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CONVERGENCE RATE FOR THE APPROXIMATION OF THE LIMIT LAW OF WEAKLY INTERACTING PARTICLES: APPLICATION TO THE BURGERS EQUATION

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Abstract. In this paper we construct a stochastic particle method for the Burgers equation with a monotone initial condition; we prove that the convergence rate is $O(1/\sqrt{N}+\sqrt{\Delta t})$ for the $L^1(I\!R\times\Omega)$-norm of the error. To obtain that result, we link the PDE and the algorithm to a system of weakly interacting stochastic particles; the difficulty of the analysis comes from the discontinuity of the interaction kernel which is equal to the Heaviside function.

In [4], we show how the algorithm and the result extend to the case of non monotone initial conditions for the Burgers equation. We also treat the case of nonlinear PDE's related to particle systems with Lipschitz interaction kernels.

Our next objective is to adapt our methodology to the (more difficult) case of the 2-D inviscid Navier-Stokes equation.

1. Introduction

In this paper and in [4], we study the convergence rate of a stochastic particle method for the numerical solving of the nonlinear McKean-Vlasov Equations

$$\left\{\begin{array}{l}\frac{d}{dt}<\mu_t,f>=<\mu_t,L_{(\mu_t)}f>\\ \mu_{t=0}=\mu_0,\end{array}\right. \tag{1}$$

where μ_t is a probability measure, f is any real function of class $\mathcal{C}^\infty$ with a compact support, and the operator $L_{(\mu)}$ is defined by

$$L_{(\mu)}f(x)\;=\;\frac{1}{2}\left(\int_{I\!R}s(x,y)d\mu(y)\right)^2f''(x)\;+\;\left(\int_{I\!R}b(x,y)d\mu(y)\right)f'(x)\;. \tag{2}$$

The method is based upon the simulation of a weakly interacting particle system. Its construction and its analysis rely on the propagation of chaos theory.

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As shown in Osada [20] and in Marchioro and Pulvirenti [18] the incompressible 2D Navier-Stokes equation describes the limit behaviour of a weakly interacting particle system with a singular interaction kernel $b(x,y)$. The numerical simulation of such a particle system coincides with the well-known Chorin's random vortex method. Thus, it might be useful to start from the propagation of chaos to analyse the convergence rate of the random vortex methods. Recent publications on these methods are those of Chorin [6], Chorin and Marsden [5], Goodman [11], Hald ([13] and [14]), Puckett [21], Roberts [23] and Long [17], e.g.; see also the bibliography in [6] and in the different contributions of [12], in particular those by Chorin and Hald; see Puckett [22] and Bernard-Talay-Tubaro [2] for a stochastic particle method for convection-reaction-diffusion equations with a nonlinear reaction term. In Bossy & Talay [4], we study the convergence rate of the empirical distribution function of a system of simulated particles to the distribution function of the solution of (1) in the case where the interaction kernels $b(\cdot,\cdot)$ and $s(\cdot,\cdot)$ are bounded and Lipschitz, and $s(\cdot,\cdot)$ is bounded below by a strictly positive constant; under additional hypotheses, an estimate is also given for an approximation of the density of the solution to (1).

In view of treating the case of singular interaction kernels in the future, in this paper we construct and we analyse a stochastic particle system for the Burgers equation

$$\left\{\begin{array}{l}\dfrac{\partial V}{\partial t}(t,x) \;=\; \dfrac{1}{2}\sigma^2\dfrac{\partial^2 V}{\partial x^2}(t,x) \;-\; V(t,x)\,\dfrac{\partial V}{\partial x}(t,x),\;\; (t,x)\;\in\;(0,T]\;, \\ V(0,x)=V_0(x)\;. \end{array}\right. \tag{3}$$

For this particle system the interaction kernel $b(x,y)$ is discontinuous: it is the Heaviside function.

We then construct an algorithm of simulation of the particle system. The error analysis of this stochastic particle method deals with a kernel which is neither smooth (as in the case considered in [4]) nor singular (as in the case of the random vortex methods for the Navier-Stokes equation). Considering the Burgers equation is natural for a second reason: in the numerical analysis literature, this equation is a common test-case for algorithms solving some nonlinear PDE's of this type (in particular the Navier-Stokes equation), especially to test their performances when the viscosity term tends to 0.

The simulation of the particles involves the discretization of a stochastic differential system. We fix a time discretization step Δt of the time interval $[0,T]$. Let $\overline{V}_t(\cdot)$ be the empirical distribution function of N simulated particles at time t. We prove the following estimate for the convergence rate in $L^1(\Omega\times\mathbb{R})$ norm: for some constant C uniform with respect to N and Δt, for all $1\le k\le \frac{T}{\Delta t}$,

$$\mathbb{E}\,\|V(k\Delta t,.)-\overline{V}_{k\Delta t}(.)\|_{L^1(\mathbb{R})}\;\le\;C\,\|V_0\;-\;\overline{V}_0\|_{L^1(\mathbb{R})}\;+\;\frac{C}{\sqrt{N}}\;+\;C\,\sqrt{\Delta t}\;.$$

Here, we suppose that the initial condition V_0 equal to a distribution function. In Bossy & Talay [4], we extend the algorithm and the preceding estimate to the case where the initial condition of the Burgers equation is non monotone.

We now fix some notation.

Consider the equation (3). In the whole paper, we suppose that the initial condition, V_0, is the distribution function of a probability measure U_0 on $\mathbb{R}$:

$$V_0(x) = \int_{-\infty}^{x} U_0(dy) \ .$$

For such an initial condition, we interpret the solution of the Burgers equation as the distribution function of the probability measure U_t solution to the following PDE of the McKean-Vlasov type:

$$\left\{ \begin{aligned} \frac{\partial U_t}{\partial t} &= \frac{1}{2}\sigma^2 \frac{\partial^2 U_t}{\partial x^2} - \frac{\partial}{\partial x}\left(\left(\int_{\mathbb{R}} H(x-y)U_t(dy)\right) U_t\right) \ , \\ U_{t=0} &= U_0 \ . \end{aligned} \right. \tag{4}$$

Note that the above PDE is understood in the distribution sense (U_t operates on smooth functions with a compact support in $]0,T[\times\mathbb{R}$); its nonlinear part makes appear the discontinuous interaction kernel $b(x,y) = H(x-y)$, where H is the Heaviside function ($H(z) = 0$ if $z < 0$, $H(z) = 1$ if $z \geq 1$).

To this McKean-Vlasov equation, is associated the nonlinear stochastic differential equation

$$\left\{ \begin{aligned} & dX_t = \sigma dw_t + \int_{\mathbb{R}} H(X_t - y)U_t(dy)\, dt \ , \text{ where } U_t(dy) \text{ is the law of } X_t \ , \\ & X_{t=0} = X_0 \ , \quad \text{of law } U_0 \ . \end{aligned} \right. \tag{5}$$

In the stochastic differential equation (5), the interaction kernel is not Lipschitz: as a matter of fact, the existence and uniqueness of a weak solution cannot be derived from classical results, and the error analysis of the stochastic particle method is much more complex than in the Lipschitz case investigated in [4].

In §2, we give a proof of the existence and uniqueness of a weak solution to (5). In §3, we show that the distribution function V_t of the law of X_t is the classical solution of the Burgers equation, i.e the solution given by the Cole-Hopf transformation [15]. In §4, we use the probabilistic interpretation of the solution of the Burgers equation, and the ideas developed in [4] to construct a stochastic particle method. Its rate of convergence is established in §5 and §6. The Appendix proves some intermediate results.

The results of numerical experiments can be found in [4] and overall in M. Bossy's thesis [3]. In particular, they show the excellent behaviour of the algorithm even when the viscosity coefficient σ tends to 0. By construction of the algorithm, the empirical measure of the particles approximates the measure $\frac{\partial V}{\partial x}(t,x)dx$ and thus, the particles are concentrated in the areas where the gradient of the solution is large.

One can also see the Burgers equation as the Fokker-Planck equation describing the limit law of a particle system with an intercation kernel b, roughly speaking, equal to a Dirac measure (see Sznitman [27]) instead of the Heaviside function. The corresponding algorithm must involve a smoothing of this kernel and its numerical analysis is complex: see Bossy [3] for a discussion.

Remark. If the initial condition of the Burgers equation is of the type

$$V_0(x) \;=\; 1 \;-\; \int_{-\infty}^{x} U_0(dy) \;=\; \int_{x}^{+\infty} U_0(dy)\;,$$

where U_0 is a probability law, we then consider the equation

$$\begin{cases} \dfrac{\partial U}{\partial t} \;=\; \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 U}{\partial x^2} \;-\; \dfrac{\partial}{\partial x}\left(U \left(\displaystyle\int_{\mathbb{R}} (1 - H(x-y))\, U_t(dy) \right)\right) \\ U_{t=0} \;=\; U_0. \end{cases}$$

If U_t denotes the law of the corresponding process, with similar arguments as above, we obtain that the function $\tilde{V}(x,t)$ defined by

$$\tilde{V}(t,x) \;=\; 1 \;-\; \int_{-\infty}^{x} U_t(dy) \;=\; \int_{x}^{+\infty} U_t(dy)$$

is a weak solution to the Burgers equation; our algorithm and our convergence rate can easily be extended to that situation.

2. Existence and uniqueness of a weak solution to (5)

2.1. **Link between (5) and the Burgers equation.** In this section, we first show that the distribution function of the law at time t of a weak solution to (5) which is unique in law is a weak solution (solution in the sense of the distribution) to the Burgers equation. Then, making an additional hypothesis on $V_0(\cdot)$, we will show that this weak solution is also a classical solution.

Proposition 2.1. *If the equation (5) has a weak solution which is unique in the sense of probability law, the law U_t of X_t is a weak solution of the McKean-Vlasov equation (4) in $[0,T]\times\mathbb{R}$, and the distribution function $V(t,x)$ of U_t is a weak solution to the Burgers equation (3).*

*Proof.*Suppose that there exists a weak and unique in law solution to (5). Then, applying Itô's formula to $f(X_T)$, $f \in C^\infty([0,T]\times\mathbb{R})$ being of compact support in $(0,T)\times\mathbb{R}$, one can easily check that U_t is a solution in the distribution sense to the McKean-Vlasov equation (4) in $(0,T[\times\mathbb{R}$.

Let $V(t,x)$ denote the distribution function of U_t and let V_0 denote the distribution function of U_0:

$$V(t,x) \;=\; \int_{-\infty}^{x} U_t(dy),\;\; \forall\,(t,x) \in\; [0,T]\times\mathbb{R},$$

$$V_0(x) \;=\; \int_{-\infty}^{x} U_0(dy),\;\; \forall\, x \;\in\; \mathbb{R}.$$

We now show that V is a weak solution to the Burgers equation. We follow arguments developed by Sznitman [26].

As $\dfrac{\partial V}{\partial x} = U$ in the sense of distributions, (4) implies that

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - V\frac{\partial V}{\partial x}\right).$$

The distributions $\dfrac{\partial V}{\partial t}$ and $\dfrac{\sigma^2}{2}\dfrac{\partial^2 V}{\partial x^2} - V\dfrac{\partial V}{\partial x}$ have the same spatial derivates, thus their difference is a distribution invariant by a translation on the x-axis (cf. Schwartz [24]). Thus, for any test function $f(t,x)$ and for any $z \in I\!\!R$, one has that

$$\begin{aligned}
&< -\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - V\frac{\partial V}{\partial x}, f > \\
&= \int_{[0,T]\times I\!\!R} V(t,x)\left(\frac{\partial f}{\partial t}(t,x+z) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x+z)\right)dtdx \\
&\quad + \int_{[0,T]\times I\!\!R} \frac{1}{2}V^2(t,x)\frac{\partial f}{\partial x}(t,x+z)dtdx \\
&= \int V(t,x-z)\left(\frac{\partial f}{\partial t}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x)\right)dtdx + \int \frac{1}{2}V^2(t,x-z)\frac{\partial f}{\partial x}(t,x)dtdx.
\end{aligned}$$

As for any t in $[0,T]$, $V(t,x)$ is bounded and tends to 0 when x tends to $-\infty$, the right hand-side term tends to 0 when z tends to $+\infty$ by the bounded convergence theorem. This implies that V solves the Burgers equation in the sense of the distribution. ■

Under an additional hypothesis on the initial law U_0, we now show that the corresponding distribution function of the law of X_t is the "classical" solution to the Burgers equation, which can be explicited by the Cole-Hopf transformation (cf. Cole [7], Hopf [15]):

$$V(t,x) = \frac{\displaystyle\int_{I\!\!R} \frac{(x-y)}{t}\exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2t} + \int_{-\infty}^{y} V_0(z)dz\right]\right)dy}{\displaystyle\int_{I\!\!R} \exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2t} + \int_{-\infty}^{y} V_0(z)dz\right]\right)dy}. \tag{6}$$

We suppose:

(H0): the initial law U_0 verifies:
(i) either U_0 is probability measure with a compact support,
(ii) or U_0 has a continuous density u_0, and there exist positive constants M, η and α such that

$$\forall\, |x| > M,\ u_0(x) \le \eta \exp\left(-\alpha\frac{x^2}{2}\right).$$

Proposition 2.2. *Under (H0), the distribution function $V(t,x)$ of the law of X_t is the classical solution of the Burgers equation obtained by the Cole-Hopf transformation.*

The proof is an adaption of the proof given in [26] for the case where the initial condition of the Burgers equation is a density. For the sake of completeness, we give it in the Appendix.

From this explicit representation we deduce an estimate concerning the first spatial derivative of V.

Lemma 2.3. *If U_0 satisfies H0-(ii), then*

$$\left\| \frac{\partial V}{\partial x}(t,x) \right\|_{L^\infty([0,T]\times I\!\!R)} \leq L_0,$$

where L_0 depends on σ, u_0 and T.

If U_0 is a Dirac measure, then for any $t \in]0,T]$ one has

$$\left\| \frac{\partial V}{\partial x}(t,.) \right\|_{L^\infty(I\!\!R)} \leq \frac{L_0}{\sqrt{t}},$$

where L_0 depends on σ and T.

Proof. The proof requires easy computations from the equality (6). ∎

2.2. **Characterization of the law of X_t.** To get the uniqueness in the sense of probability law of a solution to (5), we adapt arguments used by Méléard and Roelly in [19] for a similar equation.

We first state a result which appears in the proof of Proposition 1.1 of Méléard & Roelly [19]:

Lemma 2.4. *On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), I\!\!P)$, consider the real process defined by*

$$Y_t = Y_0 + \sigma w_t + \int_0^t C_s ds, \; 0 \leq t \leq T,$$

where Y_0 is a random variable independent of the Brownian motion (w_t) and (C_t) is a process $(\mathcal{F}_t)$-adapted and bounded. Then, for all t in $]0,T]$, the law of Y_t has a density u_t which belongs to $L^2(I\!\!R)$ and it holds that

$$\|u_t\|_{L^2(I\!\!R)} \leq \frac{C}{t^{\frac{1}{4}}} \,. \tag{7}$$

Suppose that the existence of a weak solution to (5) holds. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (w_t), (X_t))$ be a weak solution; let U_t be the law of X_t. Set

$$C_t = \int_{I\!\!R} H(X_t - y)\, U_t(dy)\,.$$

(C_t) being a bounded process, the preceding lemma shows that U_t has a density in $L^2(I\!\!R)$; we denote it by u_t. We are going to show that u_t is the unique solution in an appropriate space of the equation

$$p_t = S_t U_0 - \int_0^t S_{t-s}\left(\frac{\partial}{\partial x}\left(p_s \,.\, \int_{I\!\!R} H(x-y) p_s(y) dy\right)\right) ds, \;\; \forall\, t \in (0,T]\,. \tag{8}$$

where, for any $t > 0$ g_t denotes the density of the law of σw_t and where S_t denotes the heat semigroup: $S_t U = g_t * U$.

