Optimized Schwarz Methods for Time-Harmonic Wave Problems

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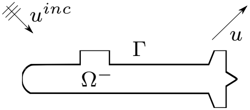
http://onelab.info/wiki/GetDDM
http://onelab.info/wiki/DDM_for_Waves

Équipe-Projet Inria NACHOS, 20/06/2017

Outline

Reference problem

Scattering of an acoustic wave on an obstacle



$$\left\{\begin{array}{rcll} (\Delta + k^2)u & = & 0 & (\mathbb{R}^3 \setminus \overline{\Omega^-}) \\ u & = & -u^{inc} & (\Gamma) \\ \multicolumn{4}{l}{\displaystyle\lim_{\|\mathbf{x}\| \to +\infty} \left(\nabla u \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} - \imath k u \right) = 0} \end{array}\right.$$

With...

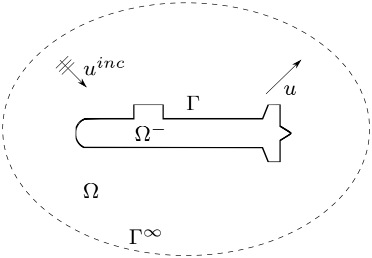
- k: wavenumber ; $u^{inc} = e^{\imath k \mathbf{x} \cdot \boldsymbol{\alpha}}$: incident plane wave
- Sommerfeld radiation condition at infinity

Practical applications

- Communication between submarines
- Electromagnetic waves in urban environment

Reference problem

FE: truncation of the domain



$$\left\{\begin{array}{rcll} (\Delta + k^2)u & = & 0 & (\Omega) \\ u & = & -u^{inc} & (\Gamma) \\ \cancel{\lim_{\|\mathbf{x}\|\to+\infty}\left(\nabla u \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} - \imath k u\right) = 0} & & & \\ \partial_{\mathbf{n}} u - \imath k u & = & 0 & (\Gamma^\infty) \end{array}\right.$$

With. . .

- $\mathbf{n}$: unit outwardly directed vector to Ω
- Simple Absorbing Boundary Condition (ABC) on Γ^∞ (not the topic here)

Domain decomposition method

Numerical solution: major problems

- **Solution is a wave**: mesh refinement (typical element size: $\pi/(5k)$)
- **High frequency** ($\lambda := \frac{2\pi}{k} \ll L$): direct solving impossible
- **Indefinite operator**: iterative solving hard if not impossible

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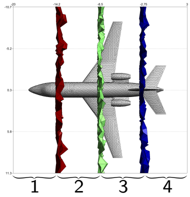
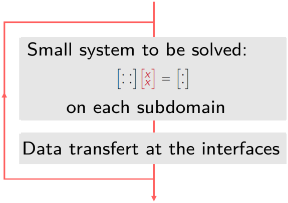
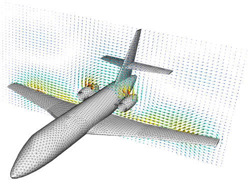
Hybrid method: Domain Decomposition Method (DDM)



Mesh decomposition

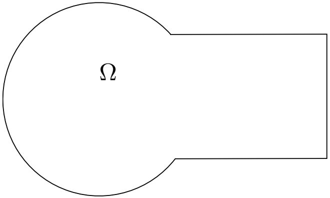


Iterative algorithm



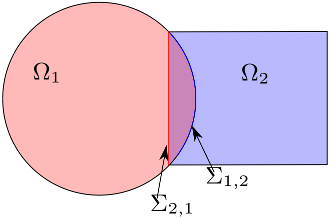
Result

Domain decomposition method: principle and origin



$$\left\{\begin{array}{rcll} -\Delta u & = & f & (\Omega) \\ u & = & 0 & (\Gamma) \end{array}\right.$$

Domain decomposition method: principle and origin



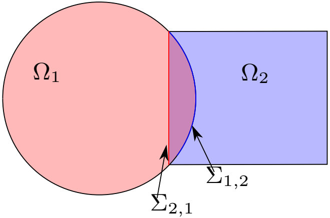
$$\left\{\begin{array}{rcll} -\Delta u & = & f & (\Omega) \\ u & = & 0 & (\Gamma) \end{array}\right.$$

Schwarz alternating method (H. Schwarz (1870))

$$\left\{\begin{array}{rcll} -\Delta u_1^{n+1} & = & f & (\Omega_1) \\ u_1^{n+1} & = & 0 & (\Gamma_1) \\ u_1^{n+1} & = & u_2^{n} & (\Sigma_{1,2}) \end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta u_2^{n+1} & = & f & (\Omega_2) \\ u_2^{n+1} & = & 0 & (\Gamma_2) \\ u_2^{n+1} & = & u_1^{n+1} & (\Sigma_{2,1}) \end{array}\right.$$

And glue the solutions in the overlap.

