x,\alpha_1} \Rightarrow |1 - \frac{\mathcal{R}(x)}{\mathcal{R}(y)}| \cdot \|\nabla(x)\| < \frac{\epsilon}{2} \tag{4}$$

$$\frac{\mathcal{R}(x)}{\mathcal{R}(y)} \leq 2 \tag{5}$$

Using (ii), there is α_2 such that :

$$y \in \mathbb{B}_{x,\alpha_2} \Rightarrow \Theta(y) \cdot \mathbf{e} < \Theta(x) \cdot \mathbf{e} + \frac{\epsilon \mathcal{R}(x)}{8}$$

Let $\alpha = \min(\alpha_1, \alpha_2, \frac{\epsilon \mathcal{R}(x)}{8})$ Then, if $y \in \mathbb{B}_{x,\alpha}$, we have :

$$(y - \Theta(y)) \cdot \mathbf{e} - (x - \Theta(x)) \cdot \mathbf{e} > -\frac{\epsilon \mathcal{R}(x)}{4}$$

or, using (4),

$$\frac{(y - \Theta(y)) \cdot \mathbf{e}}{\mathcal{R}(y)} - \frac{(x - \Theta(x)) \cdot \mathbf{e}}{\mathcal{R}(y)} > -\frac{\mathcal{R}(x)}{\mathcal{R}(y)} \frac{\epsilon}{4} \geq -\frac{\epsilon}{2}$$

that gives :

$$\nabla(y) \cdot \mathbf{e} - \nabla(x) \cdot \mathbf{e} + (1 - \frac{\mathcal{R}(x)}{\mathcal{R}(y)})(\nabla(x) \cdot \mathbf{e}) > -\frac{\epsilon}{2}$$

that is, using (5)

$$y \in \mathbb{B}_{x,\alpha} \Rightarrow \nabla(y) \cdot \mathbf{e} - \nabla(x) \cdot \mathbf{e} > -\epsilon$$

and that proves (iii). □

For a point $x \in \mathcal{O}$, an unit vector v and an angle $\theta \in [0, \pi/2]$, $K_{v,\theta}(x)$ denotes the cone of apex x, axis v and half angle θ :

$$K_{v,\theta}(x) = \{y \in R^n, y = x \ \text{ or } \ y \neq x \ \text{ and } \ \frac{v \cdot (y - x)}{\|y - x\|} \geq \cos(\theta)\}$$

In order to prove that $\nabla(x)$ is the right derivative of $t \mapsto \mathfrak{C}(t, x)$ at $t = 0$, we first show the lemma below, that is, for t sufficiently small, $\mathfrak{C}(t, x)$ remains in a cone of apex x and axis e with an arbitrary small half angle.

Lemma 4.9 *For any $x \in \mathcal{O}$ such that $\nabla(x) \neq 0$ and any $\theta > 0$ there is $\alpha > 0$ such that, for any $t \in [0, \alpha]$, $\mathfrak{C}(t, x)$ is in $K_{e,\theta}(x)$, in other words :*

$$\forall x \in \mathcal{O}, \ \nabla(x) \neq 0, \ \forall \theta > 0, \ \exists \alpha > 0,$$

$$\forall t \in [0, \alpha], \quad \mathfrak{C}(t, x) \in K_{e,\theta}(x)$$

Proof Let us define $\rho = \frac{\mathcal{R}(x)}{2}$. We have of course $\mathbb{B}_{x,\rho} \subset \mathcal{O}_\rho$.

From lemma 4.8 we know that there is a positive number α_1, $0 < \alpha_1 < \rho$ such that :

$$y \in \mathbb{B}_{x,\alpha_1} \Rightarrow \nabla(y) \cdot \mathbf{e} > \frac{\|\nabla(x)\|}{2} \tag{6}$$

Let us denote by ϕ the angle in $[0, \frac{\pi}{2}]$ defined by : $\cos(\phi) = \frac{\|\nabla(x)\|}{2}$. Because $\|\nabla(y)\| \leq 1$ and $\nabla(y) \cdot \mathbf{e} > \cos(\phi)$, any Euler scheme defined on the interval $[0, \alpha_1]$ (or any smaller interval) and starting at x will remain in the cone $K_{e,\phi}(x)$ of apex x, axis e and half angle ϕ.

Notice that $0 < \cos(\phi) \leq \frac{1}{2}$ and therefore $\frac{\pi}{3} \leq \phi < \frac{\pi}{2}$. We denote by ϵ the positive number defined by :

$$\epsilon = \frac{\sin(\theta)\tan^2(\theta)\|\nabla(x)\|}{4\tan(\phi)}$$

From lemmas 4.1 and 4.8 we know that there is a positive number $\alpha_2 > 0$, with $\alpha_2 < \alpha_1$ and $\alpha_2 < 1$ such that (lemma 4.1):

$$y \in \mathbb{B}_{x,\alpha_2}, \quad y \neq x \quad \Rightarrow (\nabla(y) - \nabla(x)) \cdot \frac{y - x}{\|y - x\|} \leq \epsilon \tag{7}$$

and (lemma 4.8):

$$y \in \mathbb{B}_{x,\alpha_2} \Rightarrow (\nabla(y) - \nabla(x)) \cdot e > -\epsilon \tag{8}$$

Let us consider a point $y^* \in \mathbb{B}_{x,\alpha_2}$ such that $y^* \in K_{e,\phi}(x) - K_{e,\theta}(x)$.

Let us define the unit vector f orthogonal to e and the angle $\psi \in (\theta, \phi]$ such that:

$$\frac{y^* - x}{\|y^* - x\|} = \cos(\psi)e + \sin(\psi)f$$

we get from (7):

$$(\nabla(y^*) - \nabla(x)) \cdot (\cos(\psi)e + \sin(\psi)f) \leq \epsilon$$

and, because $\nabla(x) \cdot f = 0$:

$$\nabla(y^*) \cdot (\sin(\psi)f) \leq \epsilon - (\nabla(y^*) - \nabla(x)) \cdot (\cos(\psi)e)$$

that gives, using (8):

$$\nabla(y^*) \cdot (\sin(\psi)f) < \epsilon(1 + \cos(\psi)) < 2\epsilon$$

which gives, using the definition of ϵ and the fact that $\psi > \theta$:

$$\nabla(y^*) \cdot f < \frac{\tan^2(\theta)\|\nabla(x)\|}{2\tan(\phi)} \tag{9}$$

This inequality is used below.