The preceding equation is natural for the following reason. By a formal differentiation of (8), one obtains that

$$\begin{aligned}
\frac{\partial p_t}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}(S_t U_0) - S_0\left(\frac{\partial}{\partial x}\left(p_t \int_{\mathbb{R}} H(x-y)p_t(y)dy\right)\right) \\
&\qquad - \int_0^t \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left(S_{t-s}\left(\frac{\partial}{\partial x}\left(p_s \int_{\mathbb{R}} H(x-y)p_s(y)dy\right)\right)\right) ds \\
&= \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[S_t U_0 - \int_0^t S_{t-s}\left(\frac{\partial}{\partial x}\left(p_s \int_{\mathbb{R}} H(x-y)p_s(y)dy\right)\right) ds\right] \\
&\qquad - \frac{\partial}{\partial x}\left(p_t \int_{\mathbb{R}} H(x-y)p_t(y)dy\right).
\end{aligned}$$

Thus, the probability measure $p_t(x)dx$ is a weak solution to (4) as well as U_t (remember Proposition 2.1).

The following lemma characterizes the density of the law of X_t as the unique solution in an appropriate space to the equation (8).

Lemma 2.5. **(i):** *For any weak solution (X_t) to (5), the density of the law of X_t is a weak solution to (8).*

(ii): *For any $0 < t \leq T$, there exists at most one function p_t in $L^1(\mathbb{R})$ weak solution to (8) and such that*

$$\exists C > 0 \ , \ \sup_{t \in]0,T]} \|p_t\|_{L_1(\mathbb{R})} \leq C.$$

*Proof.*We first show (i). Fix t in $(0,T]$ and f in $C^\infty(\mathbb{R})$ of compact support. Set

$$G(s,x) = S_{t-s}f(x) \ , \ 0 \leq s < t.$$

$G(s,x)$ solves the heat equation in backward time

$$\begin{cases} \dfrac{\partial G}{\partial s} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 G}{\partial x^2} = 0 \ , \quad 0 \leq s < t \ , \\ G(t,x) = f(x) \ . \end{cases}$$

The Itô formula implies that

$$G(t,X_t) = G(0,X_0) + \int_0^t \frac{\partial G}{\partial x}(s,X_s)dw_s + \int_0^t \frac{\partial G}{\partial x}(s,X_s)\left(\int_{\mathbb{R}} H(X_s-y)u_s(y)dy\right) ds.$$

We deduce that

$$\begin{aligned}
&\int_{I\!R} f(x)u_t(x)dx \\
&= \int_{I\!R} G(0,x)U_0(dx) \; + \int_0^t \int_{I\!R} \frac{\partial}{\partial x} G(s,x) \; \left(\int_{I\!R} H(x-y)u_s(y)dy \right) u_s(x)dx \, ds \\
&= \int_{I\!R} \left(S_t U_0\right)(x) f(x)dx + \int_0^t \int_{I\!R} \frac{\partial}{\partial x} S_{t-s} f(x) \left(\int_{I\!R} H(x-y)u_s(y)dy \right) u_s(x)dxds.
\end{aligned}$$

An integration by parts shows that

$$\begin{aligned}
&\int_{I\!R_x} \frac{\partial}{\partial x} \left(\int_{I\!R_z} g_{t-s}(x-z)f(z)dz \right) \left(\int_{I\!R_y} H(x-y)u_s(y)dy \right) u_s(x)dx \\
&\quad = \; - \int_{I\!R_x} \int_{I\!R_z} g_{t-s}(x-z)f(z) \frac{\partial}{\partial x} \left[u_s(x) \left(\int_{I\!R_y} H(x-y)u_s(y)dy \right) \right] dx \, dz \\
&\quad = \; - \int_{I\!R_z} f(z) \, S_{t-s} \left(\frac{\partial}{\partial x} \left[u_s(x) \left(\int_{I\!R_y} H(x-y)u_s(y)dy \right) \right] \right) \Big|_{x=z} dz,
\end{aligned}$$

so that we conclude that u_t solves (8) in the weak sense.

Let us now show (ii). Let u_t and v_t be two weak solutions to (8) belonging to $L^1(I\!R)$ and satisfying

$$\exists \, C > 0, \; \sup_{t \in (0,T]} \; \left(\|u_t\|_{L^1(I\!R)} + \|v_t\|_{L^1(I\!R)} \right) \; \leq \; C \; .$$

Then, for any $t \in (0,T]$, it holds that

$$\begin{aligned}
&\|u_t - v_t\|_{L^1(I\!R)} \\
&= \left\| \int_0^t S_{t-s} \frac{\partial}{\partial x} \left(u_s(x) \int_{I\!R} H(x-y)u_s(y)dy - v_s(x) \int_{I\!R} H(x-y)v_s(y)dy \right) ds \right\|_{L^1(I\!R)} \\
&\leq \; \int_0^t \left\| g_{t-s} * \frac{\partial}{\partial x} \left(u_s(x) \int_{-\infty}^x u_s(y)dy \; - \; v_s(x) \int_{-\infty}^x v_s(y)dy \right) \right\|_{L^1(I\!R)} ds \\
&\leq \; \int_0^t \left\| \frac{\partial}{\partial x} g_{t-s} \right\|_{L^1(I\!R)} \times \left\| u_s(x) \int_{-\infty}^x u_s(y)dy \; - \; v_s(x) \int_{-\infty}^x v_s(y)dy \right\|_{L^1(I\!R)} ds \\
&\leq \; \int_0^t \frac{2}{\sqrt{2\pi(t-s)\sigma^2}} \; \left\| u_s(x) \int_{-\infty}^x u_s(y)dy \; - \; v_s(x) \int_{-\infty}^x v_s(y)dy \right\|_{L^1(I\!R)} ds.
\end{aligned}$$

But one has

$$\begin{aligned}
&u_s(x) \int_{-\infty}^x u_s(y)dy - v_s(x) \int_{-\infty}^x v_s(y)dy \\
&\qquad\qquad = \; u_s(x) \int_{-\infty}^x \left(u_s(y) - v_s(y)\right) dy - \left(v_s(x) - u_s(x)\right) \int_{-\infty}^x v_s(y)dy \\
&\qquad\qquad \leq \; |u_s(x)| \, \|u_s \; - \; v_s\|_{L^1(I\!R)} \; + \; C \, |v_s(x) - u_s(x)|,
\end{aligned}$$

where C is a constant uniform with respect to t; thus,

$$\|u_t - v_t\|_{L^1(\mathbb{R})} \leq \int_0^t \frac{4\,C}{\sqrt{2\pi(t-s)\sigma^2}}\,\|u_s - v_s\|_{L^1(\mathbb{R})}\,ds.$$

As $s \to 1/\sqrt{t-s}$ is integrable on $[0,t]$, an application of Gronwall's Lemma ends the proof. ∎

2.3. **A nonlinear martingale problem.** Having supposed the existence of a weak solution to (5), we have fully characterized the law of each random variable X_t. In this section we show the existence of a weak solution and its uniqueness in the sense of probability law. A classical method is to pose the associated martingale problem.

We first fix some notation. For any space E, $\mathcal{P}(E)$ denotes the set of probability measures on E; $x(\cdot)$ is the canonical process on the space of continuous functions from $[0,T]$ to $\mathbb{R}$; for any measure $\mu \in \mathcal{P}(\mathbb{R})$, the differential operator $\mathcal{L}_{(\mu)}$ is defined by

$$\mathcal{L}_{(\mu)}f(x) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \left(\int_{\mathbb{R}} H(x-y)\mu(dy)\right)\frac{\partial f}{\partial x}(x)\,.$$

A solution to the hereafter nonlinear martingale problem (9), associated to the operator $\mathcal{L}_{(.)}$ and the initial distribution $U_0 \in \mathcal{P}(\mathbb{R})$, is an element $\mathcal{Q}$ of $\mathcal{P}(C([0,T];\mathbb{R}))$ (we denote by $\mathcal{Q}_t$, $t \in [0,T]$, its onedimensional distributions), such that

$$\text{(9)}\qquad \left.\begin{array}{ll} (i) & \mathcal{Q}_0 = U_0, \\ (ii) & \forall f \in C_K^2(\mathbb{R}),\ f(x(t)) - f(x(0)) - \displaystyle\int_0^t \mathcal{L}_{(\mathcal{Q}_s)}f(x(s))ds,\ t \in [0,T] \\ & \text{is a } \mathcal{Q} \text{ martingale.} \end{array}\right\}$$

Suppose that there exists a solution $\mathcal{Q}$ to the *nonlinear* martingale problem (9). Set

$$\widehat{C}(t,x) := \int_{\mathbb{R}} H(x-y)\mathcal{Q}_t(dy).$$

$\mathcal{Q}$ also solves the *linear* martingale problem associated to the operator $\widehat{\mathcal{L}}$ defined by

$$\widehat{\mathcal{L}}f(x) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x) + \widehat{C}(t,x)\frac{\partial f}{\partial x}(x).$$

Thus (cf. Karatzas & Shreve [16], e.g.) , there exists a $(C(0,T), \mathcal{B}_T, \mathcal{Q}, (\mathcal{F}_t) - (w_t))$ Brownian motion such that

$$x(t) = X_0 + \int_0^t \widehat{C}(s,x(s))ds + \sigma w_t,\ \mathcal{Q}-\text{a.s.}$$

As the probability measure $\mathcal{Q}_t$ is the law of $x(t)$ under $\mathcal{Q}$, we deduce that, under $\mathcal{Q}$, $x(t)$ is a weak solution to (5). Conversely, if there exists a solution in the sense of probability law to (5), then $\mathcal{Q} = \mathbb{P} \circ X^{-1}$ is a solution to the martingale problem (9).

2.4. **Uniqueness of the solution to the nonlinear martingale problem.** Let $\mathcal{Q}$ be a solution to the nonlinear martingale problem (9). Lemma 2.5 characterizes the law of $x(t)$ under $\mathcal{Q}$, so that $\mathcal{Q}_t = p_t(x)dx$. This is not enough to characterize $\mathcal{Q}$, but set

$$\widetilde{C}(t,x) := \int_{I\!R} H(x-y)\,p_t(y)dy\ ;$$

note (cf. page 327 in Karatzas & Shreve [16], e.g.) that there exists a unique solution $\widetilde{\mathcal{Q}}$ to the linear martingale problem associated to the operator $\widetilde{\mathcal{L}}$ defined by

$$\widetilde{\mathcal{L}}f(x) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x) + \widetilde{C}(t,x)\,\frac{\partial f}{\partial x}(x).$$

As $\mathcal{Q}_t = p_t(x)dx$, $\mathcal{Q}$ is a solution to this linear martingale problem; thus $\mathcal{Q} = \widetilde{\mathcal{Q}}$.

2.5. **Existence of a solution to the nonlinear martingale problem.** We now construct a solution to the martingale problem (9) as the limit of a sequence of probability measures of $\mathcal{P}(C([0,T]; I\!R))$.

Consider the functions $\left(H^k;\ k\in I\!N^*\right)$ defined by

$$H^k(x) = \begin{cases} 0\,, & \text{if } x < -\frac{1}{k},\\ kx+1\,, & \text{if } x \in]-\frac{1}{k},0[,\\ 1\,, & \text{if } x\geq 0.\end{cases}$$

Then

$$\forall\ x \in I\!R,\ \lim_{k\to\infty} H^k(x) = H(x),$$

and, for any k,

$$\left|H^k(x) - H^k(y)\right| \leq k|x-y|.$$

Substituting H^k to H in (5), we introduce the differential equation

$$\begin{cases} dX_t^k = \sigma dw_t + \displaystyle\int_{I\!R} H^k(X_t^k - y)U_t^k(dy)\,dt,\\ \qquad\qquad\qquad\qquad \text{where } U_t^k \text{ is the law of } X_t^k,\\ X_{t=0}^k = X_0 \quad \text{whose law is } U_0.\end{cases}$$

The corresponding interaction kernel $\big(b(x,y) = H^k(x-y)\big)$ is Lipschitz, so that (cf Sznitman [27] e.g.) the above equation has a unique strong solution.

For a fixed measure $\mu\in\mathcal{P}(I\!R)$ and for any $k>1$, define the operator $\mathcal{L}^k_{(\mu)}$ by

$$\mathcal{L}^k_{(\mu)}f(x) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x) + \left(\int_{I\!R} H^k(x-y)\mu(dy)\right)\frac{\partial f}{\partial x}(x).$$

The probability $\mathcal{Q}^k := I\!P\circ(X^k)^{-1}$ solves the martingale problem similar to (9), obtained by substituting $\mathcal{L}^k_{(.)}$ to $\mathcal{L}_{(.)}$ and $\mathcal{Q}^k_t = U^k_t$, for all $0\leq t\leq T$.

Proposition 2.6. *The family* $(\mathcal{Q}^k)$ *is tight.*

*Proof.*As $\mathcal{Q}^k = \mathbb{P} \circ (X^k)^{-1}$, it is enough to check that there exist strictly positive constants C_T, α and β such that

$$\sup_k \mathbb{E} \mid X_t^k - X_s^k \mid^\alpha \leq C_T (t-s)^{1+\beta}, \ \forall\, 0 \leq s \leq t \leq T.$$

We choose $\alpha = 4$, $\beta = 1$ and readily conclude. ∎

Now we show that any limit point $\mathcal{Q}^\infty$ of a convergent subsequence (still denoted by $(\mathcal{Q}^k)$) of $(\mathcal{Q}^k)$ solves the martingale problem (9), i.e that, for any f in $C_K^2(\mathbb{R})$, one has

$$\begin{aligned} &\mathbb{E}_{\mathcal{Q}^\infty}\left[f(x(t)) - f(x(s)) - \int_s^t \mathcal{L}_{(\mathcal{Q}_\tau^\infty)} f(x(\tau))\, d\tau \mid x(\theta),\ 0 < \theta \leq s\right] \\ &\quad = 0,\ 0 \leq s \leq t \leq T. \end{aligned} \tag{10}$$

Set

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \mathcal{L}_{(\mathcal{Q}_\tau^\infty)} f(x(\tau))\, d\tau.$$

Thus (10) is equivalent to

$$\mathbb{E}_{\mathcal{Q}^\infty}[(M_t - M_s)\phi(x(t_1),\dots,x(t_n))] = 0,\ \forall\, \phi \in C_b(R^n) \text{ and } 0 \leq t_1 < \dots. < t_n < s.$$

In fact, we only need to prove that for all $\varepsilon > 0$, for all $\phi \in C_b(R^n)$ and $0 < \varepsilon \leq t_1 < \dots < t_n < s$,

$$\mathbb{E}_{\mathcal{Q}^\infty}[(M_t - M_s)\phi(x(t_1),\dots,x(t_n))] = 0, \tag{11}$$

since then

$$\mathbb{E}_{\mathcal{Q}^\infty}[M_t|\mathcal{F}_\varepsilon] = M_\varepsilon,\ \forall\, \varepsilon > 0,$$

so that, as M_t is uniformly bounded on $\Omega \times [0,T]$, $\mathbb{E}_{\mathcal{Q}^\infty}[M_t|\mathcal{F}_0] = 0 = M_0$.