Domain decomposition method: principle and origin



$$\left\{\begin{array}{rcll} -\Delta u & = & f & (\Omega) \\ u & = & 0 & (\Gamma) \end{array}\right.$$

Additive Schwarz method

$$\left\{\begin{array}{rcll} -\Delta u_1^{n+1} & = & f & (\Omega_1) \\ u_1^{n+1} & = & 0 & (\Gamma_1) \\ u_1^{n+1} & = & u_2^{n} & (\Sigma_{1,2}) \end{array}\right. \qquad \left\{\begin{array}{rcll} -\Delta u_2^{n+1} & = & f & (\Omega_2) \\ u_2^{n+1} & = & 0 & (\Gamma_2) \\ u_2^{n+1} & = & u_1^{n} & (\Sigma_{2,1}) \end{array}\right.$$

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Schwarz method and Helmholtz equation

Limitations

- (Very) **slow convergence**
- Overlap is **mandatory**
- Even with overlap, the algorithm **does not converge for Helmholtz equation**

Schwarz method and Helmholtz equation

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Simple case

$$\left\{ \begin{array}{rcll} (\partial_{xx} + \partial_{yy})u_1^{n+1} + k^2 u_1^{n+1} & = & 0 & x \in (-\infty, L), y \in \mathbb{R}, \\ u_1^{n+1}(L, y) & = & u_2^n(L, y), & \end{array} \right.$$

$$\left\{ \begin{array}{rcll} (\partial_{xx} + \partial_{yy})u_2^{n+1} + k^2 u_2^{n+1} & = & 0 & x \in (0, +\infty), y \in \mathbb{R}, \\ u_2^{n+1}(0, y) & = & u_1^n(0, y). & \end{array} \right.$$

Schwarz method and Helmholtz equation

Fourier transform in the y direction ($\xi =$ Fourier variable)

$$\left\{\begin{array}{rcll} \partial_{xx}\hat{u}_1^{n+1} + (k^2-\xi^2)\hat{u}_1^{n+1} & = & 0 & x \in (-\infty, L), \xi \in \mathbb{R}, \\ \hat{u}_1^{n+1}(L,\xi) & = & \hat{u}_2^{n}(L,\xi), & \end{array}\right.$$

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Solution of the ODE

$$\left\{\begin{array}{rcl} \hat{u}_1^{n+1}(0,\mathrm{x}) & = & e^{-2\sqrt{\xi^2-k^2}L}\hat{u}_1^{n-1}(0,\mathrm{x}), \\ \hat{u}_2^{n+1}(L,\mathrm{x}) & = & e^{-2\sqrt{\xi^2-k^2}L}\hat{u}_2^{n-1}(L,\mathrm{x}), \end{array}\right.$$

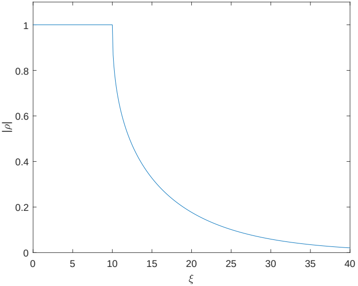
Convergence factor

$$\rho := e^{-2\sqrt{\xi^2-k^2}L} = \left\{\begin{array}{ll} e^{-2i\sqrt{k^2-\xi^2}L} & \text{if } \xi^2 \leq k^2, \\ e^{-2\sqrt{\xi^2-k^2}L} & \text{otherwise.} \end{array}\right.$$

Schwarz method and Helmholtz equation

Absolute value of the convergence factor

$$|\rho| := \begin{cases} 1 & \text{if } \xi^2 \leq k^2 \quad \text{(Propagative modes)} \\ e^{-2\sqrt{\xi^2 - k^2}L} & \text{otherwise.} \quad \text{(Evanescent modes)} \end{cases}$$



Schwarz method and Helmholtz equation

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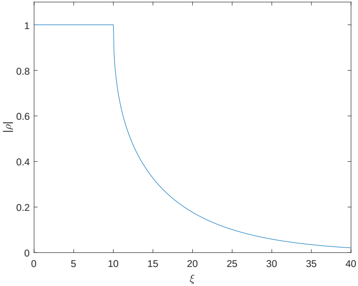
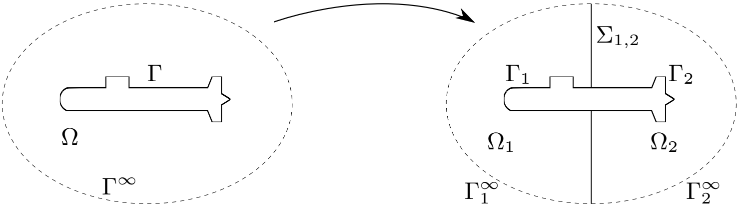


Solution

P-L. Lions algorithm: Fourrier-Robin type transmission condition

Non-overlapping domain decomposition method

Decompose the domain (here $N = 2$)



Recast the system into 2 coupled systems

$$\left\{\begin{array}{rcll} (\Delta + k^2)u_1 & = & 0 & (\Omega_1) \\ u_1 & = & -u^{inc} & (\Gamma_1) \\ (\partial_{\mathbf{n}_1} - ik)u_1 & = & 0 & (\Gamma_1^\infty) \\ (\partial_{\mathbf{n}_1} + \mathcal{S}_1)u_1 & = & (\partial_{\mathbf{n}_1} + \mathcal{S}_1)u_2 & (\Sigma_{1,2}) \end{array}\right.$$

$$\left\{\begin{array}{rcll} (\Delta + k^2)u_2 & = & 0 & (\Omega_2) \\ u_2 & = & -u^{inc} & (\Gamma_2) \\ (\partial_{\mathbf{n}_2} - ik)u_2 & = & 0 & (\Gamma_2^\infty) \\ (\partial_{\mathbf{n}_2} + \mathcal{S}_2)u_2 & = & (\partial_{\mathbf{n}_2} + \mathcal{S}_2)u_1 & (\Sigma_{2,1}) \end{array}\right.$$

$\mathcal{S}_j$: Transmission operators

Non-overlapping domain decomposition method

Parallel Schwarz algorithm

Introducing surface unknown $g_{ij} := (\partial_{\mathbf{n}_i} + \mathcal{S}_i)u_j$, the algorithm reads (iteration n to $n+1$):