For any $y \in \mathbb{B}_{x,\alpha_2}$, we use the following notation. $y^{||} = ((y-x)\cdot e)e$ and $y^{\perp} = (y-x) - y^{||}$. We have from this definition $y = x + y^{||} + y^{\perp}$

Similarly, we define : $\nabla^{||}(y) = (\nabla(y)\cdot e)e$ and $\nabla^{\perp}(y) = \nabla(y) - \nabla^{||}(y)$. We have from this definition $\nabla(y) = \nabla(y)^{||} + \nabla(y)^{\perp}$

Let us define the map $g : \mathbb{B}_{x,\alpha_2} \cap K_{e,\phi}(x) \mapsto \mathbb{R}$ by:

$$g(y) = (y^{\perp})^2 - \tan^2(\theta)(y^{||})^2$$

For any $y \in \mathbb{B}_{x,\alpha_2} \cap K_{e,\phi}(x)$, we have :

$$g(y) \leq 0 \Leftrightarrow y \in K_{e,\theta}(x)$$

Notice that, because $\alpha_2 \leq 1$, g is 2-Lipschitz.

We consider now, for any $k \in \mathbb{N}$, the Euler Scheme E_k starting at x with the h_k-sampling of $[0, \alpha_2]$ with 2^k constant steps, $t_{i+1} - t_i = h_k$:

$$h_k = 2^{-k}\alpha_2 \;\text{ and }\; t_i = i \cdot 2^{-k}\alpha_2$$

We denote by y_i the i^{th} iterate of E_k, that is $y_i = E_k(t_i)$.

We have :

$$y_{i+1}^{\perp} = y_i^{\perp} + h\nabla^{\perp}(y_i)$$

and

$$y_{i+1}^{||} = y_i^{||} + h\nabla^{||}(y_i)$$

Let us assume that, for some i, $1 \leq i \leq 2^k$, we have $g(y_i) > 0$.

We have then :

$$g(y_{i+1}) - g(y_i) = 2h(y_i^{\perp}\cdot\nabla^{\perp}(y_i)) + h^2(\nabla^{\perp}(y_i))^2 - 2h(\tan^2(\theta)y_i^{||}\cdot\nabla^{||}(y_i)) - h^2\tan^2(\theta)(\nabla^{||}(y_i))^2$$

$$\leq 2h(y_i^{\perp}\cdot\nabla^{\perp}(y_i) - \tan^2(\theta)y_i^{||}\cdot\nabla^{||}(y_i)) + h^2$$

Because $y_i \notin K_{e,\theta}(x)$, we can use the inequality (9) above with $y^* = y_i$. Note that $y_i^{\perp} = \|y_i^{\perp}\| f$ and $\nabla(y_i)\cdot f = \nabla^{\perp}(y_i)\cdot f$.

That gives :

$$y_i^{\perp}\cdot\nabla^{\perp}(y_i) < \|y_i^{\perp}\|\frac{\tan^2(\theta)\|\nabla(x)\|}{2\tan(\phi)}$$

Because $y_i \in K_{e,\phi}(x)$, we have $\|y_i^{\perp}\| \leq \tan(\phi)\|y_i^{||}\|$ and we have:

$$y_i^{\perp}\cdot\nabla^{\perp}(y_i) < \|y_i^{||}\|\frac{\tan^2(\theta)\|\nabla(x)\|}{2} \tag{10}$$

on the other hand, we have $y_i \in \mathbb{B}_{x,\alpha_1}$ and we get from (6):

$$\nabla(y_i) \cdot \mathbf{e} > \frac{\|\nabla(x)\|}{2}$$

and that gives :

$$y_i^{\|} \cdot \nabla^{\|}(y_i) > \|y_i^{\|}\| \frac{\|\nabla(x)\|}{2}$$

We can then write, using (10):

$$y_i^{\perp} \cdot \nabla^{\perp}(y_i) - \tan^2(\theta) y_i^{\|} \cdot \nabla^{\|}(y_i) < 0$$

which finally leads to :

$$g(y_{i+1}) - g(y_i) \leq h^2$$

We know that $g(y_0) = g(x) = 0$ and $g(y_1) = g(y_0 + h\nabla(x)) < 0$. Because g is 2-Lipschitz and E_k is 1-Lipschitz, if i is the first index for which $g(y_i) > 0$, $g(y_i) \leq 2h$. Then for any $i' > i$, we have:

$$g(y_{i'}) \leq 2h + (i' - i)h^2 \leq 2h + 2^k h^2 \leq (2\alpha_2 + \alpha_2^2) \cdot 2^{-k}$$

Because the sequence of Euler schemes E_i converges uniformly toward $t \mapsto \mathfrak{C}(t, x)$:

$$\forall \eta > 0, \exists k, \forall t \in [0, \alpha_2], \|\mathfrak{C}(t, x) - E_k\| < \eta \quad \text{and} \quad (2\alpha_2 + \alpha_2^2) \cdot 2^{-k} < \eta$$

and, because g is 2-Lipschitz we have:

$$\forall t \in [0, \alpha_2], \quad g(\mathfrak{C}(t, x)) < 2\eta + \eta$$

We get then

$$\forall t \in [0, \alpha_2], \quad g(\mathfrak{C}(t, x)) \leq 0$$

and that completes the proof of the lemma.

$\square$

Lemma 4.10 *The map $t \mapsto \nabla(\mathfrak{C}(t, x))$ is the right derivative of $t \mapsto \mathfrak{C}(t, x)$. In other words, we have :*

(i) $\nabla(x)$ is the right derivative of $t \mapsto \mathfrak{C}(t, x)$ at $t = 0$, that is:

$$\forall x \in \mathcal{O}, \forall \epsilon > 0, \exists \alpha > 0,$$

$$\forall t \in [0, \alpha], \quad \|\mathfrak{C}(t, x) - (x + t\nabla(x))\| < t\epsilon$$

(ii)

$$\forall x \in \mathcal{O}, \forall \epsilon > 0, \forall t \in [0, \frac{\mathcal{R}(x)}{4}], \exists \alpha > 0,$$

$$\forall t' \in [t, t+\alpha], \quad \|\mathfrak{C}(t', x) - (\mathfrak{C}(t, x) + (t'-t)\nabla(\mathfrak{C}(t, x)))\| < (t'-t)\epsilon$$

Proof If $\nabla(x) = 0$ any Euler scheme starting at x is stationary : $\forall t \geq 0, \mathfrak{C}(t, x) = x$ and the proposition (i) of the lemma is trivially satisfied. Let assume now that $\nabla(x) \neq 0$.

