Waves in Stochastic Neural Fields

Paul C Bressloff[1,2]

[1]Department of Mathematics, University of Utah

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I. Stochastic Fronts

Deterministic neural field equation

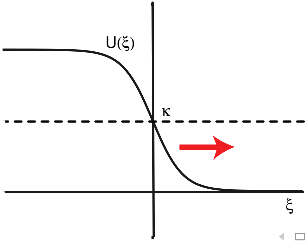
Deterministic neural field equation (Amari 1977)

$$\tau \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')F(u(x',t))dx'.$$

- $u(x,t)$ is local population activity (voltage or current)
- τ is a synaptic or membrane time constant (of order 10 msec),
- $w(x)$ denotes the spatial distribution of excitatory synaptic connections (positive, even function, monotonically decreasing function of $|x|$)

$$w(x) = \frac{1}{2\sigma} e^{-|x|/\sigma},$$

where σ determines the range of synaptic connections.



DETERMINISTIC NEURAL FIELD EQUATION

- $F(u)$ is a nonlinear firing rate function:

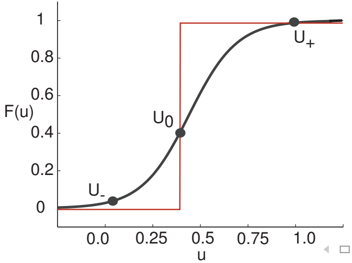
$$F(u) = \frac{1}{1 + e^{-\gamma(u-\kappa)}}$$

- In the high–gain limit $\gamma \to \infty$, this reduces to a Heaviside

$$F(u) \to H(u-\kappa) = \begin{cases} 1 & \text{if } u > \kappa \\ 0 & \text{if } u \leq \kappa \end{cases}$$

- Homogeneous fixed point solution U^*:

$$U^* = W_0 F(U^*), \quad W_0 = \int_{-\infty}^{\infty} w(y) dy.$$



TRAVELING FRONT SOLUTION (*Heavisides*)

- Assume front solution of speed c

$$u(x,t) = U(\xi), \quad \lim_{\xi \to -\infty} U(\xi) = U_+ > 0, \quad \lim_{\xi \to \infty} U(\xi) = 0.$$

with $\xi = x - ct$, and

$$U(0) = \kappa, \quad U(\xi) < \kappa \ \text{for } \xi > 0, \quad U(\xi) > \kappa \ \text{for } \xi < 0$$

- For $F(u) = H(u - \kappa)$ we have

$$-cU'(\xi) + U(\xi) = \int_\xi^\infty w(x)dx \equiv W(\xi),$$

- Integration yields

$$U(\xi) = \mathrm{e}^{\xi/c}\left[\kappa - \frac{1}{c}\int_0^\xi \mathrm{e}^{-y/c} W(y)dy\right].$$

- Boundedness in limit $\xi \to \infty$ for $c > 0$ implies

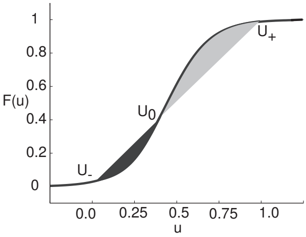
$$\kappa = \frac{1}{c}\int_0^\infty \mathrm{e}^{-y/c} W(y)dy,$$

Traveling front solution (*Sigmoids*)

- Can extend analysis to a sigmoid using a continuation method (Ermentrout and McLeod 93).
- Suppose that $\widetilde{F}(u) = -u + F(u)$ has precisely three zeros at $u = U_{\pm}, U_0$ with $U_- < U_0 < U_+$ and $\widetilde{F}'(U_{\pm}) < 0$.
- There exists a unique traveling front solution with $U(\xi) \to U_{\pm}$ as $\xi \to \mp\infty$ and speed

$$c = \frac{\Gamma}{\int_{-\infty}^{\infty} U'(\xi)^2 F'(U(\xi)) d\xi}, \quad \Gamma = \int_{U_-}^{U_+} \widetilde{F}(U) dU$$

- The sign of c is determined by the sign of the coefficient Γ.



Stochastic Neural Field Equation

- Neural field with additive noise

$$dU(x,t) \quad = \quad \left[-U(x,t) + \int_{-\infty}^{\infty} w(x-y)F(U(y,t))dy\right] dt + \varepsilon^{1/2} dW(x,t).$$

- $dW(x,t)$ is an independent Wiener process

$$\begin{aligned} \langle dW(x,t) \rangle &= 0, \\ \langle dW(x,t) dW(x',t') \rangle &= 2C([x-x']/\lambda)\delta(t-t')dtdt' \end{aligned}$$

- λ is the spatial correlation length of the noise

Separation of timescales

- Fluctuating term generates two distinct phenomena that occur on different time–scales (Geier et al 1993,Sagues, Sancho and Garcia-Ojalvo 2007)

- A diffusive–like displacement $\Delta(t)$ of the front from its uniformly translating position at long time scales, and fluctuations in the front profile around its instantaneous position at short time scales .