1. Solve the N independant problems

$$\left\{\begin{array}{rcll} (\Delta + k^2)u_1^{n+1} & = & 0 & (\Omega_1) \\ u_1^{n+1} & = & -u^{inc} & (\Gamma_1) \\ (\partial_{\mathbf{n}_1} - ik)u_1^{n+1} & = & 0 & (\Gamma_1^\infty) \\ (\partial_{\mathbf{n}_1} + \mathcal{S}_1)u_1^{n+1} & = & g_{12}^n & (\Sigma_{1,2}) \end{array}\right.$$

$$\left\{\begin{array}{rcll} (\Delta + k^2)u_2^{n+1} & = & 0 & (\Omega_2) \\ u_2^{n+1} & = & -u^{inc} & (\Gamma_2) \\ (\partial_{\mathbf{n}_1} - ik)u_2^{n+1} & = & 0 & (\Gamma_2^\infty) \\ (\partial_{\mathbf{n}_2} + \mathcal{S}_2)u_2^{n+1} & = & g_{21}^n & (\Sigma_{2,1}) \end{array}\right.$$

2. Update the surface unknown

$$\left\{\begin{array}{rcll} g_{12}^{n+1} & = & -g_{21}^n + (\mathcal{S}_1 + \mathcal{S}_2)u_2^{n+1} & (\Sigma_{1,2}) \\ g_{21}^{n+1} & = & -g_{12}^n + (\mathcal{S}_1 + \mathcal{S}_2)u_1^{n+1} & (\Sigma_{2,1}) \end{array}\right.$$

Non-overlapping domain decomposition method

Gather the surface unknown in one vector

$$g = (g_{i,j})_{i,j}$$

One iteration of the algorithm reads as:

$$g^{n+1} = \mathcal{A}g^{n} + b$$

- $\mathcal{A}$: iteration operator. Applying $\mathcal{A}$ is amount to solving N volume PDEs $+ N$ surface PDEs (with $u^{inc} = 0$)
- b: right-hand side, containing physical information (u^{inc}).

At convergence, g is solution to:

$$(\mathcal{I} - \mathcal{A})g = b \tag{1}$$

Krylov acceleration

System (1) can be solved using a Krylov subspace solver.

Non-overlapping domain decomposition method

2 subdomains and DtN

Let $\Lambda_j : H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ be the DtN (Dirichlet-to-Neumann) map associated to Ω_j:

$$\Lambda_j f = \partial_{\mathbf{n}_j} w_j, \qquad \text{on } \Sigma.$$

with w_j solution of

$$\left\{ \begin{array}{rcll} (\Delta + k^2) w_j & = & 0 & \text{in } \Omega_j, \\ w_j & = & 0 & \text{on } \Gamma_j, \\ \partial_{\mathbf{n}} w_j - ikw_j & = & 0 & \text{on } \Gamma_j^\infty, \\ w_j & = & f & \text{on } \Sigma. \end{array} \right.$$

Then, if $\mathcal{S}_j = -\Lambda_j$, the algorithm converges in 2 iterations.

Remark

Extended to N subdomains: convergencence in N iterations.

Non-overlapping domain decomposition method

One-dimensional case

$$\left\{\begin{array}{rcll} u'' + k^2 u & = & 0, & \text{in } [0,1], \\ u(0) & = & e^{\imath k x} = 1, & \\ u'(1) - \imath k u(1) & = & 0. & \end{array}\right.$$

Solution

$$u(x) = e^{\imath k x}$$

Exact DtN

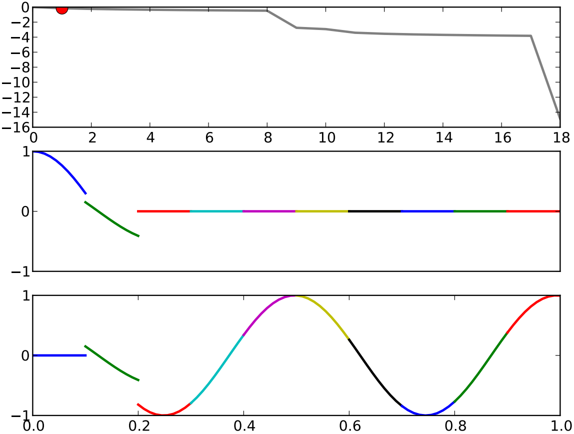
$$\Lambda = \imath k$$

Non-overlapping domain decomposition method

Residual

Solution

Error



Non-overlapping domain decomposition method

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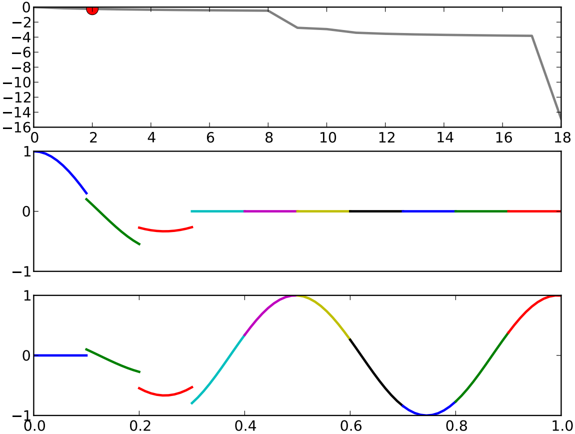