Let $\epsilon > 0$ be the positive number given in the lemma. We can assume, without loss of generality that $\epsilon < \frac{\|\nabla(x)\|}{2}$.

Let $\theta > 0$ be an angle such that $\tan(\theta) < \frac{\epsilon}{4}$ and $\cos(\theta) > \frac{1}{2}$.

From lemma 4.9 one can choose α_1 such that

$$\forall t \in [0, \alpha_1], \quad \mathfrak{C}(t, x) \in K_{e, \frac{\theta}{2}}(x)$$

And from lemma 4.8 one can choose α_2 such that

$$\forall y \in \mathbb{B}_{x, \alpha_2}, \quad \nabla(y) \cdot \mathbf{e} > \|\nabla(x)\| - \frac{\epsilon}{2} > \frac{\|\nabla(x)\|}{2} \tag{11}$$

Let E_k be a sequence of Euler schemes, each E_k being based on an h_k sampling of an interval $[0, \alpha]$, with $\alpha \leq \min(\alpha_1, \alpha_2)$ and such that :

$$\lim_{k \to \infty} h_k = 0$$

Then, we get from lemma 4.4 and lemma 4.5 that the sequence E_k converges uniformly toward $t \mapsto \mathfrak{C}(t, x)$ and that there is a constant γ such that :

$$\|E_k(x)(t) - \mathfrak{C}(t, x)\| \leq \sqrt{h_k \gamma(\mathbf{e}^{2Kt} - 1)} \tag{12}$$

On another side, from $\forall y \in \mathbb{B}_{x, \alpha_2}, \quad \nabla(y) \cdot \mathbf{e} > \frac{\|\nabla(x)\|}{2}$ we get that :

$$E_k(x)(t) \cdot e > t\frac{\|\nabla(x)\|}{2} \tag{13}$$

And, because $\forall t \in [0, \alpha_1], \quad \mathfrak{C}(t, x) \in K_{e, \frac{\theta}{2}}(x)$ we get from the inequations (12) and (13) above that for any $\eta > 0$, there is $K(\eta)$ such that

$$\forall k \geq K(\eta), \quad \forall t \in [\eta, \alpha_1], \quad E_k(t) \in K_{e, \theta}(x) \tag{14}$$

From Lemma 4.1 we can choose α_3 such that :

$$\forall y \in \mathbb{B}_{x, \alpha_3}, \quad (\nabla(y) - \nabla(x)) \cdot \frac{y - x}{\|y - x\|} \leq \frac{\epsilon}{8} \tag{15}$$

Let $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$. Let be $t \in [0, \alpha]$ and a sequence of Euler scheme E_k as above and such that, for some $i(k)$, $t_{i(k)} = t$. Let y_j be an iterate of the Euler scheme E_k such that $y_j \in \mathbb{B}_{x,\alpha} \cap K_{e,\theta}(x)$. We use, as in the proof of lemma 4.9 the decomposition of $y_j - x$ on the orthogonal unit vectors e and f :

$$\frac{y_j - x}{\|y_j - x\|} = \cos(\psi)e + \sin(\psi)f$$

with $0 \leq \psi \leq \theta$ which gives, from (15):

$$(\nabla(y_j) - \nabla(x)) \cdot (\cos(\psi)e + \sin(\psi)f) \leq \frac{\epsilon}{8}$$

and that gives :

$$\cos(\psi)\nabla(y_j) \cdot e \leq \cos(\psi)\nabla(x) \cdot e - \sin(\psi)\nabla(y_j) \cdot f + \frac{\epsilon}{8}$$

that is :

$$\nabla(y_j) \cdot e \leq \nabla(x) \cdot e - \tan(\psi)\nabla(y_j) \cdot f + \frac{\epsilon}{8\cos(\psi)}$$

Because $0 \leq \psi \leq \theta$, we have $\tan(\psi) < \frac{\epsilon}{4}$ and $\cos(\psi) > \frac{1}{2}$ and :

$$\nabla(y_j) \cdot e \leq \nabla(x) \cdot e + \frac{\epsilon}{2}$$

We have then, using (11):

$$\|\nabla(x)\| - \frac{\epsilon}{2} \leq \nabla(y_j) \cdot e \leq \|\nabla(x)\| + \frac{\epsilon}{2} \tag{16}$$

Because (hereafter i stand for $i(k)$):

$$y_i - x = \sum_{j=0,i-1} (t_{j+1} - t_j)\nabla(y_j)$$

we have

$$(y_i - x) \cdot e = \sum_{j=0,i-1} (t_{j+1} - t_j)[\nabla(y_j) \cdot e]$$

Let us choose η such that $0 < \eta < t = t_i$ and $\eta < t\frac{\epsilon}{8}$. According to (14), for any $k > K(\eta)$, and any $t_j > \eta$, if y_j is the iterate of the Euler scheme E_k corresponding to t_j, then we have $y_j \in \mathbb{B}_{x,\alpha} \cap K_{e,\theta}(x)$.

We can write :

$$(y_i - x) \cdot e = \sum_{0 \leq j \leq i-1, t_j \leq \eta} (t_{j+1} - t_j)[\nabla(y_j) \cdot e] + \sum_{0 \leq j \leq i-1, t_j > \eta} (t_{j+1} - t_j)[\nabla(y_j) \cdot e]$$

and, from (16), we have :

$$(\|\nabla(x)\| - \frac{3\epsilon}{4})t_i \leq (y_i - x) \cdot e \leq (\|\nabla(x)\| + \frac{3\epsilon}{4})t_i$$

and that entails

$$|(y_i - [x + t_i \nabla(x)]) \cdot e| \leq t_i \frac{3\epsilon}{4}$$

Because the sequence of E_k converges toward $t \mapsto \mathfrak{C}(t,x)$ the inequality remains true for the limit:

$$\forall t \in [0,\alpha], \quad |(\mathfrak{C}(t,x) - [x + t\nabla(x)]) \cdot e| \leq t\frac{3\epsilon}{4}$$

and the proposition (i) of the lemma is a consequence of this inequation and the lemma 4.9.