Set

$$M_t^k := f(x(t)) - f(x(0)) - \int_0^t \mathcal{L}^k_{(U_\tau^k)} f(x(\tau))\, d\tau.$$

As $\mathcal{Q}^k$ solves the martingale problem associated to $\mathcal{L}^k_{(.)}$, for all function $\phi \in C_b(R^n)$ and all $0 < \varepsilon \leq t_1 < \dots. < t_n < s$, one has that

$$\begin{aligned} 0 &= \mathbb{E}_{\mathcal{Q}^k}\Big[(M_t^k - M_s^k)\,\phi(x(t_1),\dots,x(t_n))\Big] \\ &= \mathbb{E}_{\mathcal{Q}^k}\Big[(f(x(t)) - f(x(s)))\,\phi(x(t_1),\dots,x(t_n)) \\ &\qquad - \phi(x(t_1),\dots,x(t_n))\int_s^t \mathcal{L}^k_{(U_\tau^k)} f(x(\tau))\, d\tau\Big]. \end{aligned} \tag{12}$$

From this equality and the weak convergence of $(\mathcal{Q}^k)$, one easily concludes that (11) is implied by

$$\begin{aligned} &\lim_{k\to\infty} \mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t \int_{\mathbb{R}} f'(x(\tau))H^k(x-y)U_\tau^k(dy)\, d\tau\right] \\ &\quad = \mathbb{E}_{\mathcal{Q}^\infty}\left[\phi(x(t_1),\dots,x(t_n))\textstyle\int_s^t \int_{\mathbb{R}} f'(x(\tau))H(x-y)\mathcal{Q}_\tau^\infty(dy)\, d\tau\right]. \end{aligned} \tag{13}$$

In order to prove this latter equality, we decompose the first term in tow parts :

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t\int_{\mathbb{R}} f'(x(\tau))H^k(x-y)U^k_\tau(dy)d\tau\right]\\
&= \mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^k(x(\tau)-y)U^k_\tau(dy)d\tau\right]\\
&\qquad - \mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H(x(\tau)-y)\mathcal{Q}^\infty_\tau(dy)\,d\tau\right]\\
&\quad +\mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H(x(\tau)-y)\mathcal{Q}^\infty_\tau(dy)\,d\tau\right]\\
&\qquad - \mathbb{E}_{\mathcal{Q}^\infty}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H(x(\tau)-y)\mathcal{Q}^\infty_\tau(dy)\,d\tau\right]\\
&:= D_1 + D_2.
\end{aligned}
$$

We first observe from lemma 2.4 that, for any $t\in]0,T]$, U^k_t has a density u^k_t in $L^2(\mathbb{R})$ satisfying (cf. (7))

$$\|u^k_t\|_{L^2(\mathbb{R})}\ \le\ \frac{C}{t^{\frac14}}. \tag{14}$$

Thus, we have

$$
\begin{aligned}
&|D_1|\ \le\\
&\int_s^t \frac{C\|\phi\|_{L^\infty(\mathbb{R})}}{\tau^{1/4}}\sqrt{\int_{\mathbb{R}} f'^2(x)\left[\int_{\mathbb{R}} H^k(x-y)U^k_\tau(dy)-\int_{\mathbb{R}} H(x-y)\mathcal{Q}^\infty_\tau(dy)\right]^2 dx}\ d\tau.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\left[\int_{\mathbb{R}} H^k(x-y)U^k_\tau(dy)\ -\ \int_{\mathbb{R}} H(x-y)\mathcal{Q}^\infty_\tau(dy)\right]^2\\
&\le 2\left[\int_{\mathbb{R}}\Big(H^k(x-y)-H(x-y)\Big)\,U^k_\tau(dy)\right]^2+2\left[\int_{-\infty}^x U^k_\tau(dy)-\int_{-\infty}^x \mathcal{Q}^\infty_\tau(dy)\right]^2.
\end{aligned}
$$

As

$$\int_{\mathbb{R}}\Big(H^k(x-y)-H(x-y)\Big)^2 dy\ \le\ \frac{1}{3k},$$

we obtain that

$$
\begin{aligned}
|D_1|\le\ &\|\phi\|_{L^\infty(\mathbb{R})}\frac{1}{\sqrt{k}}\left(\int_s^t\frac{C}{\sqrt{\tau}}\,d\tau\right)\|f'\|_{L^2(\mathbb{R})}\\
&+C\|\phi\|_{L^\infty(\mathbb{R})}\int_s^t\sqrt{\int_{\mathbb{R}} f'^2(x)\left[\int_{-\infty}^x U^k_\tau(dy)-\int_{-\infty}^x\mathcal{Q}^\infty_\tau(dy)\right]^2dx}\ \frac{d\tau}{\tau^{1/4}}.
\end{aligned}
$$

From (14), we deduce that for all function $g\in C_K(\mathbb{R})$,

$$<\mathcal{Q}^\infty_t,\ g>\le\ \frac{C}{t^{\frac14}}\|g\|_{L^2(\mathbb{R})}; \tag{15}$$

therefore, for all $t > 0$, $\mathcal{Q}_t^\infty$ has a density q_t^∞ w.r.t. the Lebesgue measure belonging to $L^2(\mathbb{R})$; this implies that the distribution function $V_t^\infty(\cdot)$ of $\mathcal{Q}_t^\infty$ is continuous, so that $V_t^k(\cdot)$ converges to $V_t^\infty(\cdot)$ everywhere and thus, D_1 tends to 0 when k tends to infinity.

Now we consider D_2; we again need to introduce a smoothing of the kernel :

$$
\begin{aligned}
D_2 \ &= \mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau\right] \\
&\quad -\mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau\right] \\
&\quad +\mathbb{E}_{\mathcal{Q}^k}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau\right] \\
&\quad -\mathbb{E}_{\mathcal{Q}^\infty}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau\right] \\
&\quad +\mathbb{E}_{\mathcal{Q}^\infty}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau\right] \\
&\quad -\mathbb{E}_{\mathcal{Q}^\infty}\left[\phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau\right].
\end{aligned}
\tag{16}
$$

From (15), we readily obtain that

$$\|q_t^\infty\|_{L^2(\mathbb{R})} \ \le \ \frac{C}{t^{\frac{1}{4}}}.$$

Thus,

$$\left[\int_{\mathbb{R}}\Big(H(x-y)-H^{k_0}(x-y)\Big)\,\mathcal{Q}_\tau^\infty(dy)\right]^2 \ \le \ \frac{C}{k_0\sqrt{t}},$$

so that, f being of compact support, one can choose k_0 uniformly in k to make arbitrarily small the first and the last differences of (16); such a k_0 being fixed, the second difference tends to 0 when k goes to infinity as a consequence of the weak convergence of $(\mathcal{Q}^k)$, since the smoothness of H^{k_0} implies that the functional

$$
\begin{aligned}
C([0,T];\mathbb{R}) \quad &\longrightarrow \quad \mathbb{R} \\
x(\cdot) \quad &\longrightarrow \quad \phi(x(t_1),\dots,x(t_n))\int_s^t f'(x(\tau))\int_{\mathbb{R}} H^{k_0}(x(\tau)-y)\mathcal{Q}_\tau^\infty(dy)\,d\tau
\end{aligned}
$$

is continuous.

Consequently we have proven that $\mathcal{Q}^\infty$ solves the nonlinear martingale problem (9). ∎

3. Algorithm and convergence rate

In all the sequel, we suppose :

(H1): the initial law U_0 satisfies :
- **(i):** either U_0 is a Dirac measure,
- **(ii):** or U_0 has a smooth density u_0, satisfying one of the two conditions:

- $u_0(\cdot)$ is a continuous function, and there exists strictly positive constants M, η and α such that
$$\forall |x| \; > \; M, \; u_0(x) \; \leq \; \eta \; \exp\left(-\alpha \frac{x^2}{2}\right),$$
- u_0 is a function with a compact support and is continuous on this support.

The existence in the sense of probability law of a solution to (5) implies the existence in the sense of probability law of a solution to

$$\left\{ \begin{array}{l} dz_t \; = \; V(t, z_t) \, dt \; + \; \sigma \, dw_t, \\ z_{t=0} \; = \; z_0. \end{array} \right. \tag{17}$$

Under (H1), Lemma 2.3 shows that $V(t, \cdot)$ is a Lipschitz function in x with a Lipschitz constant bounded from above by $\dfrac{L_0}{\sqrt{t}}$ for all $t \in (0, T]$, which implies the pathwise uniqueness of the solution to (17). Indeed, if (z_t^1) and (z_t^2) are two solutions, then
$$|z_t^1 - z_t^2| \; \leq \; \int_0^t \frac{L_0}{\sqrt{s}} |z_s^1 - z_s^2| ds,$$
so that $z_t^1 = z_t^2$ by Gronwall's Lemma.

The Markov process (z_t) with the initial distribution U_0 coincides with (X_t) and
$$V(t, x) \; = \; I\!\!E_{U_0} H(x - z_t).$$

We now construct our algorithm by successive approximations of the preceding representation.

3.1. **Approximation of the initial condition.** Choose N points in $I\!\!R$, $(y_0^1, .., y_0^N)$, such that the piecewise constant function
$$\overline{V}_0(x) = \frac{1}{N} \sum_{i=1}^{N} H(x - y_0^i)$$
approximates V_0 and denote by $\overline{U}_0 = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \delta_{y_0^i}$ the corresponding empirical measure.

When U_0 is a Dirac measure at a given point x_0, set $y_0^i = x_0$. Then, $\overline{U}_0 = U_0$ and $\overline{V}_0 = V_0$.

When U_0 satisfies (H1-ii), set
$$y_0^i \; = \; \left\{ \begin{array}{l} inf\{y; \; V_0(y) = \frac{i}{N}\}, \; i = 1, \ldots, N-1, \\ \\ inf\{y; \; V_0(y) = 1 - \frac{1}{2N}\}, \; i = N. \end{array} \right.$$

A first approximation of $V(t, .)$ is
$$V(t, x) \; \simeq \; I\!\!E_{\overline{U}_0} H(x - z_t) \; = \; \frac{1}{N} \sum_{i=1}^{N} \; I\!\!E H(x - z_t(y_0^i)).$$

3.2. **Approximation of the expectation.** Consider N independent copies $(w_t^i)_{i=1}^N$ of the Brownian motion (w_t) and the family of independent processes $(z_t^i)_{i=1}^N$ defined by

$$\left\{ \begin{array}{lll} dz_t^i & = & V(t, z_t^i)\, dt \,+\, \sigma\, dw_t^i, \\ z_0^i & = & y_0^i\ . \end{array} \right. \tag{18}$$

We now approximate $V(t,.)$ by applying the strong law of large numbers:

$$V(t,x)\ \simeq\ \frac{1}{N}\sum_{i=1}^{N}\ H(x - z_t^i)\ .$$

3.3. **Time discretization.** T being fixed, define $\Delta t > 0$ and $K \in \mathbb{N}$ such that $T = K\Delta t$. The discretization times are denoted by $t_k = k\Delta t\ , 1 \leq k \leq K$. Applying the Euler scheme to the stochastic differential equations (18), one defines independent discrete time processes $(\overline{z}_{t_k}^i)$:

$$\left\{ \begin{array}{l} \overline{z}_{t_{k+1}}^i = \overline{z}_{t_k}^i + V(t_k, \overline{z}_{t_k}^i)\Delta t + \sigma\,(w_{t_{k+1}}^i - w_{t_k}^i), \\ \\ \overline{z}_0^i = y_0^i. \end{array} \right. \tag{19}$$

Thus, at time t_k $(k = 1, .., K)$, $V(t_k, .)$ is approximated by

$$V(t_k, x)\ \simeq\ \frac{1}{N}\sum_{i=1}^{N}\ H(x - \overline{z}_{t_k}^i).$$

3.4. **Approximation of the interaction kernel.** The dynamics of the $\overline{z}^i$'s depend on the function V, which is our unknown. Thus, we are led to approximate $V(t_k, \cdot)$ by the empirical distribution function of the particles, that we denote by $\overline{V}_{t_k}(\cdot)$. This approximation leads to the consideration of a new particle system $(Y_{t_k}^i)$.

Let $Y_{t_k}^i$ be the position of the i^{th} particle at time t_k and let $\overline{U}_{t_k}$ be the corresponding empirical measure. Set

$$\overline{V}_{t_k}(x)\ =\ \int_{\mathbb{R}} H(x-y)\overline{U}_{t_k}(dy)\ =\ \frac{1}{N}\sum_{i=1}^{N}\ H(x - Y_{t_k}^i). \tag{20}$$

We substitute V in (19) for this approximation. This defines the dynamics of the particle system $(Y_{t_k}^i)_{i=1}^N$ which can be simulated on a computer:

$$\left\{ \begin{array}{ll} Y_{t_{k+1}}^i & = Y_{t_k}^i \,+\, \overline{V}_{t_k}(Y_{t_k}^i)\,\Delta t \,+\, \sigma\,\Delta w_{k+1}^i \\ & = Y_{t_k}^i \,+\, \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} H(Y_{t_k}^i - Y_{t_k}^j)\,\Delta t \,+\, \sigma\,\Delta w_{k+1}^i, \\ Y_0^i & = y_0^i, \end{array} \right.$$

where $\Delta w_{k+1}^i = w_{t_{k+1}}^i - w_{t_k}^i$.

3.5. **Convergence rate.** We now state our estimate on the convergence rate of the empirical distribution function to the solution of the Burgers equation.

Theorem 3.1. *T being fixed, let $\Delta t > 0$ be such that $T = \Delta t K$, $K \in \mathbb{N}$.*

Let $V(t_k, x)$ be the solution at time $t_k = k\Delta t$ of the Burgers equation (3) with the initial condition V_0.

Let $\overline{V}_{t_k}(x)$ be defined as in (20), N being the number of particles.

Under (H1), there exist a strictly positive constant C depending on σ, U_0 and T, such that, for all $k \in \{1,..,K\}$:

$$\mathbb{E}\, \|V(t_k,.) - \overline{V}_{t_k}(.)\|_{L^1(\mathbb{R})} \leq C\,\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} + C\,\frac{1}{\sqrt{N}} + C\,\sqrt{\Delta t} \tag{21}$$

$$\leq \frac{C}{\sqrt{N}} + C\sqrt{\Delta t}\ . \tag{22}$$

The order $\mathcal{O}\frac{1}{\sqrt{N}}$ for the error in $L^1(\mathbb{R} \times \Omega)$ cannot be improved. Indeed, it is easy to see that this convergence rate also holds for systems of *independent* particles: see Talay [28]. Besides, numerical experiments confirm this theoretical estimate (remember that the exact solution $V(t,x)$ is explicitly given by (6)): see Bossy [3].