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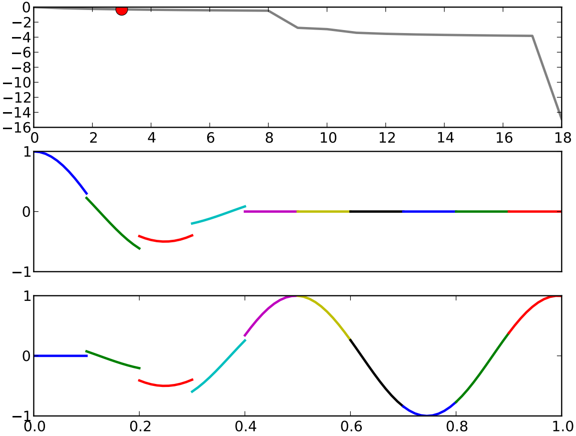

Non-overlapping domain decomposition method

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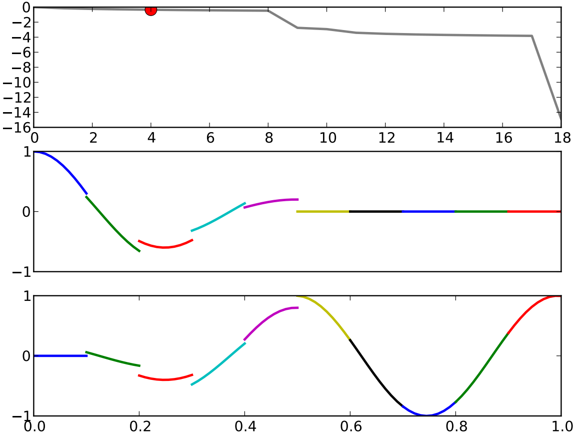

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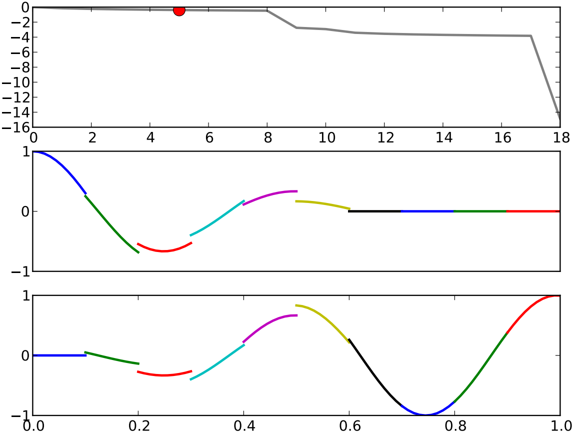

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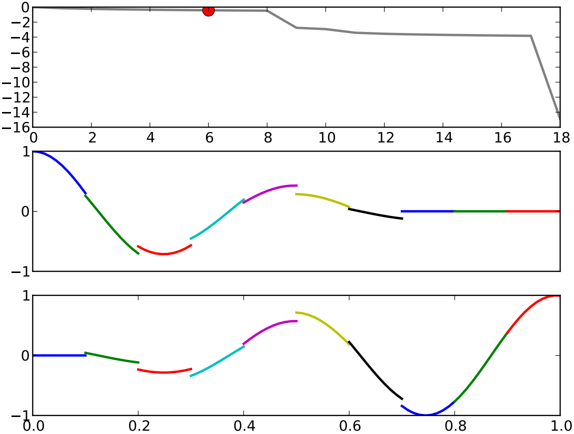

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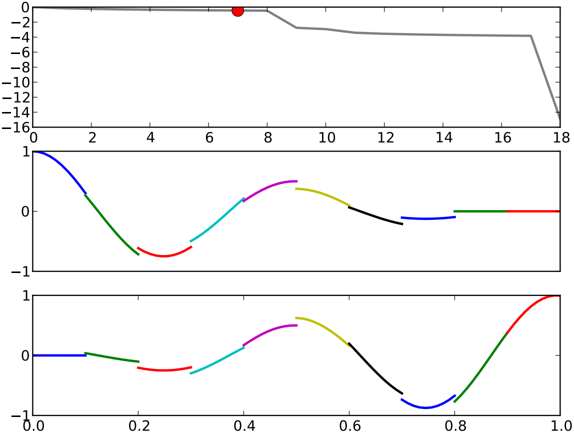

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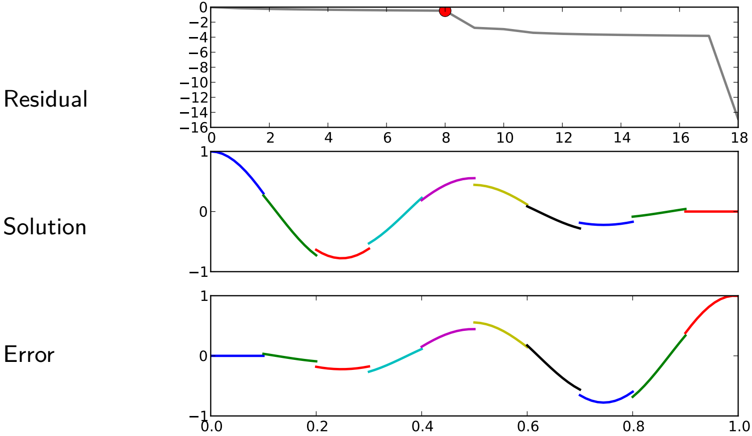

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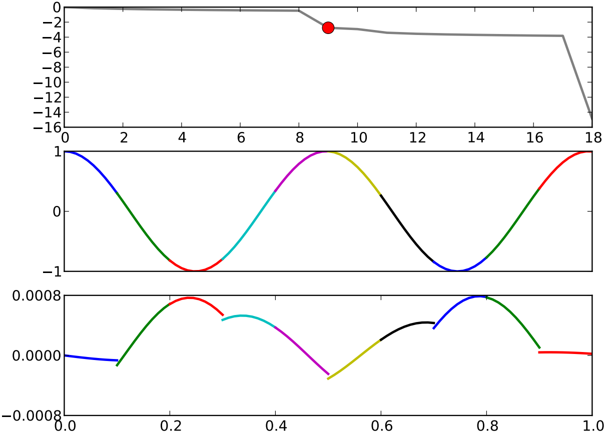