It follows from the definition of $\mathfrak{C}$, lemma 4.4 and lemma 4.5 that :

$$\forall t \in [0, \frac{\mathcal{R}(x)}{4}], \forall t' \in [t, \frac{\mathcal{R}(x)}{2}],$$

$$\mathfrak{C}(t',x) = \mathfrak{C}(t'-t, \mathfrak{C}(t,x))$$

And the proposition (ii) is obtained by applying the proposition (i) at $x' = \mathfrak{C}(t,x)$. □

The previous lemma state that the map $t \mapsto \nabla(\mathfrak{C}(t,x))$ is the right derivative of the map $t \mapsto \mathfrak{C}(t,x)$. Similarly, the lemma below claims that the map $t \mapsto \nabla(\mathfrak{C}(t,x))^2$ is the right derivative of the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$.

Lemma 4.11 *The map $t \mapsto \nabla(\mathfrak{C}(t,x))^2$ is the right derivative of the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$. In other words , we have :*

$$\forall x \in \mathcal{O}, \forall \epsilon > 0, \forall t \in [0, \frac{\mathcal{R}(x)}{4}], \exists \alpha > 0,$$

$$\forall t' \in [t, t+\alpha], \quad |\mathcal{R}(\mathfrak{C}(t',x)) - [\mathcal{R}(\mathfrak{C}(t,x)) + (t'-t)\nabla(\mathfrak{C}(t,x))^2]| < (t'-t)\epsilon$$

Proof As for lemma 4.10, it is enough to prove the proposition for $t = 0$.

Again, the proposition is trivial if $\nabla(x) = 0$. We can then assume that $\nabla(x) \neq 0$ and we define the angle ψ defined by $\cos(\psi) = -\|\nabla(x)\| = -\nabla(x) \cdot e$, with $\frac{\pi}{2} < \psi \leq \pi$.

From lemma 4.6, we can choose $\alpha_1 > 0$ such that

$$\forall t \in [0, \alpha_1], \quad \because (\mathfrak{C}(t,x)) \subset \because (x) \oplus \mathbb{B}^\circ{}_{0, |\cos(\psi)| \frac{\epsilon \mathcal{R}(x)}{8}}$$

Let us pick $t \in [0, \alpha_1]$ and $z' \in \because (\mathfrak{C}(t,x))$.

We have :

$$d(x,z') \leq \mathcal{R}(x)(1+\frac{\epsilon}{8}) \tag{17}$$

and, as in lemma 4.8

$$(z'-x)\cdot e \leq \mathcal{R}(x)\cos(\psi)(1-\frac{\epsilon}{8}) < 0 \tag{18}$$

If we call ψ' the angle between the vectors e and $z'-x$ we have :

$$\cos(\psi') = \frac{(z'-x)\cdot e}{d(x,z')}$$

and from (17) and (18) (remember that $\cos(\psi) < 0$):

$$\cos(\psi') \leq \cos(\psi)\frac{1-\frac{\epsilon}{8}}{1+\frac{\epsilon}{8}} \leq \cos(\psi)(1-\frac{\epsilon}{4}) \tag{19}$$

The minimality condition on the definition of $\mathbb{B}_{\Theta(x),\mathcal{F}(x)}$ and the compactness of $\because (x)$ imply that there exists at least one $z \in \because (x)$ such that the angle between the vector e and $z-x$ is ψ.

We have :

$$\mathcal{R}(x) = d(x,z)$$

$$\mathcal{R}(\mathfrak{C}(t,x)) \leq d(\mathfrak{C}(t,x),z)$$

and :

$$\mathcal{R}(\mathfrak{C}(t,x)) = d(\mathfrak{C}(t,x),z')$$

$$\mathcal{R}(x) \leq d(x,z')$$

which gives :

$$d(\mathfrak{C}(t,x),z')^2 - d(x,z')^2 \leq \mathcal{R}(\mathfrak{C}(t,x))^2 - \mathcal{R}(x)^2 \leq d(\mathfrak{C}(t,x),z)^2 - d(x,z)^2 \tag{20}$$

Using lemma 4.10, we can write :

$$\mathfrak{C}(t,x) = x + t\nabla(x) + o(t)$$

which gives :

$$[d(\mathfrak{C}(t,x),z)]^2 - [d(x,z)]^2$$

$$= [z-(x+t\nabla(x)+o(t))]^2 - [z-x]^2$$

$$= -2t(z-x)\nabla(x) + o(t) = -2t\mathcal{R}(x)\cos(\psi)\|\nabla(x)\| + o(t)$$

$$= 2t\mathcal{R}(x)\|\nabla(x)\|^2 + o(t)$$

Similarly, we get :

$$[d(\mathfrak{C}(t,x),z')]^2 - [d(x,z')]^2 = -2t\mathcal{R}(x)\cos(\psi')\|\nabla(x)\| + o(t)$$

and, using (19):

$$[d(\mathfrak{C}(t,x),z')]^2 - [d(x,z')]^2 \geq 2t\mathcal{R}(x)\|\nabla(x)\|^2(1-\frac{\epsilon}{4}) + o(t)$$

That is, from (20) :

$$2t\mathcal{R}(x)\|\nabla(x)\|^2(1-\frac{\epsilon}{4}) + o(t) \leq \mathcal{R}(\mathfrak{C}(t,x))^2 - \mathcal{R}(x)^2 \leq 2t\mathcal{R}(x)\|\nabla(x)\|^2 + o(t)$$

By choosing $\alpha \leq \alpha_1$ small enough, the $o(t)$ in the above equations can be made smaller than $\frac{\epsilon t}{2}$ for any $t \in [0,\alpha]$ and we have :

$$|\mathcal{R}(\mathfrak{C}(t,x))^2 - \mathcal{R}(x)^2 - 2t\mathcal{R}(x)\nabla(x)^2| < \epsilon t$$

which means that the right derivative of the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))^2$ at $t=0$ is $2\mathcal{R}(x)[\nabla(x)]^2$. We can conclude that the right derivative of the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ at $t=0$ is $\nabla(x)^2$ and that completes he proof of the lemma. □

Lemma 4.12 *We have the following equalities:*

$$\forall x \in \mathcal{O}, \quad \forall t \in [0, \mathcal{R}(x)/4]$$

$$\mathfrak{C}(t,x) = x + \int_0^t \nabla(\mathfrak{C}(\tau,x))d\tau$$

$$\mathcal{R}(\mathfrak{C}(t,x)) = \mathcal{R}(x) + \int_0^t \nabla(\mathfrak{C}(\tau,x))^2 d\tau$$

The curve image of $t \mapsto \mathfrak{C}(t,x)$ is rectifiable and its arc length is an increasing function of t given by:

$$s(t) = \int_0^t \|\nabla(\mathfrak{C}(\tau,x))\|d\tau$$

Notice that the lemma asserts as well that these expressions are well defined, that is the functions under the symbol $\int$ are Lebesgues integrable.