- Decompose solution in moving frame as

$$U(x,t) = U_0(\xi - \Delta(t)) + \varepsilon^{1/2}\Phi(\xi - \Delta(t), t)$$

  where U_0 and wave speed c are obtained from the deterministic equation

$$-c\frac{dU_0}{d\xi} + U_0(\xi) = \int_{-\infty}^{\infty} w(\xi - \xi')F(U_0(\xi'))d\xi'$$

  and $d\Delta(t) = O(\epsilon^{1/2})$.

Separation of timescales

- Substitute decomposition into NF equation and expand to $\mathcal{O}(\varepsilon^{1/2})$:

$$\begin{aligned} d\Phi(\xi-\Delta(t),t) &= \widehat{L}\circ\Phi(\xi-\Delta(t),t)dt+\varepsilon^{-1/2}U_0'(\xi-\Delta(t))d\Delta(t) \\ &\quad +d\widetilde{W}(\xi-\Delta(t),t)+\mathcal{O}(\varepsilon^{1/2}), \end{aligned}$$

  where $\widehat{L}$ is the non–self–adjoint linear operator

$$\widehat{L}\circ A(\xi) = c\frac{dA(\xi)}{d\xi}-A(\xi)+\int_{-\infty}^{\infty}w(\xi-\xi')F'(U_0(\xi'))A(\xi')d\xi'$$

  for any function $A(\xi)\in L_2(\mathbb{R})$.

- $\widetilde{W}$ is a Wiener process with $\widetilde{W}(\xi,t)=W(\xi+ct+\Delta(t),t)$.
- The linear operator $\widehat{L}$ has a 1D null space spanned by $U_0'(\xi)$ (Ermentrout and McLeod 1993)

Separation of timescales

- In terms of the inner product

$$\int_{-\infty}^{\infty} B(\xi)\widehat{L}A(\xi)d\xi = \int_{-\infty}^{\infty} \left[\widehat{L}^* B(\xi)\right] A(\xi)d\xi$$

  the adjoint operator is

$$\widehat{L}^* B(\xi) \quad = \quad -c\frac{dB(\xi)}{d\xi} - B(\xi) + F'(U_0(\xi)) \int_{-\infty}^{\infty} w(\xi - \xi')B(\xi')d\xi'$$

- $\widehat{L}^*$ also has a one-dimensional null-space spanned by some function $\mathcal{V}(\xi)$.
- Boundedness of Φ implies solvability condition

$$\int_{-\infty}^{\infty} \mathcal{V}(\xi) \left[U_0'(\xi)d\Delta(t) + \varepsilon^{1/2} d\widetilde{W}(\xi, t)\right] d\xi = 0.$$

SEPARATION OF TIMESCALES

- Thus $\Delta(t)$ satisfies the stochastic differential equation (SDE)

$$d\Delta(t) = -\varepsilon^{1/2} \frac{\int_{-\infty}^{\infty} \mathcal{V}(\xi) d\widetilde{W}(\xi, t) d\xi}{\int_{-\infty}^{\infty} \mathcal{V}(\xi) U_0'(\xi) d\xi}.$$

- Assuming that $\Delta(0) = 0$, we have

$$\langle \Delta(t) \rangle = 0, \quad \langle \Delta(t)^2 \rangle = 2D(\varepsilon)t$$

- $D(\varepsilon)$ is the effective diffusivity

$$D(\varepsilon) = \varepsilon \frac{\int_{-\infty}^{\infty} \mathcal{V}(\xi)^2 d\xi}{\left[\int_{-\infty}^{\infty} \mathcal{V}(\xi) U_0'(\xi) d\xi\right]^2}.$$

Explicit results for Heaviside rate function

- Null vector $\mathcal{V}$ satisfies the equation

$$c\mathcal{V}'(\xi) + \mathcal{V}(\xi) = -\frac{\delta(\xi)}{U_0'(0)} \int_{-\infty}^{\infty} w(\xi')\mathcal{V}(\xi')d\xi'.$$

- Has explicit solution (Bressloff 2001)