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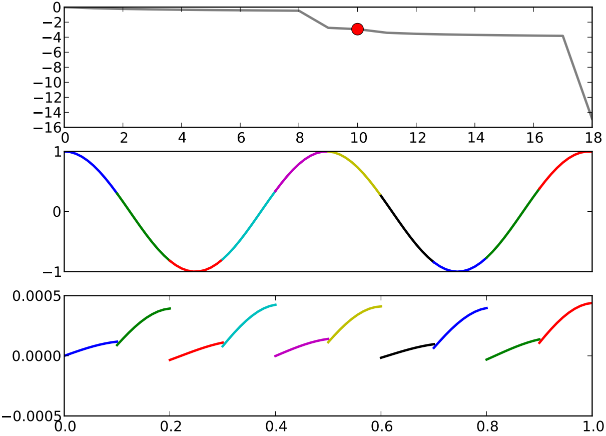

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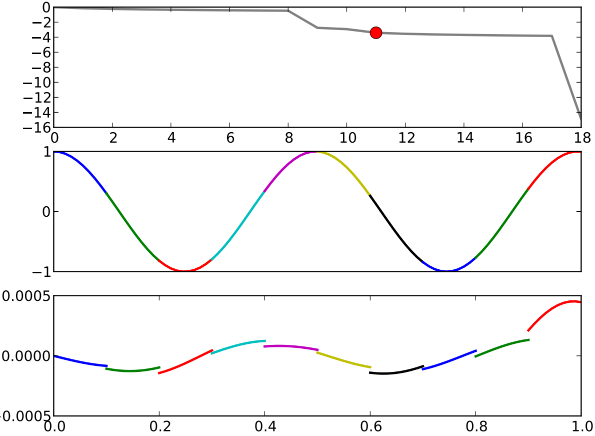

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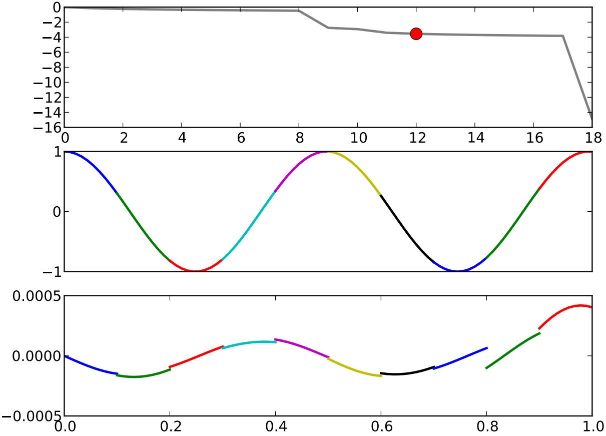

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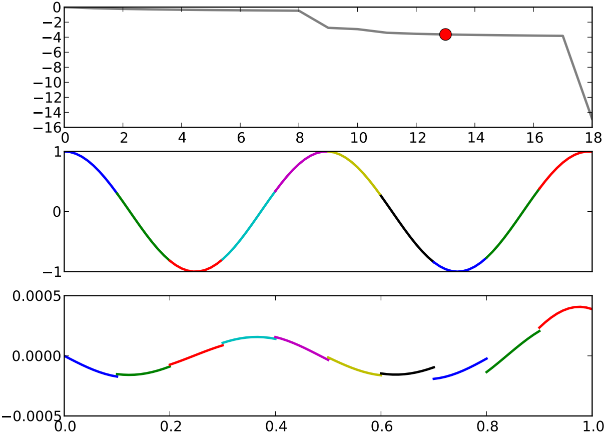

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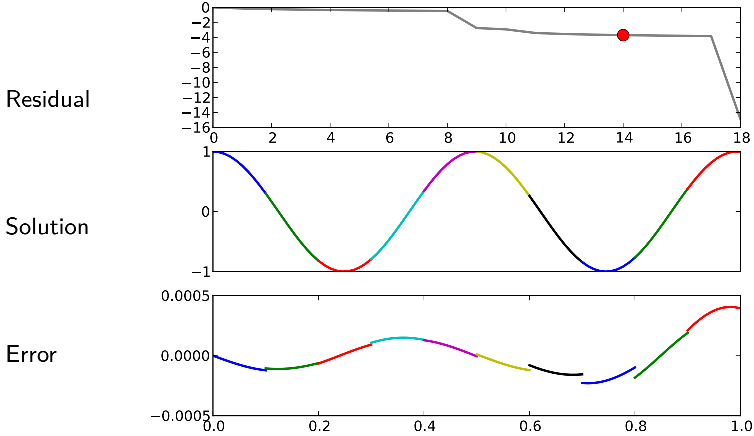

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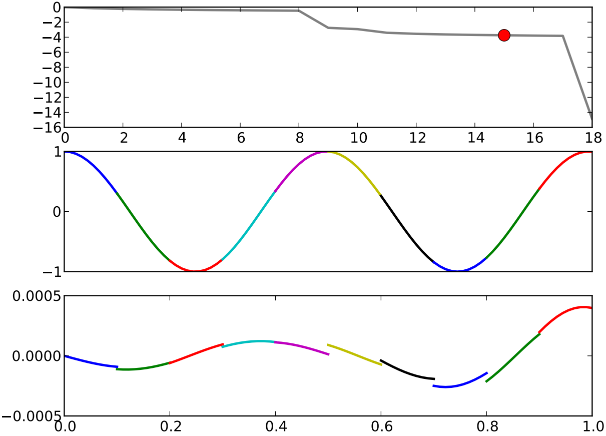

