Approximating the Stationary Solution of the Mc-Kean Vlasov Equation

Angela Ganz.

Contents

Presentation of the Problem.

We consider the Mc-Kean Vlasov Equation in $\mathbb{R}^d$:

$$\frac{\partial U(t,x)}{\partial t} = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 U(t,x)}{\partial x_i^2} - \sum_{i=1}^{d} \frac{\partial(U(t,x) b_i(x, U(t,x)))}{\partial x_i} \tag{1}$$

$$U_0(0,x) = \mu_0(x) \tag{2}$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $b_i(x, U(t,x))) = \int_{\mathbb{R}} b_i(x,y) U(dy)$.

We are interested in the limit function when the time goes to infinite, $\lim_{t\to\infty} U(t,x)$.

Presentation of the Problem .-Probabilistic Approach

- The probabilistic interpretation let us **prove existence, uniqueness** and to give **a discretization algorithm to the solution of the equation** 1.

- We have a stochastic differential equation associated to the Mc Kean Vlasov.

$$dX_t = [\int_{\mathbb{R}^d} b(X_t, y)\mu_t(dy)]dt + dW_t \quad (3)$$

$$\mathcal{L}(X_t) = \mu_t \quad \forall t, \quad (4)$$

  where W_t is a d-dimensional Wiener process.

- Now our objectives are equivalence to:

  Study the existence and the uniqueness of a stationary measure. Propose an algorithm to approximate the stationary measure. Characterize its convergence velocity.

Theoretical Framework.

1. The analysis of the stationary measure existence is generally easier than the uniqueness analysis.
2. Under restrictive hypothesis, **Tamura (1982)** proved there exists a unique stationary solution for the equation (1)and a theoretical velocity of convergence. he use a complicated norm.

$$\|U(t) - U_\infty\|_{L^2(\frac{dx}{U_\infty})} \leq a \exp(-\lambda t)$$

3. Also we can cite **Veretennikov (2004)**. He proved existence and uniqueness using weak hypothesis and treated the velocity with a especial norm (total variation). Our approach is different, because we search a result in a functional norm.

Theoretical Framework.- Our Approach

We use hypothesis stronger than Veretennikov's one and different than Tamura's one. We want to show the existence of the stationary measure of the equation 3, but using Maximum Principle techniques. Thanks of this type the technique we can bound the density in each t and the stationary density. Overall we search measure of the velocity of convergence punctually in the densities.

Discretization Framework.

Classical techniques.

System Particles equation:

$$\begin{aligned} dX_t^{i,N} &= \frac{1}{N}\sum_{j=1}^{d} b(X_t^{i,N}, X_t^{j,N})dt + dW_t^i \\ X_0^i &= \mu_0(x) \qquad \forall i \end{aligned}$$

Euler's squeme (General):

$$\begin{aligned} X_0^{(\triangle t)} &= X_0 \\ X_{t_i} &= X_{t_i-1} + b(X_{t_i-1}, t_i - 1)\triangle t + W_{\triangle t}, \end{aligned}$$

where $W_{\triangle t} \sim N(0, \triangle t)$

Discretization Framework.

- For the analysis in [0,T] we can approximations by Particle of System introduced by Talay and Bossy

  Theorem 1. *Under* H_5*, if* T *is fixed,* $0 < \Delta t < 1$*, such that* $T = \Delta t * K$*,* $K \in \mathbb{N}$*. Let* $V(t_k)$ *be the distribution function associated to* $U(t_k)$ *in time* t_k*. Let* $\bar{V}(t_k)$ *be the approximation by* N *particles then*

$$\mathbb{E}\|V(t_k,\cdot) - \bar{V}(t_k,\cdot)\|_{L^1} \leq C(T)(\|V_0 - \bar{V}_0\|_{L^1}) + \frac{1}{\sqrt{N}} + \sqrt{\Delta t}$$

Discretization.-Our Approach

We denote $\tilde{X}_t$

$$d\tilde{X}_t = a\tilde{X}_{\tau(t)}dt + \int_{\mathbb{R}} \nabla\phi(\tilde{X}_{\tau(t)} - y)\tilde{\mu}_{\tau(t)}(dy)dt + dw_t \qquad (5)$$

with $\tau(t) = \inf\{t_k | t_k < t\}$, t_k $k = 1, \dots K$ discretization times of Euler's schema

Proposition 1. *If $b(x, y) = -ax + \beta\phi'(x - y)$, a positive, ϕ has a bounded second derivative, and $\beta \leq \frac{a}{4\|\phi''\|_\infty}$. Then*

$$\mathbb{E}(X_t - \tilde{X}_t)^2 \leq C(\triangle t)^2$$

C independent on the time.

Conclusions

1. The first problem is to give the sufficient conditions to guarantee existence and uniqueness. These hypothesis let us frame the approximation.

2. The technique of the maximum principle is useful to prove bounding properties on the densities and uniqueness and existence of the stationary measure.

3. Respect to discretization, the usual techniques to prove convergence velocity are useful when the analyze is in [0,T].In our case is necessary to extend result.

Future works

We will concentrate in the velocity of convergence in a functional framework. That is to say, in the existence of the density and inequalities for densities in the stronger hypothesis case.

Approximating the Stationary Solution of the Mc-Kean Vlasov Equation

Angela Ganz.

Hypothesis

H_1. Tamura

- $b = \nabla\phi_1 + \beta\nabla_1\phi_2$
- $\phi_1 = \frac{-\alpha}{2}|x|^2 + \varphi_1(x)$ where $\varphi_1(x) \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha > 0$
- $\phi_2(x,y) = \phi_2(y,x)$
- $\phi_2 \in \mathcal{S}(\mathbb{R}^2 d)$ or $\phi_2(x,y) = \varphi_2(x-y)$, $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$
- $D^2F(U_\infty,\cdot)[u][u] > 0$, where F is the free energy functional and u is a measurable function in $\mathbb{R}^d$ such that $ess.\sup_{x\in\mathbb{R}} |\frac{u}{U_\infty}| < \infty$

Hypothesis

H_2. Veretennikov

1. $b(x,y) = b_0(x) + b_1(x,y)$
2. $\sup_y \langle b_0(x) - b_0(y), x - y \rangle \leq -C_0 |x-y|^2$, with $C_0 > 0$
3. $\lim_{|x|\to\infty} \sup_y \langle b(x,y), x \rangle = -\infty$ or $\lim_{|x|\to\infty} \sup_y \langle b(x,y), x \rangle = -r$
4. $b_1(x,y) - b_1(y,x) = 0$ and $\langle (x-y) - (z-y), b_1(x,y) - b_1(x,z) \rangle \leq 0$
5. (replacing previous hypothesis)
   $\max(|b_1(x,y) - b_1(z,y)|, |b_1(y,x) - b_1(y,z)|) \leq C_{lip}|x-z|$ and
   $C_0 > C_{lip}$ C_0 as in 2

Hypothesis

H_5. Bossy-Talay

- b uniformly bounded in $\mathbb{R}^2$ and lipschitz.
- U_0 admits a continuous density u_0 verifying : $\exists M, \eta$ positive constants such that $u_0(x) \leq \eta \exp(-\alpha \frac{x^2}{2})$ when $|x| > M$ or
- U_0 admits a density u_0 with compact support and it is continuous in its support. Or
- U_0 is a Dirac Measure.