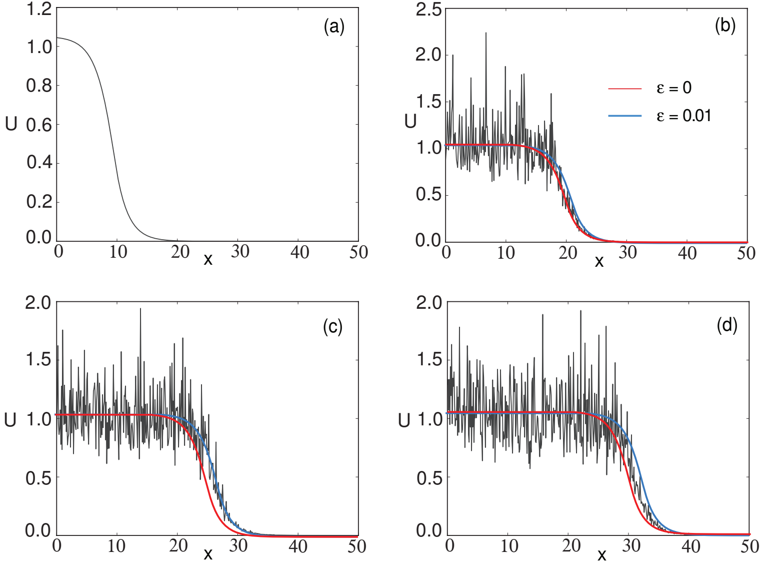
$$\mathcal{V}(\xi) = -H(\xi)\exp\left(-\xi/c\right), \quad c = \frac{\sigma}{2\kappa}(1-2\kappa).$$

- Diffusivity is

$$D(\varepsilon) = \varepsilon \frac{\int_0^{\infty} \mathrm{e}^{-2\xi/c} U_0(\xi)^2 d\xi}{\left[\int_0^{\infty} \mathrm{e}^{-\xi/c} U_0'(\xi) d\xi\right]^2} = \frac{1}{2}\varepsilon\sigma(1+\sigma/c)$$

Numerical Results

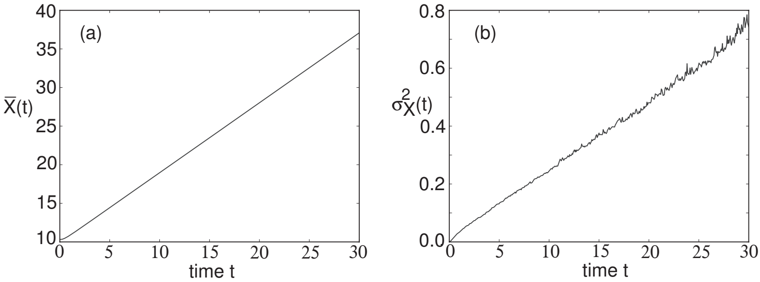
- Snapshots of stochastic front



Numerical Results

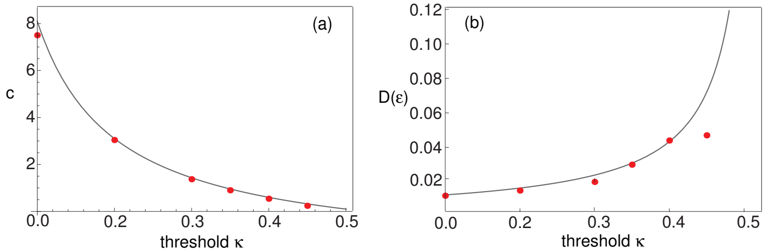
- Time evolution of mean and variance averaged over $N = 4000$ trials - use level sets.
- Determine the positions $X_a(t)$ such that $U(X_a(t), t) = a$, for various level set values $a \in (0.5\kappa, 1.3\kappa)$ and then define

$$\overline{X}(t) = \mathbb{E}[X_a(t)], \quad \sigma_X^2(t) = \mathbb{E}[(X_a(t) - \bar{X}(t))^2]$$



Numerical Results

- Plot of (a) wave speed c and (b) diffusion coefficient $D(\varepsilon)$ as a function of threshold κ



II. Simulus-locked Fronts

Existence of stimulus-locked fronts

- Moving front stimulus with speed v and amplitude $I_0 = I(-\infty) - I(\infty)$

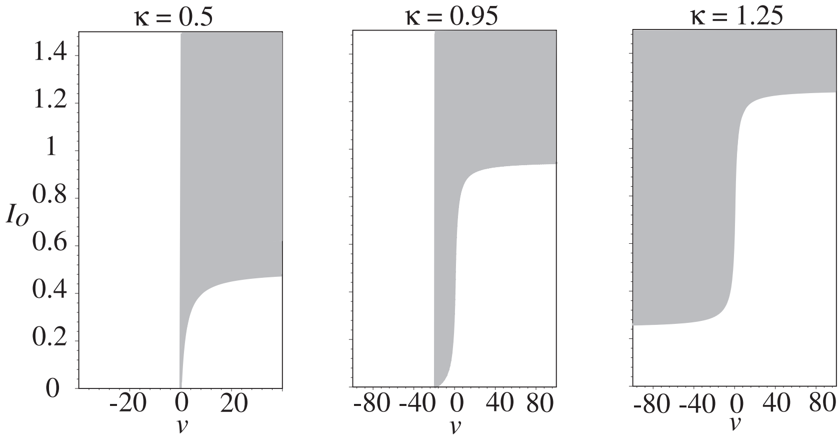
$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')F(u(x',t))dx' + I(x-vt)$$