Remark. If the initial law U_0 is a Dirac measure, then $\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} = 0$. In the other case, one can prove (see [4]) that $\|V_0(\cdot) - \overline{V}_0(\cdot)\|_{L^1(\mathbb{R})}$ converges with the order $\mathcal{O}\frac{1}{N}\sqrt{\log(N)}$. Therefore, (22) is an immediate consequence of (21).

3.6. **Propagation of chaos.** Consider N particles which at time 0 are independent with law U_0 and follow the dynamics

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} H(X_t^{i,N} - X_t^{j,N})dt + \sigma dw_t^i\ .$$

In this section, we prove the propagation of chaos for this system of particles. The propagation of chaos property explains the convergence of the algorithm: when N goes to infinity, any finite subsystem of these particles tends to behave like a system of independent particles, each one having the law $\mathcal{Q}$ defined in Section 2.4.

Theorem 3.2. *Let $\mathbb{P}^N$ be the joint law on $(C([0,T];\mathbb{R}))^N$ of the particle system $(X_\cdot^{1,N},\dots,X_\cdot^{N,N})$. For any $k \in \mathbb{N}^*$, for any continuous and bounded functions $f_1,\dots,f_k : C([0,T];\mathbb{R}) \to \mathbb{R}$, one has that*

$$\lim_{N\to+\infty} < \mathbb{P}^N, f_1 \otimes \dots \otimes f_k \otimes 1 \dots \otimes 1 > = \prod_{i=1}^{k} < \mathcal{Q}, f_i >\ ,$$

where $\mathcal{Q}$ is the solution of the nonlinear martingale problem (9) (the sequence $(\mathbb{P}^N)$ is said "$\mathcal{Q}$-chaotic").

Proof. To our knowledge, our context does not satisfy the hypotheses of the systems studied in the literature. We adapt arguments appearing in Méléard & Roelly [19] or Sznitman [27] e.g.

The $\mathcal{Q}$-chaoticity is equivalent to the convergence of the laws of the empirical measures $\mu^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X^{i,N}_\cdot}$ to $\delta_{\mathcal{Q}}$ (cf. Aldous [1] or Tanaka [29]).

When the kernels are smooth, the argument is as follows. First, one shows that the sequence of the laws of the μ^N's is tight. Let Π_1^∞ be a limit point of a convergent subsequence of $\{\mathcal{L}aw(\mu^N)\}$. Set

$$F(m) := < m, \left(f(x(t)) - f(x(s)) - \int_s^t L_{(m_\theta)} f(x(\theta))d\theta\right) g(x(s_1), \dots, x(s_k)) > ,$$

where $L_{(\mu)}$ is as in (2), $f \in C_b^2(\mathbb{R})$, $g \in C_b(\mathbb{R}^k)$, $0 < s_1 < \dots < s_k \leq s \leq T$ and m is a probability on $C([0,T];\mathbb{R})$. Then one uses two arguments:

- **(a):** first, one checks that $\lim_{N\to+\infty} \mathbb{E}[F(\mu^N)]^2 = 0$ by using the dynamics of the particles (see (23) below);
- **(b):** then, one uses the continuity of $F(\cdot)$ in $\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}))$ endowed with the Vaserstein metric to deduce that the support of Π_1^∞ is the set of solutions to the nonlinear martingale problem (9) with $L_{(\mathcal{Q}_s)}$ defined as in (2) instead of $\mathcal{L}_{(\mathcal{Q}_s)}$. One proves the uniqueness of such a solution, which implies that $\Pi_1^\infty = \delta_{\mathcal{Q}}$.

In the case of the Burgers equation, the step (a) does not need to be changed:

$$\begin{aligned}\lim_{N\to+\infty} \mathbb{E}[F(\mu^N)]^2 &\leq \lim_{N\to+\infty} \frac{C}{N^2}\mathbb{E}\left(\sum_{i=1}^N \left\{ f(X_t^{i,N}) - f(X_s^{i,N}) - \int_s^t \mathcal{L}_{(\mu_\theta^N)} f(X_\theta^{i,N})d\theta\right\}\right)^2 \\ &= \lim_{N\to+\infty} \frac{C}{N^2}\sum_{i=1}^N \mathbb{E}\left(\int_s^t \sigma dW_\theta^i\right)^2 \qquad (23) \\ &= 0 .\end{aligned}$$

But the Heaviside function being discontinuous, $F(\cdot)$ is discontinuous too and we cannot proceed as in the step (b). This leads us to use the explicit form of F.

Let ν^N be defined by

$$\nu^N := \frac{1}{N^4}\sum_{i,j,k,l=1}^N \delta_{(X^{i,N}_\cdot, X^{j,N}_\cdot, X^{k,N}_\cdot, X^{l,N}_\cdot)} .$$

First, we note that the sequence of the laws of the ν^N's is tight; indeed, a sufficient criterion due to Sznitman [25] is the tightness of the sequence of the intensity measures I^N defined by $< I^N, f >= \mathbb{E} < \nu^N, f >$, which by symmetry reduces here to the tightness of the laws $\mathbb{P}_{X^{1,N}_\cdot}$: this latter fact is implied by

$$\mathbb{E}\sup_{t\leq T}|X_t^{1,N}|^2 \leq CT .$$

Let $\Pi^\infty \in \mathcal{P}(\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4))$ be the limit of a convergent subsequence of $\{Law(\nu^N)\}$ which we still denote by $\{Law(\nu^N)\}$ in all the sequel.

We denote by ν^1 the first marginal of a measure $\nu \in \mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)$, (for all Borel sets A in $\mathcal{C}([0,T];\mathbb{R})$, $\nu^1(A) = \nu(A \times \mathcal{C}([0,T];\mathbb{R}) \times \mathcal{C}([0,T];\mathbb{R}) \times \mathcal{C}([0,T];\mathbb{R})))$. Then,

Lemma 3.3. $\Pi^\infty - a.e,\ \nu = \nu^1 \otimes \nu^1 \otimes \nu^1 \otimes \nu^1$.

*Proof.*We observe that

$$\begin{aligned} < \nu^N, f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4) > &= \frac{1}{N^4}\sum_{i_1,\dots,i_4=1}^{N} f_1(X_\cdot^{i_1,N})\dots f_4(X_\cdot^{i_4,N}) \\ &= < \nu^{N,1}, f_1 > \dots < \nu^{N,1}, f_4 > , \end{aligned}$$

where $\nu^{N,1}$ is the first marginal of ν^N. Consequently,

$$\mathbb{E}\left[< \nu^N, \prod_{j=1}^4 f_j(x_j) > - \prod_{j=1}^4 < \nu^{N,1}, f_j >\right]^2 = 0$$

from which,

$$\int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \left[< \nu, \prod_{j=1}^4 f_j(x_j) > - \prod_{j=1}^4 < \nu^1, f_j >\right]^2 d\Pi^\infty(\nu) = 0.$$

In particular, the latter equality holds for any set of functions $(f_j, j=1,\dots,4)$ in a set $\mathcal{H}$ of measure determining functions on $\mathcal{C}([0,T];\mathbb{R})$. As $\mathcal{H}$ is denumerable by definition, one has :

$$\begin{gathered} \exists\ \mathcal{N}\ ,\ \Pi^\infty(\mathcal{N}) = 0\ ,\ \forall \nu \notin \mathcal{N}\ ,\ \forall f_j \in \mathcal{H} \text{ for } j \in \{1,2,3,4\}, \\ < \nu, \prod_{j=1}^4 f_j(x_j) >= \prod_{j=1}^4 < \nu^1, f_j > \qquad . \end{gathered}$$

As $\mathcal{H}^4$ is a set of measure determining functions on $\mathcal{C}([0,T];\mathbb{R})^4$, it comes

$$\Pi^\infty - \text{a.e},\ \nu = \nu^1 \otimes \nu^1 \otimes \nu^1 \otimes \nu^1\ .\ \blacksquare$$

Coming back to the proof of Theorem 3.2, let us show that

$$\begin{aligned} &\lim_{N\to+\infty} \mathbb{E}[F(\mu^N)]^2 = \\ &\int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \left\{\int_{\mathcal{C}([0,T];\mathbb{R})^4} \left[f(x_t^1) - f(x_s^1) - \frac{\sigma^2}{2}\int_s^t f''(x_\theta^1)d\theta - \int_s^t H(x_\theta^1 - x_\theta^2)f'(x_\theta^1)d\theta\right]\right. \\ &\qquad\left. g(x_{s_1}^1,\dots,x_{s_p}^1)d\nu(x^1,x^2,x^3,x^4)\right\}^2 d\Pi^\infty(\nu)\ . \end{aligned} \tag{24}$$

An easy computation shows that, for some functionals ψ and $\tilde{\psi}$

$$\mathbb{E}\left[F(\mu^N)^2\right] = \frac{1}{N^2}\sum_{i,k=1}^{N} \mathbb{E}\psi(X^i, X^k) + \frac{1}{N^3}\sum_{i,k,l=1}^{N} \mathbb{E}\tilde{\psi}(X^i, X^k, X^l) + C_N \tag{25}$$

with

$$\begin{aligned} C_N \;:=\; & \frac{1}{N^4}\sum_{i,j,k,l=1}^{N} \int_s^t \int_s^t \int_{\mathcal{C}([0,T];\mathbb{R})^4} H(x^1_\theta - x^2_\theta) f'(x^1_\theta) g(x^1_{s_1}, \dots, x^1_{s_p}) \\ & H(x^3_\gamma - x^4_\gamma) f'(x^3_\gamma) g(x^3_{s_1}, \dots, x^3_{s_p}) d\mathbb{P}_{(X^{i,N}_{\cdot}, X^{j,N}_{\cdot}, X^{k,N}_{\cdot}, X^{l,N}_{\cdot})}(x^1, \dots, x^4) d\theta d\gamma \ . \end{aligned}$$

Let us look at the convergence of (C_N). Let τ^N be defined by

$$\tau^N := \frac{1}{N^4}\sum_{i,j,k,l=1}^{N} \delta_{(X^{i,N}_\theta, X^{j,N}_\theta, X^{k,N}_\gamma, X^{l,N}_\gamma, X^{i,N}_{s_1}, \dots, X^{i,N}_{s_p}, X^{k,N}_{s_1}, \dots, X^{k,N}_{s_p})}$$

and let $Q^N_{\theta,\gamma,s_1,\dots,s_p}$, the measure on $\mathbb{R}^{2p+4}$ defined by

$$Q^N_{\theta,\gamma,s_1,\dots,s_p}(A) = \mathbb{E}(\tau^N(A)).$$

The convergence of (a subsequence of) $\{Law(\nu^N)\}$ implies the weak convergence of $Q^N_{\theta,\gamma,s_1,\dots,s_p}$ and the limit measure on $\mathbb{R}^{2p+4}$ is defined by

$$\begin{aligned} Q_{\theta,\gamma,s_1,\dots,s_p}(A) = \int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \int_{\mathcal{C}([0,T];\mathbb{R})^4} & \mathbb{1}_A(x^1_\theta, x^2_\theta, x^3_\gamma, x^4_\gamma, x^1_{s_1}, \dots, x^1_{s_p}, x^3_{s_1}, \dots, x^3_{s_p}) \\ & d\nu(x^1, \dots, x^4) d\Pi^\infty(\nu) \ . \end{aligned}$$

This probability measure has a density w.r.t. Lebesgue's measure since, for any smooth function ϕ of compact support in $\mathbb{R}^{2p+4}$,

$$\begin{aligned} & | < Q_{\theta,\gamma,s_1,\dots,s_p}, \phi > | \\ & = \left| \lim_{N\to\infty} \frac{1}{N^4}\sum_{i,j,k,l=1}^{N} \mathbb{E}\phi(X^{i,N}_\theta, X^{j,N}_\theta, X^{k,N}_\gamma, X^{l,N}_\gamma, X^{i,N}_{s_1}, \dots, X^{i,N}_{s_p}, X^{k,N}_{s_1}, \dots, X^{k,N}_{s_p}) \right| \\ & \leq C(T, \theta, \gamma, s_1, \dots, s_p) \|\phi\|_{L^2(\mathbb{R}^{2p+4})} \end{aligned}$$

(this can be proved by using Girsanov's transformation and the boundedness of the drift term of the stochastic differential system which describes the dynamics of $(X^{i,N}, X^{j,N}, X^{k,N}, X^{l,N})$). Thus, the function ρ defined on $\mathbb{R}^{2p+4}$ by

$$\rho(x^1, \dots, x^{2p+4}) = H(x^1 - x^2) f'(x^1) g(x^5, \dots, x^{p+4}) H(x^3 - x^4) f'(x^3) g(x^{p+5}, \dots, x^{2p+4})$$

is continuous $Q_{\theta,\gamma,s_1,\dots,s_p}$-a.e., which implies

$$\frac{1}{N^4}\sum_{i,j,k,l=1}^{N}\int_{\mathcal{C}([0,T];\mathbb{R})^4} H(x^1_\theta - x^2_\theta)f'(x^1_\theta)g(x^1_{s_1},\dots,x^1_{s_p})H(x^3_\gamma - x^4_\gamma)f'(x^3_\gamma)g(x^3_{s_1},\dots,x^3_{s_p})$$
$$d\mathbb{P}_{(X^{i,N}_\cdot,X^{j,N}_\cdot,X^{k,N}_\cdot,X^{l,N}_\cdot)}(x^1,\dots,x^4)$$
$$\longrightarrow < Q_{\theta,\gamma,s_1,\dots,s_p},\rho > \quad .$$

Therefore, we have got:

$$C_N \longrightarrow \int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)}\int_s^t\int_s^t\int_{\mathcal{C}([0,T];\mathbb{R})^4} H(x^1_\theta - x^2_\theta)f'(x^1_\theta)g(x^1_{s_1},\dots,x^1_{s_p})$$
$$H(x^3_\gamma - x^4_\gamma)f'(x^3_\gamma)g(x^3_{s_1},\dots,x^3_{s_p})$$
$$d\nu(x^1,x^2,x^3,x^4)d\theta d\gamma d\Pi^\infty(\nu),$$

and by Lemma 3.3,

$$C_N \longrightarrow \int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)}\left[\int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^2)}\int_s^t H(x^1_\theta - x^2_\theta)f'(x^1_\theta)d\theta\right.$$
$$\left. g(x^1_{s_1},\dots,x^1_{s_p})d\nu^1(x^1)\otimes d\nu^1(x^2)\right]^2 d\Pi^\infty(\nu).$$

Coming back to (25) and for the first two terms of the right side using arguments similar to those developed for C_N, we deduce (24). Combining this result with (23), we have, Π^∞-a.e.,

$$\int_{\mathcal{C}([0,T];\mathbb{R})^2)}\left[f(x^1_t) - f(x^1_s) - \frac{\sigma^2}{2}\int_s^t f''(x^1_\theta)d\theta - \int_s^t H(x^1_\theta - x^2_\theta)f'(x^1_\theta)d\theta\right]$$
$$g(x^1_{s_1},\dots,x^1_{s_p})d\nu^1(x^1)\otimes d\nu^1(x^2) = 0. \tag{26}$$

Then, (26) and the uniqueness to the nonlinear martingale problem (9) imply that $\nu^1 = \mathcal{Q}$ which is equivalent to

$$\lim_{N\to\infty}(Law(\mu^N)) = \delta_{\mathcal{Q}} \ . \ \blacksquare$$

4. Proof of Theorem 3.1

4.1. **Notation.** In all the sequel, C will denote any strictly positive real number independent of N and Δt; typically it will depend on σ, T and U_0.