Proof The maps $t \mapsto \mathfrak{C}(t,x)$ and $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ are Lipschitz. Therefore they are absolutely continuous which entails ([3] page 49-54), that they are differentiable almost everywhere and that they are the integral of their derivatives. On another hand, we know that these maps are also right differentiable (everywhere).

Almost everywhere, their derivatives exist and have to be equal to the right derivative. This proves the two first equations. Notice that because $t \mapsto \mathfrak{C}(t,x)$ is Lipschitz, it defines a rectifiable arc, whose arc length is the right hand of the third equation. $\square$

It results from the previous lemma that the map $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ is increasing and this allows to show the lemma below, that is the global existence of $\mathfrak{C}$.

Lemma 4.13 *The local map $\mathfrak{C}$ extends to a global map on $\mathbb{R}^+ \times \mathcal{O}$:*

$$\mathfrak{C} : \mathbb{R}^+ \times \mathcal{O} \mapsto \mathcal{O}$$

$\mathfrak{C}$ is continuous, it is 1-Lipschitz with respect to the real variable t and, for any ρ, its restriction to $\mathbb{R}^+ \times \mathcal{O}_\rho$ is Lipschitz with respect to both variables.

Proof For any $x \in \mathcal{O}$, $\mathfrak{C}$ is defined on $[0, \mathcal{R}(x)/2] \times \mathbb{B}_{x,\mathcal{R}(x)/4}$. But, because $t \mapsto \mathcal{R}(\mathfrak{C}(t,x))$ is increasing, we know that $\mathcal{R}(\mathfrak{C}(t,x)) \geq \mathcal{R}(x)$ and it is therefore possible to extends $t \mapsto \mathfrak{C}(t,x)$ at least on $[\mathcal{R}(x)/4,\ \ \mathcal{R}(x)/4+\mathcal{R}(x)/4]$. This can be repeated an arbitrary number of time, each time extending the right end of the interval of definition of $t \mapsto \mathfrak{C}(t,x)$ of at least $\mathcal{R}(x)/4$. This means that $\mathfrak{C}$ is defined on $\mathbb{R}^+ \times \mathcal{O}$. $\square$

4.3 The map $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing

In order to get the Homotopy equivalence of $\mathcal{O}$ and $\mathcal{M}$ using the map $\mathfrak{C}$, we need to prove that the map $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing. We start with a few lemmas.

Lemma 4.14 *Let $T > 0$ and $R_0 > 0$ be positive numbers and $f : [0,T] \mapsto \mathbb{R}$ an upper semicontinuous function such that:*

$$\forall t \in [0,T], \quad f(t) \leq f(0) \quad \textit{and} \quad \exists t \in [0,T], \quad f(t) < f(0)$$

Let $\Re : [0,T] \mapsto \mathbb{R}$ be an increasing continuous map such that :

$$\forall t \in [0,T], \quad f(0) < \Re(t)$$

and satisfying the integral equation:

$$\forall t \in [0,T], \quad \Re(t) = R_0 + \int_0^t \sqrt{1 - \frac{f(u)^2}{\Re(u)^2}}du$$

Let $\Re_0 : [0,T] \mapsto \mathbb{R}$ be the increasing continuous map defined by :

$$\Re_0(t) = \sqrt{R_0^2 + 2t\sqrt{R_0^2 - f(0)^2} + t^2}$$

Then, we have :

$$\exists t \in [0,T], \quad \Re(t) > \Re_0(t)$$

Proof First, note that

$$\forall t \in [0,T], \quad f(0) < \Re_0(t)$$

and a simple computation shows that :

$$\forall t \in [0,T], \quad \Re_0(t) = R_0 + \int_0^t \sqrt{1 - \frac{f(0)^2}{\Re_0(u)^2}} du$$

The equation on $\Re$ can be expressed as a fix point of an operator ω_f . Given a continuous increasing map g defined on $[0,T]$ such that $\forall t \in [0,T], \quad f(t) < g(t)$ $\omega_f(g)$ is defined by:

$$\forall t \in [0,T], \quad \omega_f(g)(t) = g(0) + \int_0^t \sqrt{1 - \frac{f(u)^2}{g(u)^2}} du$$

And we have :

$$\Re = \omega_f(\Re)$$

Note that ω_f is monotonic in the following sense. If two continuous maps g_1 and g_2 are such that $\forall t \in [0,T], f(t) < g_1(t)$ and $f(t) < g_2(t)$ we check easily that if:

$$\forall t \in [0,T], \quad g_1(t) \leq g_2(t)$$

We have :

$$\forall t \in [0,T], \quad \omega_f(g_1)(t) \leq \omega_f(g_2)(t)$$

We start with the function $g_0 = \Re_0$ and we build the sequence of functions defined by:

$$g_{i+1} = \omega_f(g_i)$$

Recall that :

$$\forall t \in [0,T], \quad f(t) \leq f(0) \quad \text{and} \quad \exists t \in [0,T], \quad f(t) < f(0)$$

And, because f is upper semicontinuous, there is $\epsilon > 0$ and $\alpha > 0$ such that

$$\exists t_0 \in [0,T] \ \ \forall t \in [t_0 - \alpha, t_0 + \alpha], \quad f(t) \leq f(0) - \epsilon$$

and, if we compare the expressions giving $g_0 = \Re_0$ and $g_1 = \omega_f(g_0) = g(0) + \int_0^t \sqrt{1 - \frac{f(u)^2}{g_0(u)^2}} du$, this entails :

$$\forall t \in [0,T], \quad \Re_0(t) = g_0(t) \leq g_1(t) \quad \text{and} \quad \exists t \in [0,T], \quad \Re_0(t) = g_0(t) < g_1(t)$$