- Seek a traveling front solution $u(x,t) = U(\xi)$ where $\xi = x - vt$ and $U(\xi_0) = \kappa$ for some $\xi_0 \in \mathbb{R}$.

$$-v\frac{dU(\xi)}{d\xi} = -U(\xi) + \int_{-\infty}^{\xi_0} w(\xi-\xi')d\xi' + I(\xi).$$

- The threshold crossing condition $U(\xi_0) = \kappa$ determines the position ξ_0 of the front relative to the input as a function of speed v, input amplitude I_0 and threshold κ.

EXISTENCE OF STIMULUS-LOCKED FRONTS



Folias and Bressloff (SIAM J. Appl. Math. 2005)

EFFECTS OF EXTRINSIC NOISE ON STIMULUS-LOCKED FRONTS

- Incorporate an external input into the stochastic NF equation

$$dU(x,t) = \left[-U(x,t) + \int_{-\infty}^{\infty} w(x-y)F(U(y,t))dy\right] dt$$
$$\varepsilon^{1/2} I(x-vt)dt + \varepsilon^{1/2} dW(x,t)$$

- Separation of time-scales with $\xi = x - vt$:

$$U(x,t) = U_0(\xi - \Delta(t)) + \varepsilon^{1/2}\Phi(\xi - \Delta(t), t).$$

- Here U_0 satisfies the deterministic equation

$$-c\frac{dU_0}{d\xi} + U_0(\xi) = \int_{-\infty}^{\infty} w(\xi - \xi')F(U_0(\xi'))d\xi'.$$

where c is the natural speed. Assume $v = c + \sqrt{\epsilon} v_1$.

EFFECTS OF EXTRINSIC NOISE ON STIMULUS-LOCKED FRONTS

- Perturbation analysis yields inhomogeneous equation

$$\begin{aligned} d\Phi(\xi,t) &= \widehat{L}\circ\Phi(\xi,t)dt + \varepsilon^{-1/2}U_0'(\xi)d\Delta(t) + d\widetilde{W}(\xi,t) + I(\xi+\Delta(t))dt \\ &\quad + v_1 U_0'(\xi)dt \end{aligned}$$

  where $\widehat{L}$ is the non–self–adjoint linear operator

$$\widehat{L}\circ A(\xi) = v\frac{dA(\xi)}{d\xi} - A(\xi) + \int_{-\infty}^{\infty} w(\xi-\xi')F'(U_0(\xi'))A(\xi')d\xi'$$

  for any function $A(\xi)\in L_2(\mathbb{R})$.
- Let $\mathcal{V}(\xi)$ span the nullspace of the adjoint operator $\widehat{L}^*$

EFFECTS OF EXTRINSIC NOISE ON STIMULUS-LOCKED FRONTS

- The solvability condition shows that $\Delta(t)$ satisfies the SDE

$$d\Delta(t) + G(\Delta(t))dt = d\widehat{W}(t),$$

where

$$G(\Delta) = \varepsilon^{1/2} \frac{\int_{-\infty}^{\infty} \mathcal{V}(\xi)[I(\xi+\Delta) + v_1 U_0'(\xi)]d\xi}{\int_{-\infty}^{\infty} \mathcal{V}(\xi) U_0'(\xi) d\xi}$$

and

$$\widehat{W}(t) = -\varepsilon^{1/2} \frac{\int_{-\infty}^{\infty} \mathcal{V}(\xi)\widetilde{W}(\xi,t)d\xi}{\int_{-\infty}^{\infty} \mathcal{V}(\xi) U_0'(\xi) d\xi}.$$

EFFECTS OF EXTRINSIC NOISE ON STIMULUS-LOCKED FRONTS

- Suppose that there exists a unique shift $\Delta = \xi_0$ for which $G(\xi_0) = 0$ and $G'(\xi_0) > 0$. This repressnts a stable stimulus-locked state in the absence of noise.
- Taylor expanding about the fixed point by setting $Y(t) = \Delta(t) - \xi_0$ with $Y(t) = O(\epsilon^{1/2})$ yields the OU process

$$dY(t) + AY(t)dt = d\widehat{W}(t),$$

where

$$A = \sqrt{\epsilon}\frac{\int_{-\infty}^{\infty} \mathcal{V}(\xi)I'(\xi+\xi_0)d\xi}{\int_{-\infty}^{\infty} \mathcal{V}(\xi)U_0'(\xi)d\xi}$$