We also will denote by $\mathbb{E}_\mu f(z_t)$ the expectation of $f(z_t)$ when z_0 has the distribution μ, where (z_t) is the Markov process solution to (17).

4.2. **Preliminaries.** As in the case of smooth kernels (cf. [4]), we decompose the error at time t_k, $\big(V(t_k,.) - \overline{V}_{t_k}(.)\big)$, into three terms:

$$
\begin{aligned}
&\mathbb{E}\left\|V(t_k,x) \;-\; \overline{V}_{t_k}(x)\right\|_{L^1(\mathbb{R})}\\
&\quad\le \left\|\mathbb{E}_{U_0}H(x-z_{t_k}) \;-\; \mathbb{E}_{\overline{U}_0}H(x-z_{t_k})\right\|_{L^1(\mathbb{R})}\\
&\qquad + \mathbb{E}\left\|\mathbb{E}_{\overline{U}_0}H(x-z_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^{N}H(x-z^i_{t_k})\right\|_{L^1(\mathbb{R})}\\
&\qquad + \mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^{N}H(x-z^i_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^{N}H(x-Y^i_{t_k})\right\|_{L^1(\mathbb{R})}.
\end{aligned}
\tag{27}
$$

In the right handside, the first term corresponds to the approximation of the initial condition V_0 by the piecewise constant function $\overline{V}_0$. The second term corresponds to the introduction of the independent processes (z^i_t) and is a statistical error. Estimates on these two terms are obtained in Bossy & Talay [4] (Lemmae 2.4 and 2.5) where the case of smooth interaction kernels is studied :

$$
\left\|\mathbb{E}_{U_0}H(x-z_{t_k}) - \mathbb{E}_{\overline{U}_0}H(x-z_{t_k})\right\|_{L^1(\mathbb{R})} \le C\,\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})},
\tag{28}
$$

$$
\mathbb{E}\left\|\mathbb{E}_{\overline{U}_0}H(x-z_{t_k}) - \frac{1}{N}\sum_{i=1}^{N}H(x-z^i_{t_k})\right\|_{L^1(\mathbb{R})} \;\le\; \frac{C}{\sqrt{N}}.
\tag{29}
$$

The proofs of these two inequalities use the following estimates on the density of the transition probability $\gamma(t,x,y)$ of the process $(z_t(x))$:

: • if U_0 satisfies (H1-ii), Lemma 2.3 shows that V is Lipschitz in x (and similarly we can also show that V is Hölder in time with exponent $\frac{1}{2}$), so that one has the following estimates (cf. Friedman [9], p.139-150 or Chapter 1 of [8]) : for any T, there exist strictly positive constants C_0 and C_1 such that, $\forall t\in[0,T]$, $\forall x,y$, $\forall \bar\sigma>\sigma$,

$$
|\gamma_t(x,y)| \;\le\; \frac{C_0}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{2\bar\sigma^2 t}\right),
\tag{30}
$$

$$
\left|\frac{\partial}{\partial y}\gamma_t(x,y)\right| \;\le\; \frac{C_1}{t}\exp\left(-\frac{(x-y)^2}{2\bar\sigma^2 t}\right).
\tag{31}
$$

The proof of (28) is based on (31), and the proof of (29) is based on (30) (see [4]).

: • If U_0 is a Dirac measure, there is no initialisation error and we just need to prove (30) to obtain (29). In this case, Friedman's hypotheses to get (30) are not satisfied (the drift coefficient of $(z_t(x))$ is V, which is not smooth enough); nevertheless, we can prove:

Lemma 4.1. *Under (H1-i), if $\gamma_t(x,y)$ denotes the density of the law of $z_t(x)$ ($t \in (0,T]$), then there exists a constant C_0 only depending on T and σ such that*

$$\gamma_t(x,y) \;\leq\; \frac{C_0}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(y-x)^2}{4t\sigma^2}\right).$$

The proof of this lemma, postponed to the Appendix, uses a representation formula for $\gamma_t(x,y)$ given in I.I. Gihman and A.V. Skorohod [10].

Thus, it remains to treat the third term of the right handside of (27), namely

$$\mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k})\right\|_{L^1(\mathbb{R})}. \tag{32}$$

When the interaction kernel is smooth (cf. [4]) one can separately treat

$$\mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^N H(x - \overline{z}^i_{t_k})\right\|_{L^1(\mathbb{R})}$$

and

$$\mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^N H(x - \overline{z}^i_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k})\right\|_{L^1(\mathbb{R})}.$$

Here, as the kernel is equal to the Heaviside function, this method does not work and a more complex analysis must be developed. The rest of this section is devoted to the proof of the

Lemma 4.2. *There exists a constant $C > 0$ only depending on V_0, σ and T such that, for all $k = 1, \ldots, K$,*

$$\mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) - \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k})\right\|_{L^1(\mathbb{R})} \;\leq\; C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right).$$

In the proof of this estimate we use that for any $t \in (0,T]$, $V(t,\cdot)$ is Lipschitz in x with a Lipschitz constant bounded from above by $\dfrac{L_0}{\sqrt{t}}$, which is true under (H1) (cf. Lemma 2.3). In the case where U_0 is smooth, some steps of the proof can be simplified but the convergence rate is not improved.

4.3. **Proof of Lemma 4.2.** Observing that

$$\forall\, a,\; b \in \mathbb{R}, \qquad \int_{\mathbb{R}} \mid H(x-a) - H(x-b) \mid dx \;=\; |a-b|, \tag{33}$$

one gets

$$\mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^N H(x - z^i_{t_k}) \;-\; \frac{1}{N}\sum_{i=1}^N H(x - Y^i_{t_k})\right\|_{L^1(\mathbb{R})} \;\leq\; \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|z^i_{t_k} \;-\; Y^i_{t_k}\right|.$$

Our objective is to bound $\left(\dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \mathbb{E}\left|z^i_{t_k} \; - \; Y^i_{t_k}\right|\right)_{k=0,..,K}$ from above.

Observe that

$$
\begin{aligned}
\mathbb{E}\left|z^i_{t_k} - Y^i_{t_k}\right| \; & \leq \mathbb{E}|z^i_{t_{k-1}} - Y^i_{t_{k-1}}| \; + \mathbb{E}\left|\int_{t_{k-1}}^{t_k} V(s, z^i_s)ds - \Delta t\overline{V}_{t_{k-1}}(Y^i_{t_{k-1}})\right| \\
& \leq \mathbb{E}\left|z^i_{t_{k-1}} - Y^i_{t_{k-1}}\right| \; + \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_{t_{k-1}})\right| ds \\
& \qquad + \Delta t \; \mathbb{E}\left|V(t_{k-1}, z^i_{t_{k-1}}) - \overline{V}_{t_{k-1}}(Y^i_{t_{k-1}})\right|.
\end{aligned} \tag{34}
$$

For all $t > 0$, $V(t, \cdot)$ is Lipschitz with a Lipschitz constant bounded from above by $\dfrac{L_0}{\sqrt{t}}$. Therefore,

$$
\begin{aligned}
& \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_{t_{k-1}})\right| ds \\
& \leq \; \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_s)\right| ds + \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(t_{k-1}, z^i_s) - V(t_{k-1}, z^i_{t_{k-1}})\right| ds \\
& \leq \; \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_s)\right| ds + \int_{t_{k-1}}^{t_k} \frac{L_0}{\sqrt{t_{k-1}}}\mathbb{E}|z^i_s - z^i_{t_{k-1}}|ds.
\end{aligned} \tag{35}
$$

When u_0 is smooth, one can bound $\mathbb{E}\displaystyle\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_s)\right| ds$ from above by using the Hölder property of $t \to V(t, x)$; in any case, under (H1-i) or (H1-ii), one can apply Lemma 4.1 (respectively (30)) and get

$$
\begin{aligned}
\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_s)\right| ds \; & \leq \; C \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} |V(s, x) - V(t_{k-1}, x)| \, \frac{1}{\sqrt{s}} dx ds \\
& \leq \; C \int_{t_{k-1}}^{t_k} \mathbb{E}_{U_0} \int_{\mathbb{R}} \left|H(x - z_s) - H(x - z_{t_{k-1}})\right| \frac{1}{\sqrt{s}} dx ds,
\end{aligned}
$$

from which, by (33), one gets that

$$
\mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s, z^i_s) - V(t_{k-1}, z^i_s)\right| ds \; \leq \; \frac{C}{\sqrt{t_{k-1}}} \int_{t_{k-1}}^{t_k} \mathbb{E}_{U_0} \left|z_s - z_{t_{k-1}}\right| ds.
$$

As

$$
\begin{aligned}
& \mathbb{E} \mid z^i_s \; - \; z^i_{t_{k-1}} \mid \; \leq \; \Delta t \; + \; \sigma \mathbb{E} \mid w^i_s \; - \; w^i_{t_{k-1}} \mid, \\
\text{and} \quad & \mathbb{E}_{U_0} \mid z_s \; - \; z_{t_{k-1}} \mid \; \leq \; \Delta t \; + \; \sigma \mathbb{E} \mid w_s \; - \; w_{t_{k-1}} \mid,
\end{aligned}
$$

the inequality (35) becomes:

$$\begin{aligned} \mathbb{E}\int_{t_{k-1}}^{t_k} \left|V(s,z_s^i) - V(t_{k-1},z_{t_{k-1}}^i)\right| ds &\leq \left(\frac{C}{\sqrt{t_{k-1}}} + \frac{L_0}{\sqrt{t_{k-1}}}\right)(\Delta t^2 + \sigma\Delta t^{\frac{3}{2}}) \\ &\leq \frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-1}}}, \end{aligned}$$

where C is a constant depending only on T, σ and V_0. Coming back to (34) and using again that $V(t_{k-1},\cdot)$ is Lipschitz, one gets:

$$\begin{aligned} \mathbb{E}\left|z_{t_k}^i - Y_{t_k}^i\right| &\leq \left(1+\frac{L_0}{\sqrt{t_{k-1}}}\Delta t\right)\mathbb{E}\left|z_{t_{k-1}}^i - Y_{t_{k-1}}^i\right| + \frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-1}}} \\ &\qquad + \Delta t\mathbb{E}\left|V(t_{k-1},Y_{t_{k-1}}^i) - \overline{V}_{t_{k-1}}(Y_{t_{k-1}}^i)\right|, \end{aligned}$$

from which it comes :

$$\begin{aligned} &\mathbb{E}\left|z_{t_k}^i - Y_{t_k}^i\right| \\ &\leq \prod_{l=1}^{k-1}\left(1+\frac{L_0}{\sqrt{t_{k-l}}}\Delta t\right)\mathbb{E}\left|z_{\Delta t}^i - Y_{\Delta t}^i\right| + \frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-1}}} + \Delta t\mathbb{E}\left|V(t_{k-1},Y_{t_{k-1}}^i) - \overline{V}_{t_{k-1}}(Y_{t_{k-1}}^i)\right| \\ &+\sum_{l=2}^{k-1}\prod_{j=1}^{l-1}\left(1+\frac{L_0}{\sqrt{t_{k-j}}}\Delta t\right)\left(\frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-l}}} + \Delta t\mathbb{E}\left|V(t_{k-l},Y_{t_{k-l}}^i) - \overline{V}_{t_{k-l}}(Y_{t_{k-l}}^i)\right|\right). \end{aligned}$$

For all $l\in\{2,\ldots,k-1\}$,

$$\prod_{j=1}^{l-1}(1+\frac{L_0}{\sqrt{t_{k-j}}}\Delta t) \leq \exp\left(\sum_{j=k-l+1}^{k-1}\frac{L_0\Delta t}{\sqrt{j\Delta t}}\right) \leq \exp\left(\int_{t_{k-l+1}}^{t_k}\frac{L_0}{\sqrt{s}}ds\right) \leq \exp\left(2L_0\sqrt{T}\right).$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left|z_{t_k}^i - Y_{t_k}^i\right| \\ &\leq \exp\left(2L_0\sqrt{T}\right)\left(\mathbb{E}\left|z_{\Delta t}^i - Y_{\Delta t}^i\right| + \sum_{l=1}^{k-1}\Delta t\mathbb{E}\left|V(t_l,Y_{t_l}^i) - \overline{V}_{t_l}(Y_{t_l}^i)\right| + \sum_{l=1}^{k-1}\frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-l}}}\right). \end{aligned}$$

As $z_0^i = Y_0^i$, one has that $\mathbb{E}|z_{\Delta t}^i - Y_{\Delta t}^i| \leq \Delta t$, so that

$$\mathbb{E}\left|z_{t_k}^i - Y_{t_k}^i\right| \leq \exp(L_0T)\left(\Delta t + \sum_{l=1}^{k-1}\Delta t\mathbb{E}\left|V(t_l,Y_{t_l}^i) - \overline{V}_{t_l}(Y_{t_l}^i)\right| + 2C\sqrt{t_k}\sqrt{\Delta t}\right).$$

For $k=0,..,K$ set

$$E_k := \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|V(t_k,Y_{t_k}^i) - \overline{V}_{t_k}(Y_{t_k}^i)\right|. \tag{36}$$

Then,

$$\left\{\begin{array}{l}\displaystyle\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|z_{t_k}^i - Y_{t_k}^i\right| \leq C\left(\sum_{l=1}^{k-1}\Delta t\,E_l + \sqrt{t_k}\,\sqrt{\Delta t}\right),\ k=2,\dots,K,\\ \displaystyle\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|z_{\Delta t}^i - Y_{\Delta t}^i\right| \leq \Delta t,\end{array}\right. \tag{37}$$

where C is a constant depending only on T, σ and V_0.

Below (Lemma 4.3) we will prove that there exists a constant C depending only on V_0, T and σ such that, for any $k = 0,\dots,K$,

$$E_k \leq C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right). \tag{38}$$

Assuming this result, (37) becomes

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|z_{t_k}^i - Y_{t_k}^i\right| \leq C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right). \tag{39}$$

Thus, Lemma 4.2 is proved. ∎

Lemma 4.3. *There exists a constant C depending only on V_0, T and σ such that, for any $k = 0,\dots,K$,*

$$E_k = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|V(t_k, Y_{t_k}^i) - \overline{V}_{t_k}(Y_{t_k}^i)\right| \leq C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right). \tag{40}$$

Proof. First note that, when U_0 is a Dirac measure, then $V_0 = \overline{V}_0$ and thus, $E_0 = 0$.

