ABOUT THE STABILITY OF DECISION MACHINES.

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1. Introduction

In this talk we consider a decision procedure that is an algorithm which, on a certain input, associates the answer *yes* or *no*. For example the input is a real symmetric matrix A and we want to know whether A is positive definite or not ... To take such a decision we consider a decision machine. We consider here decision machines over the real numbers. They are described by four different kinds of nodes (we use here the BSS model as it is described in [5] or in [4]): an input node, the input is a vector $x \in \mathbf{R}^n$, computation nodes where real polynomials $f(x)$ are computed, branching nodes defined by polynomial inequalities $f(x) \geq 0$, and an output node where the output is *yes* or *no*, or *accept* or *reject*.

Let us give an example of such a machine. We want to know if $(x_1, x_2) \in \mathbf{R}^2$ satisfies the system of inequalities: $x_1 \geq 1$ and $x_1^2(x_1 - 1) \geq x_2^2$. This is achieved by the following decision machines:

$\mathbf{R}^n$ considered as the set of inputs for our machine is the disjoint union of two subsets denoted by $\mathcal{Y}$ and $\mathcal{N}$. We suppose that our machine decides $\mathcal{Y}$, that is: for any input in $\mathcal{Y}$ (resp. $\mathcal{N}$) the output is *yes* (resp. *no*).

We call *ill-posed* the instances in $\mathbf{R}^n$ arbitrarily close to $\mathcal{Y}$ and $\mathcal{N}$. The set of such instances is the boundary of $\mathcal{Y}$ and $\mathcal{N}$ and is denoted by

$$\Sigma = \partial\mathcal{Y} = \partial\mathcal{N}.$$

It is important to notice that this concept of ill-posedness is a geometric concept: it depends on the partition $\mathbf{R}^n = \mathcal{Y} \cup \mathcal{N}$ and not on the way we decide this set i.e. on the machine used to perform this decision.

Associated to the concept of ill-posedness is the concept of *condition number*. The condition number of a given instance $x \in \mathbf{R}^n$ is equal to the inverse of the distance to ill-posed instances: we denote it by

$$\mu(x)^{-1} = d(x, \Sigma) = \inf_{y \in \Sigma} \|x - y\|.$$

This definition is coherent with Renegar's one in [22]. According to this definition the concept of condition number is also a geometric concept, independent on the particular algorithm we are considering.

In our real machine we use the arithmetic over the real numbers. However, in real life computations, we use floating point numbers and floating point arithmetic instead of real numbers. For this reason instead of computing a polynomial like $f(x)$ we compute an approximate quantity $f(x) + \epsilon$ where ϵ is a small number.

The problem, we are studying here, is the following: we have an exact decision machine designed over the real numbers and also an approximate machine which is designed similarly as the exact one but where computations are made approximately. Suppose that we have taken a decision on input $x \in \mathbf{R}^n$ with the approximate machine. Is it a good decision?

To study this question we adopt a backward analysis viewpoint. We will show that, under a certain hypothesis, the decision taken on x by the approximate machine is identical to the decision taken by the exact machine on a nearby input $y \in \mathbf{R}^n$. We will also prove that, if the instance x is well conditioned, then the decisions taken by the exact and the approximate machines on the input x are the same.

An interesting fact appearing in this study is the concept of unstable instance. For such instances our theorem is not valid: unstable instances depend on the machine, this is not a geometric concept.

But let us be more precise.

2. Main results.

Suppose that the computation path on input $x \in \mathbf{R}^n$ in the approximate machine is described by the inequalities

$$f_i(x) + \epsilon_i \geq 0 \text{ for each } 1 \leq i \leq p,$$

and

$$f_i(x) + \epsilon_i > 0 \text{ for each } p+1 \leq i \leq m,$$

with $f : \mathbf{R}^n \to \mathbf{R}^m$ a polynomial map. To f and $x \in \mathbf{R}^n$ we associate the two following invariants:

$$\Gamma(f,x) = \sup_{k\geq 2} \left\| \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}$$

which is always finite when f is analytic and

$$\delta(f,x) = \max\left(1, \left\| (Df(x)Df(x)^* + Diag(4f_i(x)))^{-1} \right\|\right).$$

We also let $\delta(f,x) = \infty$ when the matrix $Df(x)Df(x)^* + Diag(4f_i(x))$ is singular. Here A^* denotes the adjoint of A, $\| \ \|$ is the operator norm associated with the usual Euclidean norms in $\mathbf{R}^n$ and $\mathbf{R}^m$ and $Diag(d_i)$ is the diagonal matrix with diagonal entries $d_1, d_2, \ldots$