EFFECTS OF EXTRINSIC NOISE ON STIMULUS-LOCKED FRONTS

- Have

$$\langle d\widehat{W}(t)\rangle = 0, \quad \langle d\widehat{W}(t)d\widehat{W}(t')\rangle = 2D(\varepsilon)dtdt'\delta(t-t')$$

with $D(\epsilon)$ is the same as the zero stimulus case

- Using standard properties of an Ornstein–Uhlenbeck process

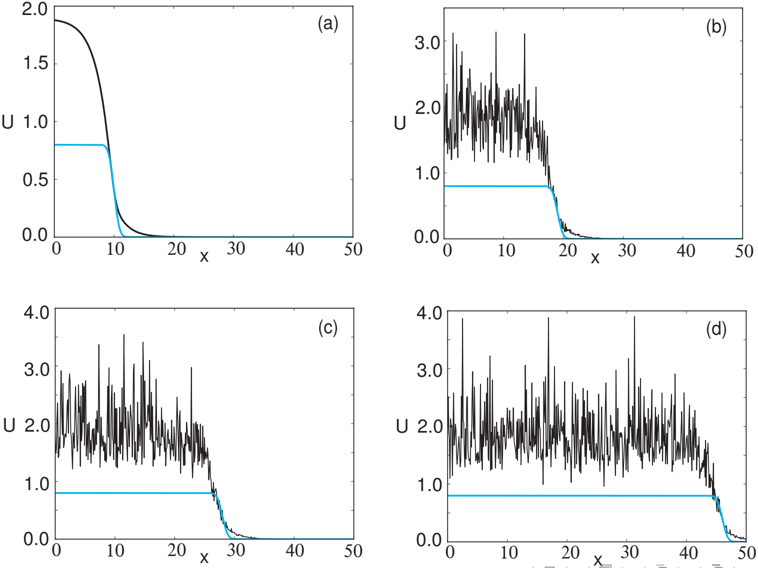
$$\langle \Delta(t)\rangle = \xi_0\left[1-\mathrm{e}^{-At}\right] + \Delta(0)\mathrm{e}^{-At},$$

$$\langle \Delta(t)^2\rangle - \langle \Delta(t)\rangle^2 = \frac{D(\varepsilon)}{A}\left[1-\mathrm{e}^{-2At}\right].$$

- Hence, $\langle \Delta(t)\rangle \to \xi_0$ as $t \to \infty$. Predicted shift ξ_0 relative to the input
- The variance approaches a constant $D(\varepsilon)/A$ in the large t limit.

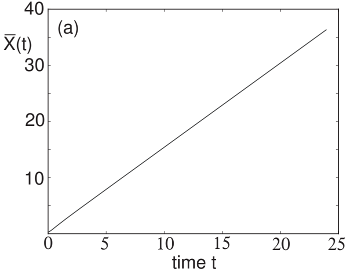
Heaviside example

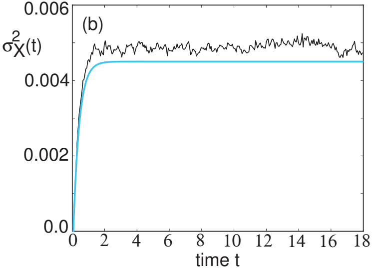
- Propagation of stochastic stimulus-locked fronts



HEAVISIDE EXAMPLE

- Mean and variance





III. Pulled Fronts

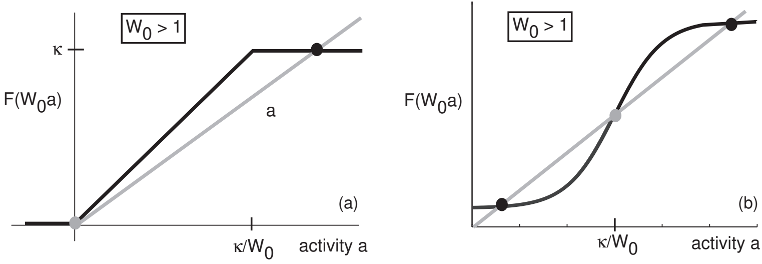
FISHER-LIKE NEURAL FIELD MODEL

- Consider activity-based NF equation

$$\tau \frac{\partial a(x,t)}{\partial t} = -a(x,t) + F\left(\int_{-\infty}^{\infty} w(x-x')a(x',t)dx'\right).$$

- Consider piecewise rate function

$$F(a) = 0 \text{ for } a \leq 0, \quad F(a) = a \text{ for } 0 < a \leq \kappa, \quad F(a) = \kappa \text{ for } a > \kappa.$$