When (H1-ii) holds, by definition of the (y_0^i)'s one has

$$\begin{aligned}E_0 &= \frac{1}{N}\sum_{i=1}^{N}\left|V(0, z_0^i) - \overline{V}_0(Y_0^i)\right|\\ &= \frac{1}{N}\sum_{i=1}^{N-1}\left|V_0(V_0^{-1}(\frac{i}{N})) - \frac{i}{N}\right| + \left|V_0(V_0^{-1}(1-\frac{1}{2N})) - 1\right| = \frac{1}{2N}.\end{aligned}$$

Now fix $k \in \{1, \ldots, K\}$ and decompose E_k into three terms :

$$
\begin{aligned}
E_k &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|V(t_k, Y^i_{t_k}) - \frac{1}{N}\sum_{j=1}^N H(Y^i_{t_k} - Y^j_{t_k})\right| \\
&\leq \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|V(t_k, Y^i_{t_k}) - V(t_k, z^i_{t_k})\right| + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|V(t_k, z^i_{t_k}) - \frac{1}{N}\sum_{j=1}^N H(z^i_{t_k} - z^j_{t_k})\right| \\
&\qquad + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N H(z^i_{t_k} - z^j_{t_k}) - \frac{1}{N}\sum_{j=1}^N H(Y^i_{t_k} - Y^j_{t_k})\right| \\
&\leq \frac{L_0}{\sqrt{t_k}}\,\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|z^i_{t_k} - Y^i_{t_k}\right| \;+\; \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|V(t_k, z^i_{t_k}) - \frac{1}{N}\sum_{j=1}^N H(z^i_{t_k} - z^j_{t_k})\right| \qquad (41) \\
&\qquad + \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\left|H(z^i_{t_k} - z^j_{t_k}) - H(Y^i_{t_k} - Y^j_{t_k})\right|.
\end{aligned}
$$

We now use the following arguments:

- We have just seen (cf. (37)) how we can bound $\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|z^i_{t_k} - Y^i_{t_k}\right|$ from above in terms of the E_l's ($l = 0, \ldots, k-1$); note that we cannot use (39) since we use (40) to get it.
- In the next subsection (Lemma 4.4), we will prove the following upper bound for the second term of the right handside of (41) : there exists a constant C, depending only on T and σ such that, for all $t \in [0, T]$ and any $i = 1, \ldots, N$, one has

$$
\mathbb{E}\left|V(t, z^i_t) - \frac{1}{N}\sum_{j=1}^N H(z^i_t - z^j_t)\right| \leq \frac{C}{\sqrt{N}}. \tag{42}
$$

- Set

$$
F_k := \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\left|H(z^i_{t_k} - z^j_{t_k}) - \frac{1}{N}\sum_{j=1}^N H(Y^i_{t_k} - Y^j_{t_k})\right|.
$$

  In the next subsection (Lemma 4.5), we will prove that there exists a constant C, depending only on T, σ and V_0 such that one has

$$
F_k \leq \begin{cases} C\left(\sqrt{\Delta t} + \dfrac{1}{N} + \displaystyle\sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}}\left(E_l + \frac{\Delta t}{\sqrt{t_l}}\sum_{q=1}^{l-1} E_q\right)\right) & \text{for } k = 3, \ldots, K, \\ C\left(\sqrt{\Delta t} + \dfrac{1}{N}\right) \text{ for } k = 1, 2. & \end{cases} \tag{43}
$$

Assume the above estimates; for $k \geq 3$, one then has :

$$\begin{aligned} E_k \;\leq\; & \frac{C}{\sqrt{t_k}} \left(\sum_{l=1}^{k-1} \Delta t \, E_l \;+\; \sqrt{t_k}\,\sqrt{\Delta t} \right) \;+\; \frac{C}{\sqrt{N}} \\ & + \; C \left[\sqrt{\Delta t} \;+\; \frac{1}{N} \;+\; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}} \left(E_l \;+\; \frac{\Delta t}{\sqrt{t_l}} \sum_{q=1}^{l-1} E_q \right) \right]. \end{aligned}$$

Besides, as $I\!\!E\left|z^i_{\Delta t} - Y^i_{\Delta t}\right| \leq \Delta t$ and $I\!\!E\left|z^i_{2\Delta t} - Y^i_{2\Delta t}\right| \leq 2\Delta t$, the inequalities (42) and (43) imply that $E_1 \leq C\left(\sqrt{\Delta t} + \dfrac{1}{\sqrt{N}}\right)$ and $E_2 \leq C\left(\sqrt{\Delta t} + \dfrac{1}{\sqrt{N}}\right)$. Thus,

$$\left\{ \begin{aligned} & E_k \leq \; C \left[\sum_{l=1}^{k-1} \frac{\Delta t}{\sqrt{t_k}} \, E_l \;+\; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}} \, E_l \;+\; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}\sqrt{t_l}} \, \sum_{q=1}^{l-1} \Delta t E_q \right] \\ & \qquad\quad + C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right), \; k = 3, \ldots, K, \\ & E_0 \leq \frac{1}{2N}, \; E_1 \;\leq\; C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right), \; E_2 \;\leq\; C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right), \end{aligned} \right.$$

where C is a constant depending only on T, σ and V_0.

We are now in a position to prove (40).

For all $t \in [0, T]$, define the function $\varepsilon(t)$ by

$$\varepsilon(t) \;:=\; \sum_{k=0}^{K-1} \mathbb{1}_{\,[t_k, t_{k+1})}(t)\, E_k, \;\; \varepsilon(T) \;:=\; E_K.$$

This function is measurable, positive and bounded by 1 (remember the definition (36)).

The function $s \longrightarrow \dfrac{1}{\sqrt{t_k - s}}$ being increasing on $(0, t_k)$, one has that

$$\sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}} E_l \;\leq\; \int_0^{t_k} \frac{\varepsilon(s)}{\sqrt{t_k - s}} ds.$$

The function $s \longrightarrow \dfrac{1}{\sqrt{s}\sqrt{t_k - s}}$ being decreasing on $(0, t_k/2)$ and increasing on $(t_k/2, t_k)$, one has that

$$\begin{aligned} \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}\sqrt{t_l}} \sum_{q=1}^{l-1} \Delta t E_q \quad & \leq \quad \int_0^{t_k} \epsilon(s) ds \; \sum_{l=2}^{k-1} \frac{\Delta t}{\sqrt{t_k - t_l}\sqrt{t_l}} \\ & \leq \quad \int_0^{t_k} \epsilon(s) ds \; \int_0^{t_k} \frac{1}{\sqrt{t_k - s}\sqrt{s}} ds \;\leq\; 4 \; \int_{\Delta t}^{t_k} \varepsilon(s) ds. \end{aligned}$$

Thus, the function $\varepsilon(t)$ satisfies:

$$\varepsilon(t_k) \leq C\left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right) + \int_0^{t_k} C\left(\frac{1}{\sqrt{t_k}} + \frac{1}{\sqrt{t_k - s}} + 1\right)\varepsilon(s)\,ds.$$

We conclude by applying Gronwall's lemma :

$$\varepsilon(T) \ \leq \ C\ \left(\sqrt{\Delta t} + \frac{1}{\sqrt{N}}\right). \ \blacksquare$$

4.4. **Technical lemmae.** We now prove the estimates (42) and (43).

Lemma 4.4 (Proof of (42))**.** *There exists a constant C depending only on T and σ such that, for all $t \in [0,T]$ and for any $i = 1, \ldots, N$, one has*

$$\mathbb{E}\left|V(t, z_t^i) - \frac{1}{N}\sum_{j=1}^{N} H(z_t^i - z_t^j)\right| \leq \frac{C}{\sqrt{N}}.$$

Proof. For $i \in \{1, \ldots, N\}$ and $t \in [0,T]$ fixed, consider

$$\begin{aligned}
&\mathbb{E}\left|V(t, z_t^i) - \frac{1}{N}\sum_{j=1}^{N} H(z_t^i - z_t^j)\right| \\
&\leq \mathbb{E}\left|V(t, z_t^i) - \mathbb{E}_{\overline{U}_0} H(x - z_t)\Big|_{x = z_t^i}\right| \\
&\qquad + \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N} \mathbb{E} H(x - z_t^j)\Big|_{x = z_t^i} - \frac{1}{N}\sum_{j=1}^{N} H(z_t^i - z_t^j)\right| \\
&:= \ A + B.
\end{aligned}$$

Let us first treat A.

If U_0 is a Dirac measure, then $U_0 = \overline{U}_0$ and A is 0; if (H1-ii) holds, then we can easily use the arguments of the proof of Lemma 3.1 in [4] to prove

$$\left|V(t,x) - \mathbb{E}_{\overline{U}_0} H(x - z_t)\right| \ \leq \ C\ \left\|V_0(\cdot) - \overline{V}_0(\cdot)\right\|_{L^\infty(\mathbb{R})}, \ \forall\, x \ \in \ \mathbb{R}.$$

By definition of $\overline{V}_0$, it comes that

$$A \ \leq \ \frac{1}{N}.$$

Now consider B:

$$
\begin{aligned}
&\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H(x-z_t^j)\Big|_{x=z_t^i} - H(z_t^i - z_t^j)\right)\right)^2 \\
&\quad = \frac{1}{N^2}\sum_{j=1}^{N}\mathbb{E}\left(\mathbb{E}H(x-z_t^j)\Big|_{x=z_t^i} - H(z_t^i - z_t^j)\right)^2 \\
&\quad + \frac{1}{N^2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\mathbb{E}\left[\left(\mathbb{E}H(x-z_t^j)\Big|_{z_t^i} - H(z_t^i - z_t^j)\right)\right. \\
&\qquad\qquad \left.\times\left(\mathbb{E}H(x-z_t^k)\Big|_{z_t^i} - H(z_t^i - z_t^k)\right)\right].
\end{aligned}
$$

The $(z_t^j, j=1,\ldots,N)$ being independent, one gets that

$$
\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H(x-z_t^j)\Big|_{x=z_t^i} - H(z_t^i - z_t^j)\right)\right)^2 \leq \frac{1}{N},
$$

which ends the proof of the lemma. ∎

Lemma 4.5 (Proof of (43)). *There exists a constant C, depending only on T, σ and V_0 such that, for any $k \in \{3,\ldots,K\}$, one has :*

$$
\begin{aligned}
F_k &:= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbb{E}\left|H(z_{t_k}^i - z_{t_k}^j) - H(Y_{t_k}^i - Y_{t_k}^j)\right| \\
&\leq C\left[\sqrt{\Delta t} + \frac{1}{N} + \sum_{l=2}^{k-1}\frac{\Delta t}{\sqrt{t_k - t_l}}\left(E_l + \frac{\Delta t}{\sqrt{t_l}}\sum_{q=1}^{l-1}E_q\right)\right].
\end{aligned}
$$

For $k=0$, $F_k = 0$ and for $k=1,2$, $F_k \leq C\left(\sqrt{\Delta t} + \dfrac{1}{N}\right)$.

Proof. As $z_0^i = Y_0^i$, it is clear that the left handside is 0 when $k=0$.

When $k=1$, $Y_{\Delta t}^i = \overline{z}_{\Delta t}^i$, thus for $i \neq j$,

$$
\begin{aligned}
&\mathbb{E}\left|H(z_{\Delta t}^i - z_{\Delta t}^j) - H(\overline{z}_{\Delta t}^i - \overline{z}_{\Delta t}^j)\right| \\
&\leq \mathbb{E}\left|H(z_{\Delta t}^i - z_{\Delta t}^j) - H(\overline{z}_{\Delta t}^i - z_{\Delta t}^j)\right| + \mathbb{E}\left|H(\overline{z}_{\Delta t}^i - z_{\Delta t}^j) - H(\overline{z}_{\Delta t}^i - \overline{z}_{\Delta t}^j)\right|.
\end{aligned}
$$

Integrate w.r.t. the law of $z^j_{\Delta t}$ and apply Lemma 4.1 or (30) and use (33):

$$\begin{aligned}
&\mathbb{E}\left|H(z^i_{\Delta t}-z^j_{\Delta t})-H(\overline{z}^i_{\Delta t}-z^j_{\Delta t})\right|\\
&\leq\ \mathbb{E}\int_{\mathbb{R}}\frac{C}{\sqrt{\Delta t}}\left|H(z^i_{\Delta t}-x)-H(\overline{z}^i_{\Delta t}-x)\right|dx\ \leq\ \frac{C}{\sqrt{\Delta t}}\mathbb{E}\left|z^i_{\Delta t}-\overline{z}^i_{\Delta t}\right|\\
&\leq\ C\sqrt{\Delta t}.
\end{aligned}$$

Besides, for $i\neq j$:

$$\begin{aligned}
&\mathbb{E}\left|H(\overline{z}^i_{\Delta t}-z^j_{\Delta t})-H(\overline{z}^i_{\Delta t}-\overline{z}^j_{\Delta t})\right|\\
&\leq\ \mathbb{E}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\left|H(x-z^j_{\Delta t})-H(x-\overline{z}^j_{\Delta t})\right|dx\ \leq\ C\sqrt{\Delta t}.
\end{aligned}$$

Thus (43) holds for $k=1$ and, similarly, for $k=2$.

Now fix $k\in\{3,\dots,K\}$. The difficulty is to find the appropriate relation between the left handside of (43) and the E_j's.

For all $x\in\mathbb{R}$, $i=1,\dots,N$ and $l=0,\dots,K$, define the process $(z^{i,l}_t(x))$ by

$$z^{i,l}_t(x)\ :=\ x\ +\ \int_0^t V\left(t_l+s,z^{i,l}_s(x)\right)ds\ +\ \sigma\left(w^i_{t+t_l}\ -\ w^i_{t_l}\right),\ t\in[0,T-t_l]. \tag{44}$$

Then

$$\mathbb{E}\left|H(Y^i_{t_k}-Y^j_{t_k})-H(z^i_{t_k}-z^j_{t_k})\right|=\mathbb{E}\left|H(Y^i_{t_k}-Y^j_{t_k})-H(z^{i,0}_{t_k}(y^i_0)-z^{j,0}_{t_k}(y^j_0))\right|,$$

from which

$$\begin{aligned}
&\mathbb{E}\left|H(Y^i_{t_k}-Y^j_{t_k})-H(z^i_{t_k}-z^j_{t_k})\right|\ \leq \\
&\sum_{l=0}^{k-1}\mathbb{E}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)-H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right|.
\end{aligned}\tag{45}$$

Fix l and $i\neq j$.

$$\begin{aligned}
&\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)-H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right|\ \leq\\
&\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)-H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right|\ +\\
&\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)-H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right|\\
&=:\ A\ +\ B.
\end{aligned}$$

We first bound $\mathbb{E}A$ from above.

Let $(\mathcal{F}_t)$ the σ-field generated by $(w^i_t\ ,\ 1 \le i \le N)$. Then, denoting by Δw^i_p the quantity $w^i_p - w^i_{p-1}$, one has :

$$
\begin{aligned}
\mathbb{E}A &= \mathbb{E}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&= \mathbb{E}\mathbb{E}^{\mathcal{F}_{t_{k-(l+1)}}}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) - \right. \\
&\qquad\qquad \left. -H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&= \mathbb{E}\mathbb{E}^{\mathcal{F}_{t_{k-(l+1)}}}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t\overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + \Delta w^i_{k-l}) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)\right. \\
&\quad \left. - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t\overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + \Delta w^i_{k-l}) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right|.
\end{aligned}
$$

Let $g_{\sigma^2\Delta t}(\cdot)$ denote the Gaussian density of mean 0 and variance $\sigma^2\Delta t$.