Theorem 1. *Let $x \in \mathbf{R}^n$ and $\epsilon \in \mathbf{R}^m$ be given and such that*

$$\|\epsilon\| \leq \frac{1}{12(1+\Gamma(f,x))\delta(f,x)}.$$

If

$$f_i(x) + \epsilon_i \geq 0 \text{ for each } 1 \leq i \leq p,$$

and

$$f_i(x) + \epsilon_i > 4\delta(f,x)\|\epsilon\|^2 \text{ for each } p+1 \leq i \leq m,$$

then there exists $y \in \mathbf{R}^n$ with

$$f_i(y) \geq 0 \text{ for each } 1 \leq i \leq p,$$

and

$$f_i(y) > 0 \text{ for each } p+1 \leq i \leq m,$$

and such that

$$\|x - y\| \leq 2\delta(f,x)^{1/2}\|\epsilon\|.$$

The signification of this theorem is the following: when $\delta(f,x)$ is finite and ϵ small enough then, the decision taken on input x by the approximate machine is identical to the decision taken by the exact machine on a nearby input y. Moreover, the path followed by x in the approximate machine and the path followed by y in the exact machine are the same. Notice that this theorem makes precise the words "ϵ small enough" and "nearby input". An important consequence of this theorem is the following

Corollary 2. *Under the same hypothesis, if the condition number at x satisfies*

$$\mu(x) < \frac{1}{2\delta(f,x)^{1/2}\|\epsilon\|}$$

then the decisions taken on input x by the exact and by the approximate machine are the same.

3. An example.

In the following example the set $\mathcal{Y} \subset \mathbf{R}^2$ we want to decide is defined by the following inequalities:

$$x_1 \geq 1 \text{ and } x_1^2(x_1 - 1) - x_2^2 \geq 0.$$

We consider the two machines previously described:

First machine: we first test whether $x_1^2(x_1 - 1) - x_2^2 \geq 0$ and then $x_1 \geq 1$,

Second machine: we first test whether $x_1 \geq 1$ and then $x_1^2(x_1 - 1) - x_2^2 \geq 0$.

Let us now consider the input $(x_1, x_2) = (0, 0)$. We have $x_1^2(x_1 - 1) - x_2^2 = 0$ so that a small perturbation of this computation gives either a nonnegative answer or a negative answer. In the first case the answer to the first test is *yes* then the answer to the second test $x_1 = 0 \geq 1$ is *no* so that $(0, 0)$ is rejected. This computation path is described by two polynomials

$$f_1(x_1, x_2) = x_1^2(x_1 - 1) - x_2^2 \text{ and } f_2(x_1, x_2) = x_1 - 1.$$

An easy computation shows that $\delta(f, x) = \infty$ so that our theorem doesn't apply in this case. In the second case, the answer to the first test is *no* and $(0, 0)$ is rejected. The computation path is described by the polynomial

$$f(x_1, x_2) = x_1^2(x_1 - 1) - x_2^2$$

and we also have $\delta(f, x) = \infty$.

In both cases our theorem doesn't apply. This is coherent with the fact that a positive perturbation of $x_1^2(x_1 - 1) - x_2^2$ at $(x_1, x_2) = (0, 0)$ cannot be the value of this polynomial on any nearby vector. Here $(0, 0)$ is an isolated point of the cubic $x_1^2(x_1 - 1) - x_2^2 = 0$ and for any (x_1, x_2) with $\|(x_1, x_2)\| < 1$ and different from $(0, 0)$ one has

$$x_1^2(x_1 - 1) - x_2^2 < 0.$$

On the second machine the computation path on $(0, 0)$ and on any nearby input on both the approximate and the exact machine is given by $x_1 - 1 < 0$ so that (x_1, x_2) is rejected. Here $f(x_1, x_2) = x_1 - 1$ and $\delta(f, x)$ is finite for any x.

The conclusion of this academic (but interesting) example is the following: we call *backward unstable* any input $x \in \mathbf{R}^n$ such that $\delta(f, x) = \infty$ for the map f defined by the computation path on input x in the approximate machine. The previous example shows that this is not a geometric concept: it is related to the considered approximate machine.

A second remark is the following: an ill-posed input may be backward stable. In our examples the ill-posed instances are defined by

$$x_1^2(x_1 - 1) = x_2^2 \text{ and } x_1 \geq 1.$$

They are backward stable for the two machines.

A third and last remark about our example: the decision taken on input $(0, 0)$ is the right decision in both machines. This input is well posed and consequently *forward stable*.

4. A word about the proof.

Theorem 1 is a consequence of a theorem about the stability of analytic systems of inequalities which may be found in [10].