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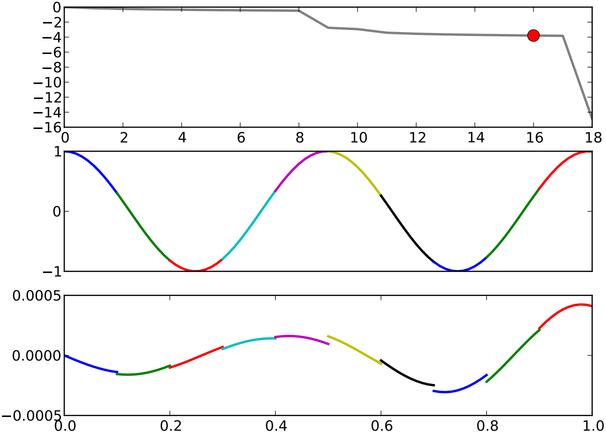

Non-overlapping domain decomposition method

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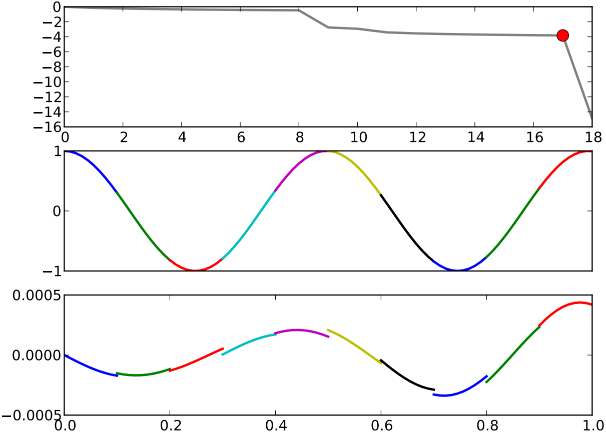

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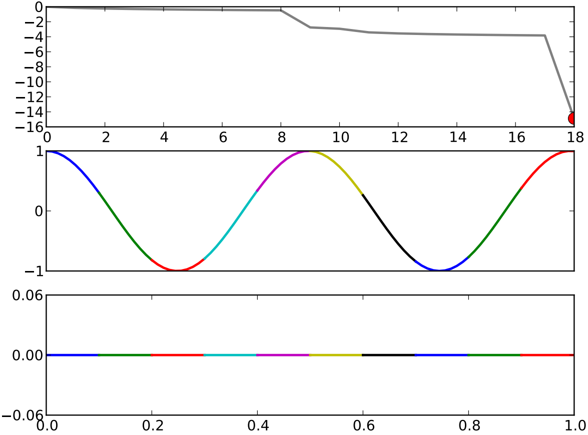

Non-overlapping domain decomposition method

Two major investigation fields

1. **Transmission condition:** find a suitable approximation of $-\Lambda_j$
2. **Coarse space:** decrease the linear convergence rate (in terms of N)

Non-overlapping domain decomposition method

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Problem

The DtN map is **non-local** and therefore is not suitable for FE framework.

Non-overlapping domain decomposition method

Two major investigation fields

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Problem

The DtN map is **non-local** and therefore is not suitable for FE framework.

Available methods

- **Local approaches**: Taylor, Padé, ...
- Integral Equation (Joly et. al)
- PML (Vion and Geuzaine)

Transmission Operators for Helmholtz equation

Half-space case with straight interface Σ

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3_+ = \{\mathbf{x} \in \mathbb{R}^3; x_1 > 0\}, \\ u = g & \text{on } \Sigma, \\ u \text{ is outgoing,} & \end{cases}$$

Fourier transform (variable ξ along Σ)

$$\partial_{\mathbf{n}} u(0, \xi) = \mathcal{F}^{-1}_{\boldsymbol{\xi}}(\sigma(\boldsymbol{\xi})\hat{u}(0, \boldsymbol{\xi}))_{|\Sigma}.$$

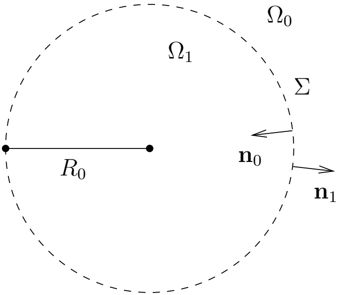
Symbol of the DtN

$$\sigma^{\mathrm{sq}}(\boldsymbol{\xi}) = \imath k \sqrt{1 - \frac{|\boldsymbol{\xi}|^2}{k^2}}.$$

DtN map

$$\Lambda^{\mathrm{sq}} := \mathrm{Op}\left(\sigma^{\mathrm{sq}}\right) = \imath k \sqrt{1 + \frac{\Delta_\Sigma}{k^2}}.$$

Model problem for convergence analysis



Model problem with two subdomains and a circular interface.

$$\begin{cases} (\Delta + k^2)u_0 & = & 0 & (\Omega_0) \\ \partial_{\mathbf{n}_0} u_0 + \mathcal{S}u_0 & = & g_0 & (\Sigma) \\ \lim\limits_{|\mathbf{x}|\to+\infty} |\mathbf{x}|^{1/2}(\partial_{|\mathbf{x}|} u_0 - iku_0) = 0 \end{cases} \qquad \begin{cases} (\Delta + k^2)u_1 & = & 0 & (\Omega_1) \\ \partial_{\mathbf{n}_1} u_1 + \mathcal{S}u_1 & = & g_1 & (\Sigma) \end{cases}$$