The random variables $z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})$ and $z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})$ are independent of Δw^i_{k-l}; besides, $z^{i,k-l}_{l\Delta t}(x)$ is independent of Δw^i_{k-l}; therefore,

$$
\begin{aligned}
\mathbb{E}A = \mathbb{E}\int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)&\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t\overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z) - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)\right. \\
&\left. - H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-(l+1)}} + \Delta t\overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z) - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| dz.
\end{aligned}
$$

Remember that $\gamma^{i,k}_t(x,\cdot)$ denotes the density of the law of $z^{i,k}_t(x)$ defined in (44); as $i \ne j$, it comes :

$$
\begin{aligned}
\mathbb{E}A \ &= \ \mathbb{E}\int_{\mathbb{R}}\left|H\left(y - z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right) \ - \ H\left(y - z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&\qquad \times \int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)\,\gamma^{i,k-l}_{l\Delta t}\left(y, Y^i_{t_{k-(l+1)}} + \Delta t\overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z\right) dzdy.
\end{aligned}
$$

We apply Lemma 4.1 or (30) and get

$$
\gamma^{i,k-l}_t(x,y) \ \le \ \frac{C}{\sqrt{2\pi t\sigma^2}}\exp\left(-\frac{(x-y)^2}{4t\sigma^2}\right). \tag{46}
$$

Thus,

$$
\int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)\,\gamma^{i,k-l}_{l\Delta t}\left(y, Y^i_{t_{k-(l+1)}} + \Delta t\overline{V}_{t_{k-(l+1)}}(Y^i_{t_{k-(l+1)}}) + z\right)\,dz \ \le \ \frac{C}{\sqrt{t_{l+1}}}.
$$

Finally using (33) once more, we get :

$$
\mathbb{E}A \ \le \ \frac{C}{\sqrt{t_{l+1}}}\,\mathbb{E}\left|z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}}) \ - \ z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right|. \tag{47}
$$

To bound $\mathbb{E}B$ from above, we follow the same way and obtain

$$
\begin{aligned}
\mathbb{E}B \ &\le \ \mathbb{E}\int_{\mathbb{R}}\frac{C}{\sqrt{t_{l+1}}}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - y\right) - H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}}) - y\right)\right| dy \\
&\le \ \frac{C}{\sqrt{t_{l+1}}}\,\mathbb{E}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}}) - z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})\right|. \qquad (48)
\end{aligned}
$$

From(47) and (48), it comes :

$$\begin{aligned}
&\mathbb{E}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)-H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&\quad\leq \frac{C}{\sqrt{t_{l+1}}}\left(\mathbb{E}\left|z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right|\right. \\
&\qquad\qquad\left.+\,\mathbb{E}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})\right|\right).
\end{aligned}\tag{49}$$

Below (Lemma 4.6), we will prove that for any $i=1,\ldots,N$ and for any $l=0,\ldots,k-3$,

$$\begin{aligned}
&\mathbb{E}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})\right| \\
&\quad\leq C\Delta t\,\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right| \\
&\qquad+\frac{C\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)}\Delta t\mathbb{E}\left|V(t_q,Y^i_{t_q})-\overline{V}_{t_q}(Y^i_{t_q})\right|+\sqrt{\Delta t}\right).
\end{aligned}\tag{50}$$

We will also prove that for $l=k-2$ and $l=k-1$,

$$\left(\mathbb{E}\left|z^{i,2}_{t_{k-2}}(Y^i_{2\Delta t})-z^{i,1}_{t_{k-1}}(Y^i_{\Delta t})\right|+\mathbb{E}\left|z^{i,1}_{t_{k-1}}(Y^i_{\Delta t})-z^{i,0}_{t_k}(y^i_0)\right|\right)\leq C\Delta t.$$

Using these estimates in (49), we get for all $l\in\{0,\ldots,k-3\}$:

$$\begin{aligned}
&\mathbb{E}\left|H\left(z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{j,k-l}_{l\Delta t}(Y^j_{t_{k-l}})\right)-H\left(z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})-z^{j,k-(l+1)}_{(l+1)\Delta t}(Y^j_{t_{k-(l+1)}})\right)\right| \\
&\quad\leq\frac{C\,\Delta t}{\sqrt{t_{l+1}}}\left\{\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^j_{t_{k-(l+1)}}\right)-V\left(t_{k-(l+1)},Y^j_{t_{k-(l+1)}}\right)\right|\right. \\
&\qquad+\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right| \\
&\qquad+\frac{\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)}\mathbb{E}\left|V(t_q,Y^j_{t_q})-\overline{V}_{t_q}(Y^j_{t_q})\right|\right) \\
&\qquad\left.+\frac{\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)}\mathbb{E}\left|V(t_q,Y^i_{t_q})-\overline{V}_{t_q}(Y^i_{t_q})\right|\right)+\frac{\sqrt{\Delta t}}{\sqrt{t_{k-(l+1)}}}\right\}.
\end{aligned}$$

For $l=k-2$, one has

$$\mathbb{E}\left|H\left(z^{i,2}_{t_{k-2}}(Y^i_{2\Delta t})-z^{j,2}_{t_{k-2}}(Y^j_{2\Delta t})\right)-H\left(z^{i,1}_{t_{k-1}}(Y^i_{1\Delta t})-z^{j,1}_{t_k})(Y^j_{\Delta t})\right)\right|\leq\frac{C\Delta t}{\sqrt{t_{k-1}}}\leq C\sqrt{\Delta t}\,.$$

For $l=k-1$, one has

$$\mathbb{E}\left|H\left(z^{i,1}_{t_{k-1}}(Y^i_{\Delta t})-z^{j,1}_{t_{k-1}}(Y^j_{\Delta t})\right)-H\left(z^i_{t_k}-z^j_{t_k})\right)\right|\leq\frac{C}{\sqrt{t_k}}\Delta t\leq C\sqrt{\Delta t}.$$

Thus, for $k \geq 3$, using the definition of E_k (cf. (36)), we get

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \mathbb{E}\left|H(z^i_{t_k}-z^j_{t_k})-\frac{1}{N}\sum_{j=1}^N H(Y^i_{t_k}-Y^j_{t_k})\right|\\ &\leq C\sqrt{\Delta t}+\sum_{l=0}^{k-3}\frac{C\Delta t}{\sqrt{t_{l+1}}}\left\{E_{k-(l+1)}+\frac{\Delta t}{\sqrt{t_{k-(l+1)}}}\sum_{q=1}^{k-(l+2)}E_q+\frac{\sqrt{\Delta t}}{\sqrt{t_{k-(l+1)}}}\right\}+\frac{1}{N}.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{l=0}^{k-3}\frac{\Delta t^{\frac{3}{2}}}{\sqrt{\Delta t(l+1)}\sqrt{\Delta t(k-(l+1))}} &= \sum_{l=1}^{k-2}\frac{\Delta t^{\frac{3}{2}}}{\sqrt{l\Delta t}\sqrt{\Delta t(k-l)}}\\ &\leq \sum_{l=1}^{[k/2]}\frac{\Delta t^{\frac{3}{2}}}{\sqrt{l\Delta t}\sqrt{\Delta t(k-[k/2])}} + \sum_{[k/2]+1}^{k-2}\frac{\Delta t^{\frac{3}{2}}}{\sqrt{\Delta t([k/2]+1)}\sqrt{\Delta t(k-l)}}\\ &\leq \Delta t\left(\tfrac{1}{\sqrt{\Delta t(k-[k/2])}}+\tfrac{1}{\sqrt{\Delta t([k/2]+1)}}\right)\int_0^{[k/2]}\tfrac{1}{\sqrt{x}}dx \;\leq\; 4\sqrt{\Delta t}.\end{aligned}$$

We deduce that, for $k \geq 3$, we have that

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \mathbb{E}\left|H(z^i_{t_k}-z^j_{t_k})-\frac{1}{N}\sum_{j=1}^N H(Y^i_{t_k}-Y^j_{t_k})\right|\\ &\qquad\leq \sum_{l=0}^{k-3}\frac{C\,\Delta t}{\sqrt{t_{l+1}}}\left\{E_{k-(l+1)}+\frac{\Delta t}{\sqrt{t_{k-(l+1)}}}\sum_{q=1}^{k-(l+2)}E_q\right\}+C\sqrt{\Delta t}+\frac{1}{N}.\end{aligned}$$

The inequality (43) is proved. ∎

Lemma 4.6 (Proof of (50))**.** *For all $i = 1,\dots,N$ and for all $l = 0,\dots,k-3$, one has that*

$$\begin{aligned}&\mathbb{E}\left|z^{i,k-l}_{l\Delta t}(Y^i_{t_{k-l}})-z^{i,k-(l+1)}_{(l+1)\Delta t}(Y^i_{t_{k-(l+1)}})\right|\\ &\qquad\leq C\Delta t\,\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y^i_{t_{k-(l+1)}}\right)-V\left(t_{k-(l+1)},Y^i_{t_{k-(l+1)}}\right)\right|\\ &\qquad\quad+\frac{C\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)}\Delta t\mathbb{E}\left|V(t_q,Y^i_{t_q})-\overline{V}_{t_q}(Y^i_{t_q})\right|+\sqrt{\Delta t}\right).\end{aligned}\tag{51}$$

For $l = k-2$ and $l = k-1$, it holds that

$$\left(\mathbb{E}\left|z^{i,2}_{t_{k-2}}(Y^i_{2\Delta t})-z^{i,1}_{t_{k-1}}(Y^i_{\Delta t})\right|+\mathbb{E}\left|z^{i,1}_{t_{k-1}}(Y^i_{\Delta t})-z^{i,0}_{t_k}(y^i_0)\right|\right)\leq C\Delta t,\tag{52}$$

where C is a positive constant depending only on T, σ and V_0.

Proof. We already noticed the strong uniqueness of the solution to (17): for all $k = 0,\dots,K$ and $i = 1,\dots,N$,

$$z^{i,k}_{t+\Delta t}(x) = z^{i,k+1}_t\left(z^{i,k}_{\Delta t}(x)\right).$$

An easy computation also shows that

$$\mathbb{E}\left|z_t^{i,k}(x) \;-\; z_t^{i,k}(y)\right| \;\leq\; \exp(L_0\, 2\sqrt{T})\; |x \;-\; y|\,. \tag{53}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left|z_{l\Delta t}^{i,k-l}(Y_{t_{k-l}}^i) - z_{(l+1)\Delta t}^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right| \\
&\qquad = \;\mathbb{E}\left|z_{l\Delta t}^{i,k-l}(Y_{t_{k-l}}^i) - z_{l\Delta t}^{i,k-l}\left(z_{\Delta t}^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right)\right| \\
&\qquad = \;\mathbb{E}\mathbb{E}^{\mathcal{F}_{t_{k-l}}}\left|z_{l\Delta t}^{i,k-l}(Y_{t_{k-l}}^i) - z_{l\Delta t}^{i,k-l}\left(z_{\Delta t}^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right)\right| \\
&\qquad \leq \;\exp(L_0\, 2\sqrt{T})\;\mathbb{E}\left|Y_{t_{k-l}}^i - z_{\Delta t}^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right|. \qquad (54)
\end{aligned}$$

We then obtain (52) from

$$\mathbb{E}\left|Y_{\Delta t}^i \;-\; z_{\Delta t}^{i,0}(y_0^i)\right| \;=\; \mathbb{E}\left|\Delta t\overline{V}_0\left(y_0^i\right) \;-\; \int_0^{\Delta t} V\left(s, z_s^0(y_0^i)\right) ds\right| \;\leq\; \Delta t$$

and

$$\mathbb{E}\left|Y_{2\Delta t}^i \;-\; z_{\Delta t}^{i,1}(Y_{\Delta t}^i)\right| \;=\; \mathbb{E}\left|\Delta t\overline{V}_{\Delta t}\left(Y_{\Delta t}^i\right) \;-\; \int_0^{\Delta t} V\left(\Delta t + s, z_s^{i,1}(Y_{\Delta t}^i)\right) ds\right| \;\leq\; \Delta t.$$

Now, for $l \in \{0, \dots, k-3\}$:

$$\begin{aligned}
&\mathbb{E}\left|Y_{t_{k-l}}^i \;-\; z_{\Delta t}^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right| \\
&= \mathbb{E}\left|\Delta t\overline{V}_{t_{k-(l+1)}}\left(Y_{t_{k-(l+1)}}^i\right) - \int_0^{\Delta t} V\left(t_{k-(l+1)} + s, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right) ds\right| \\
&\leq \;\Delta t\mathbb{E}\left|\overline{V}_{t_{k-(l+1)}}\left(Y_{t_{k-(l+1)}}^i\right) \;-\; V\left(t_{k-(l+1)}, Y_{t_{k-(l+1)}}^i\right)\right| \qquad (55)\\
&\quad + \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)} + s, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right) - V\left(t_{k-(l+1)}, Y_{t_{k-(l+1)}}^i\right)\right| ds.
\end{aligned}$$

As V is Lipschitz, it comes that

$$\begin{aligned}
&\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)} + s, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right) - V\left(t_{k-(l+1)}, Y_{t_{k-(l+1)}}^i\right)\right| ds \\
&\leq \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)} + s, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right) - V\left(t_{k-(l+1)}, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right)\right| ds \\
&\quad + \int_0^{\Delta t}\frac{L_0}{\sqrt{t_{k-(l+1)}}}\mathbb{E}\left|z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i) - Y_{t_{k-(l+1)}}^i\right| ds \\
&\leq \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)} + s, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right) - V\left(t_{k-(l+1)}, z_s^{i,k-(l+1)}(Y_{t_{k-(l+1)}}^i)\right)\right| ds \\
&\quad + \frac{L_0}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t \;+\; \sigma\sqrt{\Delta t}\right).
\end{aligned}$$

We introduce the random variable $z_s^{i,k-(l+1)}(z^i_{t_{k-(l+1)}}) =: z^i_{t_{k-(l+1)}+s}$.

$$
\begin{aligned}
&\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| ds \\
&\leq\ \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(z^i_{t_{k-(l+1)}})\right)\right| ds \\
&\ \ + \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z^i_{t_{k-(l+1)}+s}\right) - V\left(t_{k-(l+1)}, z^i_{t_{k-(l+1)}+s}\right)\right| ds \\
&\ \ + \mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}, z_s^{i,k-(l+1)}(z^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right)\right| ds \\
&\ \ + \frac{L_0}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t\ +\ \sigma\sqrt{\Delta t}\right).
\end{aligned}
$$

Using that $V(t,\cdot)$ is Lipschitz, (53) and (46), we get:

$$
\begin{aligned}
&\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| ds \\
&\leq\ \frac{C}{\sqrt{t_{k-(l+1)}}}\ \Delta t\ \mathbb{E}\left|z^i_{t_{k-(l+1)}}\ -\ Y^i_{t_{k-(l+1)}}\right| \\
&\ \ +\ \int_0^{\Delta t}\int_{\mathbb{R}}\left|\mathbb{E}_{U_0}H(x - z_{t_{k-(l+1)}+s})\ -\ \mathbb{E}_{U_0}H(x - z_{t_{k-(l+1)}})\right|\frac{C}{\sqrt{t_{k-(l+1)}}}dx\ ds \\
&\ \ +\ \frac{C}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t\ +\ \sigma\sqrt{\Delta t}\right).
\end{aligned}
$$

Applying (33), it comes that

$$
\begin{aligned}
&\mathbb{E}\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| ds \\
&\leq\ \frac{C}{\sqrt{t_{k-(l+1)}}}\ \Delta t\ \mathbb{E}\left|z^i_{t_{k-(l+1)}}\ -\ Y^i_{t_{k-(l+1)}}\right| \\
&\ \ +\ \frac{C}{\sqrt{t_{k-(l+1)}}}\Delta t\left(\Delta t\ +\ \sigma\sqrt{\Delta t}\right).
\end{aligned}
$$

Finally, we bound $I\!\!E\left|z^i_{t_{k-(l+1)}} - Y^i_{t_{k-(l+1)}}\right|$ from above as in (37) and we get

$$
\begin{aligned}
& I\!\!E\int_0^{\Delta t}\left|V\left(t_{k-(l+1)}+s, z_s^{i,k-(l+1)}(Y^i_{t_{k-(l+1)}})\right) - V\left(t_{k-(l+1)}, Y^i_{t_{k-(l+1)}}\right)\right| ds \\
\leq\ & \frac{C\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)} \Delta t I\!\!E\left|V(t_q, Y^i_{t_q}) - \overline{V}_{t_l}(Y^i_{t_q})\right| + \sqrt{\Delta t}\right) \\
& + \frac{C\Delta t^{\frac{3}{2}}}{\sqrt{t_{k-(l+1)}}} \\
\leq\ & \frac{C\Delta t}{\sqrt{t_{k-(l+1)}}}\left(\sum_{q=1}^{k-(l+2)} \Delta t I\!\!E\left|V(t_q, Y^i_{t_q}) - \overline{V}_{t_q}(Y^i_{t_q})\right| + \sqrt{\Delta t}\right).
\end{aligned}
$$

We conclude by using this estimate in (55) and then by considering (54). ∎

5. Conclusion

We have constructed a stochastic particle method for the one-dimensional Burgers equation and given its convergence rate for the $L^1(I\!\!R \times \Omega)$-norm of the error.