We have then, from the monotony of ω_f and by induction on i:

$$\forall i \in \mathbb{N},$$

$$\forall t \in [0,T], \quad g_i(t) \leq g_{i+1}(t) \quad \text{and} \quad \exists t \in [0,T], \quad \Re_0(t) < g_i(t)$$

On another hand, we have:

$$\forall i \in \mathbb{N}, \forall t \in [0,T], \quad g_i(t) \leq R_0 + t$$

Then, for a fixed t, the real number sequence $i \mapsto g_i(t)$ is increasing and bounded. Therefore, g_i converges pointwise toward a map g. From the Beppo Levi theorem of monotone convergence ([3] page 36, [4] page 18) we get that:

$$\forall t \in [0,T], \lim_{i \to \infty} \omega_f(g_i)(t) = \omega_f(g(t))$$

which gives

$$g = \omega_f(g)$$

and of course,$\forall i, \quad \forall t, \quad g_i(t) \leq g(t)$ and:

$$\forall t \in [0,T], \quad \Re_0(t) \leq g(t) \quad \text{and} \quad \exists t \in [0,T], \quad \Re_0(t) < g(t)$$

In order to complete the proof of the Lemma, we still have to prove that $\Re = g$, that is the uniqueness of the solution of the fix point equation.

Recall that $\forall t \in [0,T]$

$$0 \leq f(t) \leq f(0), \quad g(t) \geq g(0) = R_0 \geq 0 \quad \text{and} \quad \Re(t) \geq \Re(0) = R_0 \geq 0$$

Because g and $\Re$ are continuous and f upper semicontinuous, $t \mapsto \frac{f(t)}{g(t)}$ and $t \mapsto \frac{f(t)}{\Re(t)}$ are upper semicontinuous. Therefore, there is $\epsilon > 0$ such that [5]:

$$\forall t \in [0,T], \quad \frac{f(t)}{g(t)} < 1 - \epsilon$$

[5]Recall that an upper semicontinuous function defined on a compact set reaches its least upper bound.

and

$$\forall t \in [0,T], \quad \frac{f(t)}{\Re(t)} < 1 - \epsilon$$

On another hand, there is an uniform bound K to the derivative of the map:

$$G \mapsto \sqrt{1 - \frac{f(t)^2}{G^2}} \quad \text{for} \quad G \in [\frac{1}{1-\epsilon} f(t), \infty) \quad \text{and} \quad t \in [0,T]$$

Then, for any real number α, $0 < \alpha < 1$, and t, $0 \leq t < T$, if $t + \frac{\alpha}{K} \leq T$ we have :

$$\sup_{h \in [0, \frac{\alpha}{K}]} |\sqrt{1 - \frac{f(t+h)^2}{\Re(t+h)^2}} - \sqrt{1 - \frac{f(t+h)^2}{g(t+h)^2}}| \leq K \sup_{h \in [0, \frac{\alpha}{K}]} |\Re(t+h) - g(t+h)|$$

and therefore :

$$\sup_{h \in [0, \frac{\alpha}{K}]} | \int_t^{t+h} \sqrt{1 - \frac{f(\tau)^2}{\Re(\tau)^2}} d\tau - \int_t^{t+h} \sqrt{1 - \frac{f(\tau)^2}{g(\tau)^2}} d\tau | \leq \alpha \sup_{h \in [0, \frac{\alpha}{K}]} |\Re(t+h) - g(t+h)|$$

Because $g = \omega_f(g)$ and $\Re = \omega_f(\Re)$, this inequality entails that :

$$\Re(t) = g(t) \Rightarrow \sup_{h \in [0, \frac{\alpha}{K}]} |\Re(t+h) - g(t+h)| \leq \alpha \sup_{h \in [0, \frac{\alpha}{K}]} |\Re(t+h) - g(t+h)|$$

which gives, because $\alpha < 1$:

$$\Re(t) = g(t) \Rightarrow \sup_{h \in [0, \frac{\alpha}{K}]} |\Re(t+h) - g(t+h)| = 0$$

that is :

$$\Re(t) = g(t) \Rightarrow \forall h \in [0, \frac{\alpha}{K}], \quad \Re(t+h) = g(t+h)$$

We have of course $g(0) = \Re(0) = R_0$ and that entails $g = \Re$ which ends the proof.

$\square$

Lemma 4.15 *We have, for any $x, y \in \mathcal{O}$, such that $\nabla(x) \neq 0$:*

$$\mathcal{R}(y) \leq \sqrt{\mathcal{R}(x)^2 + 2d(x,y)\sqrt{\mathcal{R}(x)^2 - \mathcal{F}(x)^2} + d(x,y)^2}$$

Proof Let be $z \in \therefore (x)$ and for $y \neq x$, we define $d = d(x,y)$. Using the unit vector $u = \frac{y-x}{d}$ we can write :

$$\|z - y\|^2 = \|z - (x + du)\|^2 = ((z-x) - du)^2 = (z-x)^2 - 2d(z-x) \cdot u + d^2$$

which gives

$$\mathcal{R}(y)^2 \leq \inf_{z \in \because(x)} \|z - y\|^2 = \mathcal{R}(x)^2 + d^2 - 2d \sup_{z \in \because(x)} (z - x) \cdot u$$

For a unit vector u, $\|u\| = 1$, we define $r(u) = \sup_{z \in \because(x)} (z - x) \cdot u$.

Because $\nabla(x) \neq 0$, for some unit vector u, $r(u) < 0$ and there is a ball of radius $\sqrt{\mathcal{R}(x)^2 - r(u)^2}$ enclosing $\because (x)$. Among these balls, the one with the smallest radius is $\mathbb{B}_{\Theta(x),\mathcal{F}(x)}$ and it corresponds to the unit vector $e = \frac{\nabla(x)}{\|\nabla(x)\|}$ which then minimize $u \mapsto r(u)$:

$$\forall u, \quad \|u\| = 1, \quad r(e) \leq r(u)$$

We have then :

$$\mathcal{R}(y)^2 \leq \mathcal{R}(x)^2 + d^2 - 2dr(u) \leq \mathcal{R}(x)^2 + d^2 - 2dr(e)$$

And, because $r(e) = -\sqrt{\mathcal{R}(x)^2 - \mathcal{F}(x)^2}$, we have:

$$\mathcal{R}(y)^2 \leq \mathcal{R}(x)^2 + 2d(x, y)\sqrt{\mathcal{R}(x)^2 - \mathcal{F}(x)^2} + d(x, y)^2$$

and we obtain the inequality of the lemma.