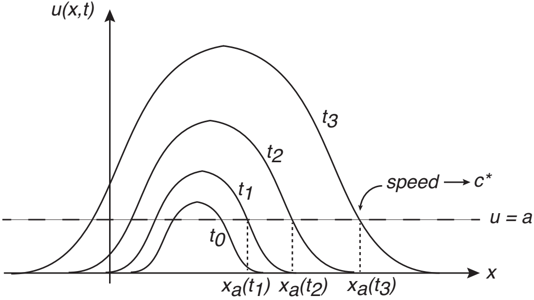


PULLED FRONTS

- Consider a traveling front propagating into an unstable state
- Analogous to invading fronts in a nonlocal version of the F-KPP equation

$$\tau \frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} + \mu p(x,t)\left(1 - \int_{-\infty}^{\infty} K(x-x')p(x',t)dx'\right).$$

- Continuum of front velocities - pulled fronts.
- Linear spreading velocity v^*: asymptotic rate at which an initial localized perturbation spreads into an unstable state



LINEAR SPREADING VELOCITY

- Consider a traveling wave solution $\mathcal{A}(x-ct)$ with $\mathcal{A}(\xi) \to \kappa$ as $\xi \to -\infty$ and $\mathcal{A}(\xi) \to 0$ as $\xi \to \infty$.
- Assume that $\mathcal{A}(\xi) \approx \mathrm{e}^{-\lambda\xi}$ for sufficiently large ξ. Linearized in traveling wave coordinates (with $\tau = 1$) takes the form

$$-c\frac{d\mathcal{A}(\xi)}{d\xi} = -\mathcal{A}(\xi) + \int_{-\infty}^{\infty} w(\xi-\xi')\mathcal{A}(\xi')d\xi'.$$

- Need to restrict the integration domain of ξ' to the leading edge of the front. Suppose, for example that $w(x)$ is given by the Gaussian distribution

$$w(x) = \frac{W_0}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-x^2/2\sigma^2}.$$

- Introduce a cut-off X with $\sigma \ll X \ll \xi$, so that

$$-c\frac{d\mathcal{A}(\xi)}{d\xi} = -\mathcal{A}(\xi) + \int_{\xi-X}^{\xi+X} w(\xi-\xi')\mathcal{A}(\xi')d\xi'.$$

Linear spreading velocity

- Substituting the exponential solution $\mathcal{A}(\xi) \approx \mathrm{e}^{-\lambda\xi}$ into (1) then yields the dispersion relation $c = c(\lambda)$ with

$$c(\lambda) = \frac{1}{\lambda}\left[\int_{-X}^{X} w(y)\mathrm{e}^{-\lambda y} dy - 1\right].$$

- Take the limit $X \to \infty$ with $w(y)$ an even function

$$c(\lambda) = \frac{1}{\lambda}\left[\widehat{W}(\lambda) + \widehat{W}(-\lambda) - 1\right],$$

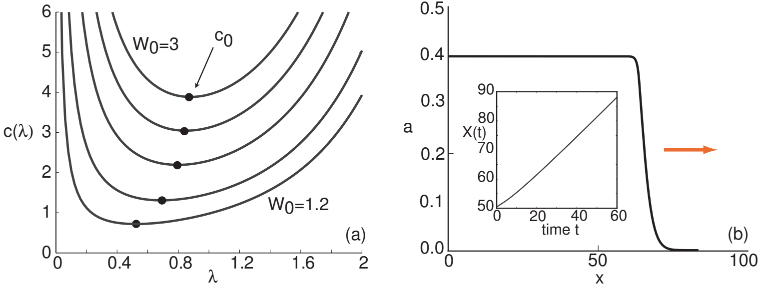
  where $\widehat{W}(\lambda)$ is the Laplace transform of $w(x)$:

$$\widehat{W}(\lambda) = \int_{0}^{\infty} w(y)\mathrm{e}^{-\lambda y} dy.$$

- If $W_0 > 1$ (necessary for the zero activity state to be unstable) then $c(\lambda)$ is a positive unimodal function with $c(\lambda) \to \infty$ as $\lambda \to 0$ or $\lambda \to \infty$ and a unique minimum at $\lambda = \lambda^*$.

DISPERSION CURVE

- A sufficiently localized initial perturbation (one that decays faster than $e^{-\lambda^* x}$) will asymptotically approach the traveling front solution with the minimum wave speed $c^* = c(\lambda^*)$. Note that $c^* \sim \sigma$ and $\lambda^* \sim \sigma^{-1}$.