Here, the initial condition is supposed equal to a distribution function. It is not too hard to extend the method and the theoretical estimate of the convergence rate to non monotonic initial conditions: this is done in Bossy & Talay [4] and in Bossy [3].

Our next objective is to extend the algorithm and our error analysis to treat the 2-D inviscid Navier-Stokes equation, which would permit to give new error estimates for the Chorin's random vortex methods. The additional difficulty is due to the fact that the corresponding interaction kernel is singular.

Appendix A. Appendix

A.1. **Proof of the proposition 2.2.** We again stress that this proof is adapted from [26]; we give the essential arguments, the details of the computations can be found in M. Bossy's thesis [3].

We start with an easy lemma.

Lemma A.1. *Under $(H0)$, the function $V(t,x) = I\!\!E H(x - X_t)$ is integrable in $-\infty$; more precisely, it satisfies: there exist strictly positive constants C, γ, δ such that, for all $t \in [0,T]$,*

$$\forall x < -M, V(t,x) \ \leq\ C\ \exp\left(-\frac{(x+\delta)^2}{\gamma}\right).$$

*Proof.*The proof only requires easy computations from

$$\begin{aligned} V(t,x) &= \mathbb{P}(X_t \le x) = \mathbb{P}\left(X_0 + \int_0^t \left(\int_{\mathbb{R}} H(X_s - y) U_s(dy)\right) ds + \sigma w_t \le x\right) \\ &\le \mathbb{P}(\sigma w_t + X_0 \le x), \end{aligned}$$

and the estimate

$$\forall x \in \mathbb{R}, \quad \int_{|x|}^{+\infty} \exp\left(-\frac{y^2}{2}\right) dy \;\le\; C \exp\left(-\frac{x^2}{2}\right). \quad \blacksquare$$

We are now in a position to prove Proposition 2.2.

For $(t,x) \in [0,T]$, set

$$F(t,x) \;=\; \int_{-\infty}^{x} V(t,y)dy$$

(which is well defined in view of the above lemma) and

$$W(t,x) \;=\; \exp\left(-\frac{1}{\sigma^2} \int_{-\infty}^{x} V(t,y)dy\right).$$

As V is a weak solution of the Burgers equation in $(0,T[\times \mathbb{R}$, F satisfies the following equality in the sense of the distribution:

$$\frac{\partial}{\partial x}\left(-\frac{\partial F}{\partial t} \;+\; \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}\right) \;=\; \frac{1}{2}\frac{\partial}{\partial x}(V^2\,) \quad \text{ in } (0,T[\times \mathbb{R}.$$

The distributions $(-\dfrac{\partial F}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F}{\partial x^2}\,)$ and $\frac{1}{2}V^2$ have the same spatial derivatives, therefore their difference is a distribution invariant by translations along the x-axis. Then, for any test function Φ and any $z \in \mathbb{R}$, one has

$$\begin{aligned} &< -\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} - \frac{1}{2}V^2, \Phi > \\ &= \int F(t,x)\left(\frac{\partial \Phi}{\partial t}(t,x+z) + \frac{1}{2}\sigma^2 \frac{\partial^2 \Phi}{\partial x^2}(t,x+z)\right) dtdx - \int \frac{1}{2}V^2(t,x)\Phi(t,x+z)dtdx \\ &= \int F(t,x-z)\left(\frac{\partial \Phi}{\partial t}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(t,x)\right) dtdx - \int \frac{1}{2}V^2(t,x-z)\Phi(t,x)dtdx. \end{aligned}$$

Using the preceding lemma, the bounded convergence theorem and the fact that $V(t,.)$ is a distribution function, one can check that the right handside tends to 0 when z tends to $+\infty$. Thus,

$$-\frac{\partial F}{\partial t} \;+\; \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} \;=\; \frac{1}{2}V^2 \quad \text{ in } (0,T[\times \mathbb{R},$$

in the sense of the distribution.

Denote by (Φ_k) a sequence of smoothing functions in $\mathbb{R}^2$, define $\bar{F}$ and $\bar{V}$ in $\mathbb{R}^2$ by

$$\bar{F}(t,x) \;= \begin{cases} F(t,x) & \text{if } (t,x) \in (0,T] \times \mathbb{R} \\ 0 & \text{if } (t,x) \in \mathbb{R}^2 \backslash (0,T] \times \mathbb{R}\,, \end{cases}$$

and

$$\bar{V}(t,x) \;=\; \begin{cases} V(t,x) & \text{if } (t,x) \in (0,T] \times \mathbb{R} \\ 0 & \text{if } (t,x) \in \mathbb{R}^2 \backslash (0,T] \times \mathbb{R}\ . \end{cases}$$

Define the functions F_k, V_k and W_k on $(0,T] \times \mathbb{R}$ by

$$F_k(t,x) := \left(\Phi_k * \bar{F}\right)(t,x),\;\; V_k(t,x) := \left(\Phi_k * \bar{V}\right)(t,x),$$
$$W_k(t,x) := \exp\left(-\frac{1}{\sigma^2}\, F_k(t,x)\right).$$

First, we note that (W_k) converges to W in the distribution sense: let ϕ be a test function and let K be such that $Supp\, \phi \subset (0,T) \times (-K,K)$; for any k such that $Supp\, \Phi_k \subset (-K,K)^2$, one has

$$\begin{aligned}
&\sigma^2 \int_{(0,T]\times \mathbb{R}} |W_k(t,x) - W(t,x)| \cdot |\phi(t,x)| dtdx \\
&\quad \le \int_{(0,T]\times \mathbb{R}} |F_k(t,x) - F(t,x)|\, |\phi(t,x)|\, dtdx \\
&\quad \le \int_{Supp\phi} \left|F(t,x) - \mathbb{1}_{(-2K,2K)}(x) F(t,x)\right| |\phi(t,x)| dtdx \\
&\qquad + \int_{Supp\phi} \left|\mathbb{1}_{(-2K,2K)}(x) F(t,x) - \left(\mathbb{1}_{(-2K,2K)} F\right)_k (t,x)\right| |\phi(t,x)| dtdx \\
&\qquad + \int_{Supp\phi} \left|\Phi_k * \left(\bar{F} - \mathbb{1}_{(-2K,2K)} \bar{F}\right)(t,x)\right| |\phi(t,x)| dtdx \\
&\quad = \int_{Supp\phi} \left|\mathbb{1}_{(-2K,2K)}(x) F(t,x) - \left(\mathbb{1}_{(-2K,2K)} F\right)_k (t,x)\right| |\phi(t,x)| dtdx.
\end{aligned}$$

Lemma A.1 shows that the function $\mathbb{1}_{(-2K,2K)} F$ belongs to $L^1((0,T) \times \mathbb{R})$, which implies that the sequence $(\mathbb{1}_{(-2K,2K)} F)_k$ converges to $\mathbb{1}_{(-2K,2K)} F$ in $L^1((0,T) \times \mathbb{R})$.

Besides, denoting $(V^2)_k := \Phi_k * V^2$, one can check that

$$\frac{\partial W_k}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 W_k}{\partial x^2} \;=\; \frac{1}{2\sigma^2}\left[(V^2)_k - (\frac{\partial F_k}{\partial x})^2\right] W_k \;=\; \frac{1}{2\sigma^2}\left[(V^2)_k - (V_k)^2\right] W_k\ .$$

Then, letting k go to infinity, easy computations show that W satisfies the heat equation so that, for $0 < s < t \le T$,

$$W(t,x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_{\mathbb{R}} W(s,y)\, \exp\left(-\frac{(x-y)^2}{2\sigma^2(t-s)}\right)\, dy.$$

We now make s tend to zero. Lemma A.1 and the bounded convergence theorem imply that $F(s,x)$ converges to $F(0,x)$ when s tends to 0. Consequently, we get that

$$W(t,x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\mathbb{R}} \exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2t} \;+\; \int_{-\infty}^{y} V_0(z)dz\right]\right)\, dy. \;\blacksquare$$

A.2. **Proof of Lemma 4.1.** In Chapter 13 of [10], I. I. Gihman and A. V. Skorohod give a representation of the transition density of a process (z_t) solution to

$$dz_t = b(t, z_t) + \sigma dw_t,$$

under the condition that the derivatives $\partial_x b(t,x)$ and $\partial_t b(t,x)$ are well defined and that the function B defined by

$$B(t,x) = -\frac{1}{2\sigma^2} b^2(t,\sigma x) - \frac{1}{2}\frac{\partial b}{\partial x}(t,\sigma x) - \int_0^x \frac{1}{\sigma}\frac{\partial b}{\partial t}(t,\sigma z)dz$$

satisfies :

$$\overline{\lim_{x\to\infty}} \frac{\sup_{0\le t\le T} B(t,x)}{1+x^2} = 0. \tag{56}$$

The formula for the density $\gamma_t(x,y)$ of the law of $z_t(x)$ is as follows:

$$\begin{aligned}\gamma_t(x,y) &= \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(y-x)^2}{2t\sigma^2}\right) \exp\left\{\frac{1}{\sigma^2}\left(\int_0^y b(t,z)dz - \int_0^x b(0,z)dz\right)\right\} \\ &\quad\times I\!\!E \exp\left\{t\int_0^1 B\left(ut, \frac{y}{\sigma} + (w(tu)-w(t)) + \frac{u}{\sigma}(x-y)\right)du\right\}.\end{aligned}$$

In our case b is equal to V and the condition (56) seems difficult to check for the process $(z_t(x))$ because of the discontinuity of the derivatives of V at $t=0$ when U_0 is a Dirac measure. Thus, we introduce an intermediate process $(z_t^\varepsilon(x))$ with $\varepsilon > 0$, and we will get the desired result by making ε decrease to 0.

Let $(z_t^\varepsilon(x))$ be defined by

$$z_t^\varepsilon(x) = x + \int_0^t V(s+\varepsilon, z_s^\varepsilon(x))ds + \sigma w_t .$$

Set

$$B^\varepsilon(t,x) = -\frac{1}{2\sigma^2}V^2(t+\varepsilon,\sigma x) - \frac{1}{2}\frac{\partial V}{\partial x}(t+\varepsilon,\sigma x) - \int_0^x \frac{1}{\sigma}\frac{\partial V}{\partial t}(t+\varepsilon,\sigma z)dz.$$

As V is the solution of the Burgers equation,

$$B^\varepsilon(t,x) = \frac{1}{2}\frac{\partial V}{\partial x}(t+\varepsilon,0) - \frac{\partial V}{\partial x}(t+\varepsilon,\sigma x) - \frac{1}{2\sigma^2}V^2(t+\varepsilon,0).$$

We already proved that for all $t\in(0,T]$, $\left\|\frac{\partial V}{\partial x}(t,.)\right\|_{L^\infty(I\!\!R)} \le \frac{C}{\sqrt{t}}$. Thus, for all (t,x) in $(0,T]\times I\!\!R$, one has

$$|B^\varepsilon(t,x)| \le C\left(1+\frac{1}{\sqrt{t+\varepsilon}}\right).$$

The condition (56) is satisfied, so that, $\gamma_t^\varepsilon(x,y)$ denoting the law of $z_t^\varepsilon(x)$,

$$\begin{aligned}\gamma_t^\varepsilon(x,y) &\le \frac{C}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(y-x)^2}{2t\sigma^2}\right) \exp\left\{\frac{1}{\sigma^2}\left(\int_0^y V(t+\varepsilon,z)dz - \int_0^x V(s,z)dz\right)\right\} \\ &\quad\times \exp\left(C(\sqrt{t}+t\right).\end{aligned}$$

Using the fact that $V(t+\varepsilon,x) = \mathbb{E}_{U_0} H(x - z_{t+\varepsilon})$, we can easily show that

$$\left| \int_0^y V(t+\varepsilon,z)dz \; - \; \int_0^x V(s,z)dz \right| \; \leq \; |y-x| + C\sqrt{t}.$$

Thus,

$$\gamma_t^\varepsilon(x,y) \; \leq \; \frac{C}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(y-x)^2 \; - \; 2t|y-x|}{2t\sigma^2} \right) \exp\left(C(\sqrt{t}+t \right).$$

For all $\gamma > \sigma$, an easy computation shows that

$$\exp\left(-\frac{(|y-x|-t)^2}{2t\sigma^2} \right) \; \leq \; \exp\left(-\frac{(y-x)^2}{2t\gamma^2} \right) \; \exp\left(\frac{t^2}{2(\gamma^2-\sigma^2)} \right).$$

Choose $\gamma = \sqrt{2}\,\sigma$:

$$\gamma_t^\varepsilon(x,y) \; \leq \; \frac{C}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(y-x)^2}{4t\sigma^2} \right).$$

Using Lemma (2.3), one easy obtains that $(z_t^\varepsilon(x))$ converges to $(z_t(x))$ in $L^1(\Omega)$ when $\varepsilon \to 0$. Thus, for any positive, continuous and bounded function f, it holds that

$$\int_{\mathbb{R}} f(y)\gamma_t^\varepsilon(x,y)dy \overset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}} f(y)\gamma_t(x,y) \; .$$

Set

$$g_t(x,y) \; := \; \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(y-x)^2}{4t\sigma^2} \right).$$

As

$$\int_{\mathbb{R}} f(y)\gamma_t^\varepsilon(x,y)dy \; \leq \; \int_{\mathbb{R}} f(y)g_t(x,y)dy \; ,$$

it comes that

$$\int_{\mathbb{R}} f(y)\gamma_t(x,y)dy \; \leq \; \int_{\mathbb{R}} f(y)g_t(x,y)dy \; ,$$

which implies that $\gamma_t(x,y) \leq g_t(x,y)$. ∎

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