Theorem 3. *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be an analytic function. Let $x \in \mathbf{R}^n$ and $\epsilon \in \mathbf{R}^m$ be given and such that*

$$\|\epsilon\| \leq \frac{1}{12(1+\Gamma(f,x))\delta(f,x)}.$$

If $f_i(x) + \epsilon_i \geq 0$ for each $i \in I = \{1, \ldots, p\}$ and $f_i(x) + \epsilon_i > 4\delta(f,x)\|\epsilon\|^2$ for each $i \in J = \{p+1, \ldots, m\}$ then there exists $y \in \mathbf{R}^n$ with $f_i(y) \geq 0$ for each $i \in I$, $f_i(y) > 0$ for each $i \in J$ and such that

$$\|x - y\| \leq 2\delta(f,x)^{1/2}\|\epsilon\|.$$

To prove this last theorem we associate to the system of inequalities $f(y) \geq 0$ the underdetermined system $F_i(y,t) = f_i(y) - t_i^2$, $1 \leq i \leq m$, which is a system of m equations in $m+n$ unknowns: $(x,t) \in \mathbf{R}^n \times \mathbf{R}^m$. We see easily that $F(y,t) = 0$ implies $f(y) \geq 0$. To prove the existence of a zero (y,t) for F with y close to x we show that Newton's sequence $(x_{k+1}, t_{k+1}) = N_F(x_k, t_k)$ starting at $(x_0, t_0) = (x, \sqrt{f(x) + \epsilon})$ is converging. Its limit is a zero for F. This process is interesting because it provides a very efficient way to compute y.

Newton's method for underdetermined systems of equations has been introduced for the first time in 1966 by Ben-Israel [3]. This iteration is defined by

$$N_F(z) = z - DF(z)^\dagger F(z), \quad z_{k+1} = N_F(z_k),$$

where z_0 is given. Here we suppose $DF(z)$ onto and we denote by $DF(z)^\dagger$ its Moore-Penrose inverse. More generally, for a surjective linear operator $L : \mathbf{E} \to \mathbf{F}$ between two Euclidean spaces, we define its Moore-Penrose inverse by $L^\dagger = L^*(LL^*)^{-1}$. For any $b \in \mathbf{F}$, $L^\dagger b$ is the minimum norm solution z of the equation $L(z) = b$.

To prove the convergence of the sequence $(x_{k+1}, t_{k+1}) = N_F(x_k, t_k)$ we use a theorem due to M. Shub and S. Smale 1996 [24].

5. Relations with other results.

We only consider here the case of $\geq$ inequalities. We can measure how far is $f(x)$ from positivity in considering the quantity $(\max(0, -f_i(x)))_{1 \leq i \leq m}$. We denote this quantity by $f(x)^-$ ($f(x)^+$ is defined similarly). The inequality $f(x) \geq 0$ is satisfied exactly if and only if $f(x)^- = 0$. Taking $\epsilon = f(x)^-$ in Theorem 1 gives

Corollary 4. *Let $x \in \mathbf{R}^n$ be such that*

$$\|f(x)^-\| \leq \frac{1}{12(1+\Gamma(f,x))\delta(f,x)}.$$

There exists $y \in \mathbf{R}^n$ with $f(y) \geq 0$ and

$$\|x - y\| \leq 2\delta(f,x)^{1/2}\|f(x)^-\|.$$

An interesting and classical case is given by linear inequalities: $f(x) = b - Ax$ with $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. Here $\Gamma(f,x) = 0$ and we write $\delta(A,b,x)$ instead of $\delta(f,x)$. In this context we get:

Corollary 5. *Let $x \in \mathbf{R}^n$ be such that*

$$\|(Ax-b)^+\| \leq \frac{1}{12\delta(A,b,x)}.$$

There exists $y \in \mathbf{R}^n$ with $Ay \leq b$ and

$$\|x-y\| \leq 2\delta(A,b,x)^{1/2}\|(Ax-b)^+\|.$$

It is interesting to compare this last result with Hoffman's Theorem. It was published for the first time in 1952 [15], and reconsidered by Güler, Hoffman and Rothblum in 1995 [13].

Theorem 6. *(Hoffman) Let $A \in \mathbf{R}^{m\times n}$. Then there exists a scalar $K(A)$, such that for each $b \in \mathbf{R}^m$ for which the set $\{x' \in \mathbf{R}^n \ : \ Ax' \leq b\} \neq \emptyset$ and for each $x \in \mathbf{R}^n$ there exists $y \in \mathbf{R}^n$ such that $Ay \leq b$ and*

$$\|x-y\| \leq K(A)\|(Ax-b)^+\|.$$

The conclusions in Hoffman's Theorem and in our Corollary 5 are slightly different.