STOCHASTIC MODEL

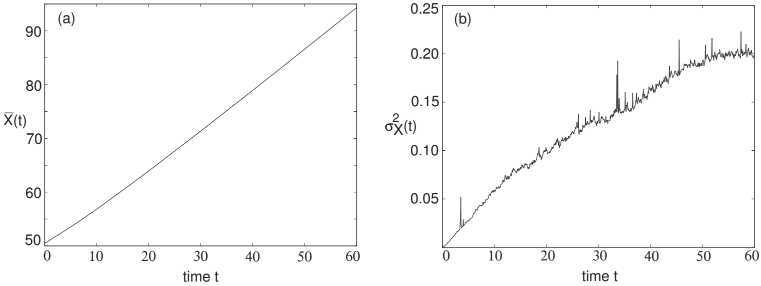
- Stochastic activity-based NF equation

$$dA(x,t) = \left[-A(x,t) + F\left(\int_{-\infty}^{\infty} w(x-y)A(y,t)dy \right) \right] dt + \varepsilon^{1/2} dW(x,t)$$

- Introduce slow/fast decomposition

$$A(x,t) = A_0(\xi - \Delta(t)) + \varepsilon^{1/2}\Phi(\xi - \Delta(t), t)$$

- Perturbative analysis breaks down. Find wandering of front is sub diffusive



IV. Hamilton-Jacobi theory of sharp fronts

Slowly varying heterogeneity

- Consider the heterogeneous neural field equation

$$\frac{\partial a(x,t)}{\partial t} = -a(x,t) + F\left(\int_{-\infty}^{\infty} w(x-x')J(\varepsilon x')a(x',t)dx'\right).$$

- Slow (non–periodic) spatial modulation $J(\varepsilon x)$ of the synaptic weight distribution with $\varepsilon \ll 1$.
- Rescale space and time according to $t \to t/\varepsilon$ and $x \to x/\varepsilon$

$$\varepsilon\frac{\partial a(x,t)}{\partial t} = -a(x,t) + F\left(\frac{1}{\varepsilon}\int_{-\infty}^{\infty} w([x-x']/\varepsilon)J(x')a(x',t)dx'\right).$$

- Under the hyperbolic rescaling, the front region where the activity $a(x,t)$ rapidly increases as x decreases from infinity becomes a step as $\varepsilon \to 0$

WKB APPROXIMATION

- Introduce the WKB approximation

$$a(x,t) \sim \mathrm{e}^{-G(x,t)/\varepsilon}$$

with $G(x,t) > 0$ for all $x > x(t)$ and $G(x(t),t) = 0$.

- The point $x(t)$ determines the location of the front and $c = \dot{x}$.
- Substituting into NF equation gives

$$-\partial_t G(x,t) = -1 + \frac{1}{\varepsilon}\int_{-\infty}^{\infty} w([x-x']/\varepsilon)J(x')\mathrm{e}^{-[G(x',t)-G(x,t)]/\varepsilon}dx'.$$

- We have used the fact that for $x > x(t)$ and $\varepsilon \ll 1$, the solution is in the leading edge of the front so that we can take F to be linear.
- Evaluating integral using steepest descents

$$-\partial_t G(x,t) = -1 + \widetilde{w}(i\partial_x G(x,t))J(x).$$

where $\widetilde{w}$ is Fourier transform of w.

Hamilton-Jacobi equation

- Equivalent to the Hamilton–Jacobi equation

$$\partial_t G + H(\partial_x G, x) = 0$$

- The Hamiltonian is

$$H(p,x) = -1 + \widetilde{w}(ip)J(x)$$

  where

$$\widetilde{w}(ip) = \widehat{W}(p) + \widehat{W}(-p) \equiv \mathcal{W}(p)$$

  with $\widehat{W}(p)$ the Laplace transform of $w(x)$.

- Hamilton–Jacobi equation can be solved in terms of the Hamilton equations

$$\frac{dx}{ds} = \frac{\partial H}{\partial p} = J(x)\mathcal{W}'(p) = J(x)[\widehat{W}'(p) - \widehat{W}'(-p)]$$

$$\frac{dp}{ds} = -\frac{\partial H}{\partial x} = -J'(x)\mathcal{W}(p).$$

HAMILTON-JACOBI EQUATION

- Let $X(s;x,t), P(s;x,t)$ denote the solution with $x(0)=0$ and $x(t)=x$. We can then determine $G(x,t)$ according to

$$G(x,t) = -E(x,t)t + \int_0^t P(s;x,t)\dot{X}(s;x,t)ds.$$

- Here

$$E(x,t) = H(P(s;x,t), X(s;x,t)),$$

which is independent of s due to conservation of "energy," that is, the Hamiltonian is not an explicit function of time.