□

Lemma 4.16 *If $\nabla(\mathfrak{C}(T, x)) \neq 0$, The arc length map $t : [0, T] \mapsto [0, s(T)]$, $t \mapsto s(t)$ given in lemma 4.12 is strictly increasing and we can define the inverse function: $t : [0, s(T)] \mapsto [0, T]$, $t : s \mapsto t(s)$.*

In other words, one can parameterize the curve by the arc length. Using this parameterization, one have :

$$\mathcal{R}(\mathfrak{C}(t(s), x)) = \mathcal{R}(x) + \int_0^s \|\nabla(\mathfrak{C}(t(\sigma), x))\| d\sigma$$

Proof For a given $x \in \mathcal{O}$, let be T such that $\|\nabla(\mathfrak{C}(T, x))\| \neq 0$. If, for some $t_0 < T$, we had $\|\nabla(\mathfrak{C}(t_0, x))\| = 0$, the map $t \mapsto \mathfrak{C}(t, x)$ would be stationary for $t \geq t_0$ which contradicts $\|\nabla(\mathfrak{C}(T, x))\| \neq 0$. This means that as far as $t < T$, we have $\|\nabla(\mathfrak{C}(t, x))\| > 0$. From the lemma 4.12 we get that, on $[0, T]$, the arc length map s is strictly increasing.

If we denote by S the arc length of the image of $[0, T]$ by $t \mapsto \mathfrak{C}(t, x)$, one can define the map $t : [0, S] \mapsto [0, T]$ inverse of the arc length map s.

From the right derivatives of the maps $t \mapsto s(t)$ and $t \mapsto \mathcal{R}(\mathfrak{C}(t, x))$ as given in lemmas 4.11 and 4.12, one can check the right differentiability and compute the right derivative of the map $s \mapsto \mathcal{R}(\mathfrak{C}(t(s), x))$.

We get :

$$\mathcal{R}(\mathfrak{C}(t(s + h), x)) = \mathcal{R}(\mathfrak{C}(t(s), x)) + h\|\nabla(\mathfrak{C}(t, x))\| + o(h)$$

As in the proof of lemma 4.12, because this map is Lipschitz, it is differentiable almost everywhere. It is then the indefinite integral of its right derivative, which completes the proof.

□

Lemma 4.17 *For fixed x, $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is an increasing, right continuous function.*

Proof Let be $x \in \mathcal{O}$ and $T > 0$ such that $\nabla(\mathfrak{C}(T,x)) \neq 0$ and let us define:

$$f_{\max} = \sup_{t \in [0,T]} \mathcal{F}(\mathfrak{C}(t,x))$$

From the upper semi-continuity of $\mathcal{F}$ we know that

$$\exists t^* \in [0,T] \text{ s.t. } \mathcal{F}(\mathfrak{C}(t,x)) = f_{\max}$$

Let us denote by t, $t : s \mapsto t(s)$, the inverse of the arc length function defined in lemma 4.16 of the arc defined by the map $t \mapsto \mathfrak{C}(t,x)$ starting at $\mathfrak{C}(t^*,x)$. If S denote the length of the arc image of $[t^*,T]$ by $t \mapsto \mathfrak{C}(t,x)$, we have :

$$t(0) = t^*, \quad t(S) = T,$$

$$\forall s \in [0,S], \quad \mathcal{R}(\mathfrak{C}(t(s),x)) = \mathcal{R}(\mathfrak{C}(t^*,x)) + \int_0^s \|\nabla(\mathfrak{C}(t(\sigma),x))\| d\sigma$$

$$= \mathcal{R}(\mathfrak{C}(t^*,x)) + \int_0^s \sqrt{1 - \frac{\mathcal{F}(\mathfrak{C}(t(\sigma),x))^2}{\mathcal{R}(\mathfrak{C}(t(\sigma),x))^2}} d\sigma$$

And, because $s \geq d(\mathfrak{C}(t^*,x), \mathfrak{C}(t(s),x))$ we get from lemma 4.15:

$$\mathcal{R}(\mathfrak{C}(t(s),x)) \leq \sqrt{\mathcal{R}(\mathfrak{C}(t^*,x))^2 + 2s\sqrt{\mathcal{R}(\mathfrak{C}(t^*,x))^2 - \mathcal{F}(\mathfrak{C}(t^*,x))^2} + s^2} \tag{21}$$

If we assume now that there is t_0, $t^* < t_0 \leq T$ such that $\mathcal{F}(\mathfrak{C}(t_0,x)) < f_{\max}$ the map $\Re$ defined by

$$\Re(s) = \mathcal{R}(\mathfrak{C}(t(s),x))$$

meets the conditions of lemma 4.14 with :

$$f(s) = \mathcal{F}(\mathfrak{C}(t(s),x))$$

and

$$\Re_0(s) = \sqrt{\mathcal{R}(\mathfrak{C}(t^*,x))^2 + 2s\sqrt{\mathcal{R}(\mathfrak{C}(t^*,x))^2 - \mathcal{F}(\mathfrak{C}(t^*,x))^2} + s^2}$$

and we get a contradiction between the statement of lemma 4.14 and inequality (21) above. We can conclude that $s \mapsto f(s) = \mathcal{F}(\mathfrak{C}(t(s),x))$ can not be decreasing on $[t^*,T]$.

This means that, for any $T > 0$ such that $\nabla(\mathfrak{C}(T,x)) \neq 0$, $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ reaches its maximum at $t = T$. $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is then increasing on $[0,T]$.

If there is some $T_{\max}$ which is the supremum (the least upper bound) of the T for which $\nabla(\mathfrak{C}(T,x)) \neq 0$, we know that, from the lower semicontinuity of $\|\nabla(\mathfrak{C}(T,x))\|$, we must have $\nabla(\mathfrak{C}(T_{\max},x)) = 0$. $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is then increasing on $[0, T_{\max})$ and stationary on $[T_{\max}, \infty)$. From the upper semi-continuity of $\mathcal{F}$ we get that :

$$\mathcal{F}(\mathfrak{C}(T_{\max},x)) \geq \lim_{t \to T_{\max}, t < T_{\max}} \mathcal{F}(\mathfrak{C}(t,x))$$

And this finally proves that $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing on $\mathbb{R}^+$.