Model problem for convergence analysis

Rewrite $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{T}_0 \\ \mathcal{T}_1 & 0 \end{pmatrix}, \qquad \mathcal{T}_j g_j^n = -g_j^n + 2\mathcal{S} u_j^{n+1}.$$

Modal decomposition

$$u_0 = \sum_m \alpha_m H_m^{(1)}(kr) e^{im\theta}, \qquad u_1 = \sum_m \beta_m J_m(kr) e^{im\theta},$$

$$\mathcal{S} = \sum_m S_m e^{im\theta}, \qquad \mathcal{T}_j = \sum_m T_{j,m} e^{im\theta}, \qquad g_j = \sum_m g_{j,m}(r) e^{im\theta}.$$

Recurrence relation

$$g_{j,m}^{n+1} = T_{0,m} T_{1,m} g_{j,m}^{n-1}.$$

Convergence factor

$$\forall m, \qquad \rho_m := T_{0,m} T_{1,m} = \left[\frac{-kZ_{0,m} + S_m}{kZ_{0,m} + S_m}\right] \cdot \left[\frac{-kZ_{1,m} + S_m}{kZ_{1,m} + S_m}\right],$$

$Z_{0,m} = -\frac{{H_m^{(1)}}'(kR_0)}{H_m^{(1)}(kR_0)}$ and $Z_{1,m} = \frac{J_m(kR_0)}{J_m(kR_0)}$. Remark: $(S_m = 0) \Rightarrow (\rho_m = 1)$

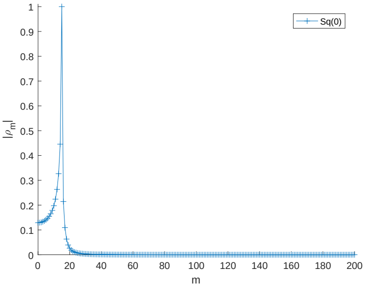
Model problem for convergence analysis

Square root operator

$$\mathcal{S}^{\mathrm{sq}} = -\Lambda^{\mathrm{sq}} = -ik\sqrt{1 - \frac{\Delta_\Sigma}{k^2}}.$$

Modal decomposition

$$S_m^{sq} = -ik\sqrt{1 - \frac{m^2}{k^2 R_0^2}}.$$



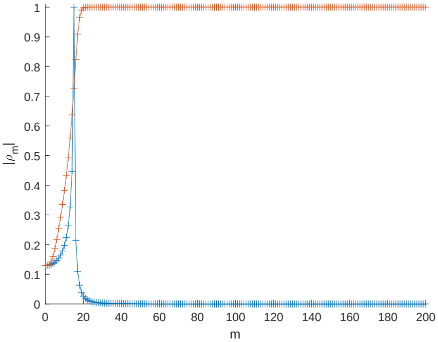
Remark : if $m^2 = k^2 R_0^2$ then $\rho_m^{\mathrm{sq}} = 1$.

Transmission Operators for Helmholtz equation

Impedance Boundary Condition (IBC) [Després, 1991]

Low frequency approximation ($\xi \to 0$):

$$\sigma^{\mathrm{sq}}(\xi) = \imath k \sqrt{1 - \frac{|\boldsymbol{\xi}|^2}{k^2}} \approx ik. \qquad \mathcal{S}^{\mathsf{IBC}} u = -\imath k u.$$



Transmission Operators for Helmholtz equation

Optimized Order 2 [Gander, Magoulès and Nataf, 2002]

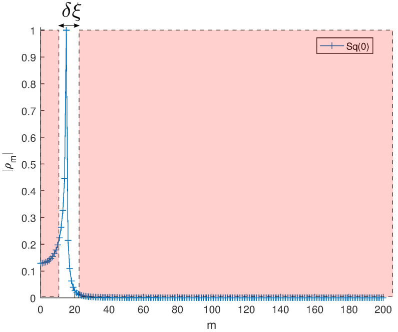
$$\sigma^{\mathrm{sq}}(\xi) = \imath k \sqrt{1 - \frac{|\boldsymbol{\xi}|^2}{k^2}} \approx a(\delta\xi) - b(\delta\xi)\xi^2,$$

where a and b are solution of the min-max problem

$$\min_{\alpha,\beta\in\mathbb{C}} \left(\max_{\xi_{min}\in(0,k-\delta\xi)\cup(k+\delta\xi,\xi_{max})} |\widetilde{\rho}(\xi;\alpha,\beta)| \right),$$

where $\widetilde{\rho}$ is the convergence factor in the case $(-\infty, 0] \times \mathbb{R}$ and $[0, +\infty) \times \mathbb{R}$:

$$\widetilde{\rho}(\xi) = \left| \frac{\sigma^{sq}(\xi) - \sigma^{oo2}(\xi; a, b)}{\sigma^{sq}(\xi) + \sigma^{oo2}(\xi; a, b)} \right|^2.$$

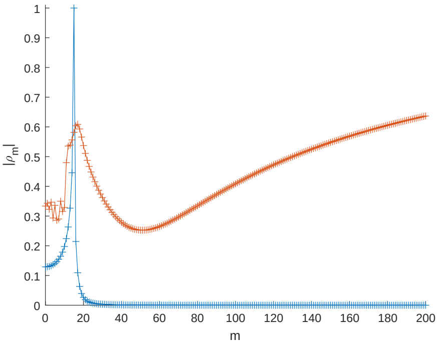


Transmission Operators for Helmholtz equation

Optimized Order 2 [Gander, Magoulès and Nataf, 2002]

$$\sigma^{\mathrm{sq}}(\xi) = \imath k \sqrt{1 - \frac{|\boldsymbol{\xi}|^2}{k^2}} \approx a(\delta\xi) - b(\delta\xi)\xi^2.$$

$$\mathcal{S}^{\mathsf{OO2}} u = a(\delta\xi) u + b(\delta\xi)\Delta_\Sigma u,$$



Transmission Operators for Helmholtz equation

Modified DtN [Boubendir, Antoine and Geuzaine, 2012]

$$\mathcal{S}^{\mathrm{sq},\varepsilon} u = -\imath k \sqrt{1 + \frac{\Delta_\Sigma}{k_\varepsilon^2}} u,$$

where $k_\varepsilon = k + \imath\varepsilon$ and $\varepsilon > 0$.