1. Both Corollary 5 and Theorem 6 give an estimate of $\min_{Ay\leq b}\|x-y\|$ in terms of $\|(Ax-b)^+\|$. This estimate is uniform in b and x in Theorem 6 and depends on b and x in Corollary 5.
2. In Corollary 5 we prove the existence of a $y \in \mathbf{R}^n$ with $Ay \leq b$. This existence is assumed in Hoffman's Theorem.
3. In Corollary 5 we assume a limitation on the size of the perturbation $\epsilon = (Ax-b)^+$. There is no such limitation in Hoffman's Theorem provided the feasible set is nonempty.

There have been a number of generalizations of Hoffman's Theorem to nonlinear cases: Robinson 1975 [23] extended this bound to a system of convex inequalities defining a convex and bounded set with nonempty interior, Mangasarian 1985 [21] considered a closed convex set defined by a system of finitely many differentiable convex inequalities, Auslender and Crouzeix 1988 [1] extended Mangasarian's result to convex nondifferentiable functions. A recent paper in these directions is "Error Bounds for Convex Inequality Systems" by Lewis and Pang 1998 [17]. It contains a very detailled bibliography.

Another generalization of Hoffman's Theorem to a nonlinear and nonconvex case may be obtained via Lojasiewiecz's Inequality. This result has been proved for the first time by Lojasiewiecz 1964 [18] for semianalytic or semialgebraic sets and functions and then extended to the subanalytic case by Hironaka 1973 [14]. A good exposé of such questions is contained in Bierstone-Milman 1988 [6].

Theorem 7. *(Lojasiewiecz) Let K be a compact and subanalytic set contained in $\mathbf{R}^n$. Let $u, v : K \to \mathbf{R}$ be continuous subanalytic functions. If $u^{-1}(0) \subset v^{-1}(0)$, there exist $\alpha > 0$ and an integer $N > 0$ such that, for all $x \in K$,*

$$\alpha\ |v(x)|^N \leq |u(x)|.$$

Taking $v(x) = dist(x, f^{\geq 0})$ and $u(x) = \|f(x)^-\|$ gives, using Lojasiewiecz's Inequality,

$$\alpha\ dist(x, f^{\geq 0})^N \leq \|f(x)^-\|.$$

This is a very general and powerful method but it gives a Hölderian inequality. It has already been used in the context of optimization theory by Dedieu 1992 [11] (penalty functions), Luo-Luo 1994 [19] (polynomial inequalities), Luo-Pang 1995 [20] (analytic inequalities).

Hoffman's Theorem has been extended by Ioffe (1979) [16] to locally Lipschitz functions using Clarke's subgradient, and more recently by Azé, Corvellec and Lucchetti (1999) [2] who consider the case of lower semicontinuous functions defined over Banach spaces.

Let us first define Clarke's subgradient [7].

Definition 8. Let $g : R^n \to R$ be locally Lipschitz. For any x and $v \in R^n$ the directional derivative of f at x in the direction v is :

$$Dg(x,v) = \limsup_{\substack{y \to x \\ \lambda \to 0_+}} \frac{g(y+\lambda v) - g(y)}{\lambda}.$$

The generalized gradient $\partial g(x)$ of g at x is the set of vectors x^* in R^n satisfying

$$< v \mid x^* > \leq Dg(x,v)$$

for each $v \in R^n$.

We now recall Rademacher's Theorem. It asserts that a function which is Lipschitz on an open subset of R^n is differentiable almost everywhere on that subset. Based on this result, the generalized gradient has the following caracterisation: for any set S of measure zero, we have

$$\partial g(x) = co\{\lim_{i\to\infty} \nabla g(x_i) \mid g \ is \ differentiable \ at \ x_i, \ x_i \notin S, \ x_i \to x\}.$$

Theorem 9. *(Ioffe) Let $g : R^n \to R$ be locally Lipschitz, let A be the set of zeros of g and $a \in A$. Let us denote $\|x^*\|_1 = \sum_{i=1}^n |x_i^*|$. Suppose there exist ϵ and $c > 0$ such that for each $x \in \bar{B}(a,\epsilon) \setminus A$ and $x^* \in \partial g(x)$ the inequality $\|x^*\|_1 \geq c$ holds. Then, for each $x \in \bar{B}(a,\epsilon/2)$ we have*

$$cdist(x,A) \leq |g(x)|.$$

Taking $g(x) = \|f(x)^-\|$ gives, using Ioffe's Theorem,

$$cdist(x, f^{\geq 0}) \leq \|f(x)^-\|.$$

6. Conclusion.

A first draught of this talk is contained in an unpublished paper by F. Cucker (Hong Kong) and J.-P. Dedieu [8]. The difficulty we try to analyse is the difference between exact and approximate computations in presence of branching nodes. Some serious difficulties aren't studied here: the exact machine on input x may return an output after a finite number of steps while the approximate machine loops forever. We don't study this problem here: our main hypothesis is the existence of an output on the approximate machine.

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