Calculation of wave speed

- Suppose that there exists a small amplitude, slow modulation of the synaptic weights $J(x) = 1 + \beta f(x)$ with $\beta \ll 1$.
- Introduce the perturbation expansions

$$x(s) = x_0(s) + \beta x_1(s) + \mathcal{O}(\beta^2), \quad p(s) = p_0(s) + \beta p_1(s) + \mathcal{O}(\beta^2)$$

- Taylor expand $f(x)$ about x_0 and $\mathcal{W}(p) = \widehat{W}(p) + \widehat{W}(-p)$ about p_0
- Obtain a hierarchy of equations, the first two of which are

$$\dot{p}_0(s) = 0, \quad \dot{x}_0(s) = \mathcal{W}'(p_0),$$

and

$$\dot{p}_1(s) = -f'(x_0)\mathcal{W}(p_0), \quad \dot{x}_1(s) = \mathcal{W}''(p_0)p_1(s) + f(x_0)\mathcal{W}'(p_0),$$

- These are supplemented by the Cauchy conditions $x_0(0) = 0, x_0(t) = x$ and $x_n(0) = x_n(t) = 0$ for all integers $n \geq 1$.

Calculation of wave speed II

- Lowest order equations have solutions of the form

$$p_0(s) = \lambda, \quad x_0(s) = \mathcal{W}'(\lambda)s + B_0$$

with λ, B_0 independent of s. Imposing the Cauchy data then implies that $B_0 = 0$ and λ satisfies the equation

$$\mathcal{W}'(\lambda) = x/t.$$

- At the next order we have the solutions

$$p_1(s) = -\mathcal{W}(\lambda)\frac{t}{x}f(xs/t) + A_1,$$
$$x_1(s) = -\mathcal{W}''(\lambda)\mathcal{W}(\lambda)\frac{t^2}{x^2}\int_0^{xs/t} f(y)dy + \int_0^{xs/t} f(y)dy + \mathcal{W}''(\lambda)A_1 s + B_1,$$

with A_1, B_1 independent of s.

- Imposing the Cauchy data then implies that $B_1 = 0$ and

$$A_1 = A_1(x,t) = \mathcal{W}(\lambda)\frac{t}{x^2}\int_0^x f(y)dy - \frac{1}{t\mathcal{W}''(\lambda)}\int_0^x f(y)dy.$$

Calculation of wave speed III

- Given these solutions, the energy function $E(x,t)$ is

$$E(x,t) = -1 + [1 + \beta f(x_0 + \beta x_1 + ...)]\mathcal{W}(\lambda + \beta p_1 + \ldots)$$
$$= -1 + \mathcal{W}(\lambda) + \beta[\mathcal{W}'(\lambda)p_1(s) + f(x_0(s))\mathcal{W}(\lambda)] + \mathcal{O}(\beta^2).$$

- Substituting for $x_0(s)$ and $p_1(s)$ and using the condition $\mathcal{W}'(\lambda) = x/t$, we find that

$$E(x,t) = -1 + \mathcal{W}(\lambda) + \beta \frac{x}{t} A_1(x,t) + \mathcal{O}(\beta^2),$$

which is independent of s as expected.

- Similarly,

$$\int_0^t p(s)\dot{x}(s)ds = \lambda x + \beta \mathcal{W}'(\lambda) \int_0^t p_1(s)ds + \mathcal{O}(\beta^2)$$
$$= \lambda x + \beta \frac{\mathcal{W}'(\lambda)}{\mathcal{W}''(\lambda)} \int_0^t \left[\dot{x}_1(s) - \mathcal{W}'(\lambda) f(\mathcal{W}'(\lambda)s)\right] ds + \mathcal{O}(\beta^2)$$
$$= \lambda x - \beta \frac{\mathcal{W}'(\lambda)}{\mathcal{W}''(\lambda)} \int_0^x f(y)dy + \mathcal{O}(\beta^2).$$

Calculation of wave speed IV

- Hence, to first order in β,

$$G(x,t) = t - \mathcal{W}(\lambda)t + \lambda x - \beta \mathcal{W}(\lambda)\frac{t}{x}\int_0^x f(y)dy.$$

- Determine the wave speed c by imposing $G(x(t),t) = 0$ and performing the perturbation expansions $x(t) = x_0(t) + \beta x_1(t) + \mathcal{O}(\beta^2)$ and $\lambda = \lambda_0 + \beta\lambda_1 + \mathcal{O}(\beta^2)$.
- Leads to the following result:

$$x(t) = c_0 t + \frac{\beta \mathcal{W}(\lambda_0)}{c_0\lambda_0}\int_0^{c_0 t} f(y)dy + \mathcal{O}(\beta^2).$$

  Here c_0 is the wave speed of the homogeneous neural field ($\beta = 0$).

- Finally, differentiating both sides with respect to t and inverting the hyperbolic scaling yields

$$c \equiv \dot{x}(t) = c_0 + \frac{\beta \mathcal{W}(\lambda_0)}{\lambda_0} f(\varepsilon c_0 t) + \mathcal{O}(\beta^2).$$

NUMERICAL RESULTS

- Propagating front in a network with a linear heterogeneity in the synaptic weights, $J(x) = 1 + \varepsilon(x - l)$, $l = 10$, and $\varepsilon^2 = 0.005$