The right continuity of $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is an immediate consequence of the fact that it is upper semicontinuous and increasing. □

4.4 Main theorem

Proposition 4.18 *If D is an upper bound of the diameter of $\mathcal{O}$, then : $\forall x \in \mathcal{O}$, $\mathfrak{C}(D,x) \in \mathcal{M}$.*

Proof Let us assume that for some $x \in \mathcal{O}$, $\mathfrak{C}(D,x) \notin \mathcal{M}$, in other words, $\mathcal{F}(\mathfrak{C}(D,x)) = 0$. Because the map $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing, this implies that $\forall t \in [0, D]$, $\mathcal{F}(\mathfrak{C}(t,x)) = 0$, and we have then, using lemma 4.12 :

$$\mathcal{R}(\mathfrak{C}(D,x)) = \mathcal{R}(x) + D$$

and this contradicts the fact that D is an upper bound of the diameter of $\mathcal{O}$. □

The last proposition allows us to define the map $f : \mathcal{O} \mapsto \mathcal{M}$ as $f(x) = \mathfrak{C}(D,x)$. We can now state the main theorem.

Theorem 4.19 *Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^n$ and $\mathcal{M}$ its Medial Axis.*
$\mathcal{O}$ and $\mathcal{M}$ have same homotopy type.

Proof From the continuity of $\mathfrak{C}$ (lemma 4.13), we know that the map H,

$$H : [0,1] \times \mathcal{O} \mapsto \mathcal{O}$$

given by :

$$H(t,x) = \mathfrak{C}(Dt,x)$$

defines an homotopy between the identity of $\mathcal{O}$ and f.

On another hand, $t \mapsto \mathcal{F}(\mathfrak{C}(t,x))$ is increasing (lemma 4.17) which implies that, if $x \in \mathcal{M}$ then, $\forall t$, $\mathfrak{C}(t,x) \in \mathcal{M}$.

It follows that the restriction $H_{\mathcal{M}} : [0,1] \times \mathcal{M} \mapsto \mathcal{M}$ of H to $\mathcal{M}$ defines an homotopy between the identity of $\mathcal{M}$ and the restriction $f_{\mathcal{M}}$ of f to $\mathcal{M}$.

In other words:

- $f : \mathcal{O} \mapsto \mathcal{M}$ is homotopic to the identity in $\mathcal{O}$.

- $f_{\mathcal{M}} : \mathcal{M} \mapsto \mathcal{M}$, restriction of f to $\mathcal{M}$ is homotopic (in $\mathcal{M}$) to the identity in $\mathcal{M}$.

Let us recall the characterization of homotopy equivalence given in [11] [6] and [12][7] :

Two spaces X and Y are said to have the same homotopy type if there are continuous maps $f : X \mapsto Y$ and $g : Y \mapsto X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y.

We take here for g the canonical inclusion map (i.e. $g : \mathcal{M} \mapsto \mathcal{O}$ is the identity in $\mathcal{M} \subset \mathcal{O}$) and we have both $g \circ f = f$ and $f \circ g = f_{\mathcal{M}}$.

□

5 Applications

In this section, we sketch shortly what kind of application one could expect from this work. The theoretical framework of the proof covers in a single description the case of the interior of a (non necessary smooth) solid and the case of the complement of a finite number of points [8].

Surface reconstruction algorithms aim to compute an approximation of a Solid from a set of sampled points on its boundary. For smooth boundaries some authors [5, 6], prove that the reconstructed surface is homeomorphic to the sampled one when the sampling density is small enough with respect to the lfs (local feature size) that is the distance between the boundary and the Medial Axis. However, this distance vanishes on non smooth objects. In order to get some results on the topology of the reconstructed surface for the non smooth case, it may be helpful to get hints on the kind of convergence that one can expect from the Medial Axis of the (complement of) the sampled points (that is the Voronoi diagram) toward the Medial Axis of the object when the sampling density increases, without any assumption on the smoothness or the lfs of the object.

Notes that $\mathcal{M}$ could be equivalently defined by $\mathcal{M} = \{x \in \mathcal{O}, \mathcal{F}(x) > 0\}$. We denote by $\mathcal{M}_\epsilon$ the set defined by :

$$\mathcal{M}_\epsilon = \{x \in \mathcal{O}, \mathcal{F}(x) \geq \epsilon\}$$

We have of course the following :

$$\mathcal{M} = \bigcup_{\epsilon > 0} \mathcal{M}_\epsilon$$

and :

$$\mathcal{M}_\epsilon \subset \mathcal{O}_\epsilon$$

[6] pages 171-172
[7] pages 108
[8] At least in a large bounded cavity.

The map $\mathcal{F}$ is upper semicontinuous (corollary 4.7) which means that

$$\forall \epsilon \in \mathbb{R}, \quad \{x \in X, \mathcal{F}(x) < \epsilon\} \quad \text{is open}$$

that is $\mathcal{M}_\epsilon^c$ is open and $\mathcal{M}_\epsilon$ is therefore a closed set.

A well known problem with the computation of approximations of the Medial Axis is its sensitivity to noise : arbitrary small perturbations in the boundary may result in large modification of the Medial Axis. With respect to this problem, $\mathcal{M}_\epsilon$ is more stable and we think that it could be more relevant than $\mathcal{M}$ for effective computations and that it may help to give convergence results for the above mentioned surface reconstruction algorithms.

Another object that comes naturally from the framework introduced in the proof could be used for shape classification and recognition. Let us call it the *center* of $\mathcal{O}$, and denote it by $\mathcal{C}$, defined by:

$$\mathcal{C} = \bigcap_{t \in \mathbb{R}} \mathfrak{C}(t, \mathcal{O})$$

It seems to us that $\mathcal{C}$ carries the essence of the shape.

6 Conclusion

This work establishes for the first time the Homotopy equivalence between any bounded open subset of $\mathbb{R}^n$ and its Medial Axis. This result give a theoretically sound justification to the significance of the Medial Axis in shape classification and reconstruction. Moreover, some object introduced in the proof may help in giving variant of the Medial Axis Transform that would be more stable or that would capture essential features of the object and therefore more relevant for shape classification and recognition.

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