Optimal ε

Searching for ε_{opt} such that:

$$\min_{\varepsilon>0} \max_m \left|\rho_m^{\mathrm{sq},\varepsilon}\right|.$$

Assume that $\max_m \left|\rho_m^{\mathrm{sq},\varepsilon}\right|$ is reached on $m = kR_0$, then we can prove that

$$\varepsilon_{opt} \approx 0.39 k^{1/3} R_0^{-2/3}.$$

Formally extended to other curves by ($\mathcal{H}$: local mean curvature):

$$\varepsilon_{opt} \approx 0.39 k^{1/3} \mathcal{H}^{2/3}.$$

Transmission Operators for Helmholtz equation

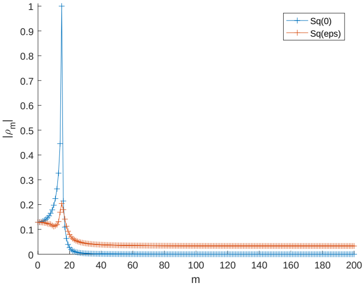
Modified DtN [Boubendir, Antoine and Geuzaine, 2012]

$$\mathcal{S}^{\mathrm{sq},\varepsilon} u = -\imath k \sqrt{1 + \frac{\Delta_\Sigma}{k_\varepsilon^2}}\, u,$$

where $k_\varepsilon = k + \imath\varepsilon$ and $\varepsilon = 0.39 k^{1/3} \mathcal{H}^{2/3}$ ($\mathcal{H}$: local mean curvature).

Modal decomposition

$$S_m^{\mathrm{sq},\varepsilon} = -\imath k \sqrt{1 - \frac{m^2}{k_\varepsilon^2 R_0^2}} \qquad \text{and} \qquad |\rho_m^{\mathrm{sq},\varepsilon}| < 1, \quad \forall m.$$



Transmission Operators for Helmholtz equation

Classical Padé approximants on square root

$$\sqrt{1+X} \approx R_{N_p}(X) = c_0 + \sum_{\ell=1}^{N_p} \frac{a_\ell X}{1+b_\ell X},$$

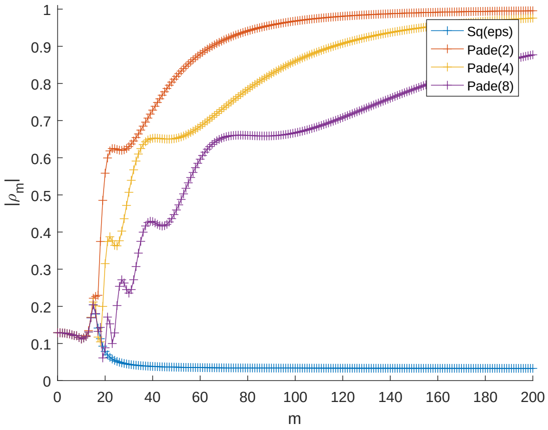
N_p is the number of Padé approximants.

Localization of the nonlocal operator $\mathcal{S}^{\mathrm{sq},\varepsilon} u = -\imath k \sqrt{1+\frac{\Delta_\Sigma}{k_\varepsilon^2}}\, u$

$$\mathcal{S}^{\mathsf{GIBC}(N_p,\,\varepsilon)} u = -ikc_0 u - ik \sum_{\ell=1}^{N_p} a_\ell \mathsf{div}_\Sigma \left(\frac{1}{k_\varepsilon^2}\nabla_\Sigma\right)\left(\mathcal{I} + b_\ell \mathsf{div}_\Sigma\left(\frac{1}{k_\varepsilon^2}\nabla_\Sigma\right)\right)^{-1} u.$$

Transmission Operators for Helmholtz equation

Modal decomposition for different number of N_p



Transmission Operators for Helmholtz equation

Vanishing modes are not well approximated

$$S_m^{\mathrm{sq},\varepsilon} = -ik\sqrt{1-\left(\frac{m^2}{k_\varepsilon^2 R_0^2}\right)}.$$

Complex Padé approximants on square root (α: rotation of the branch cut)

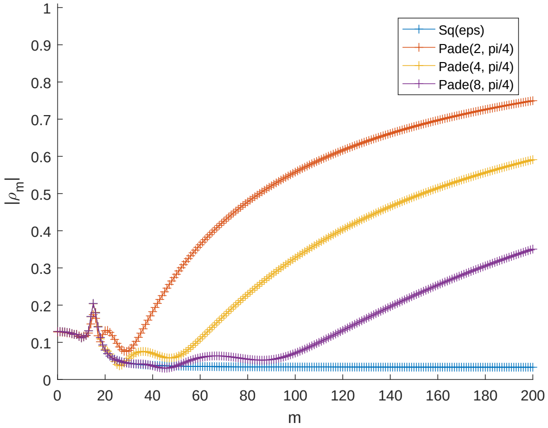
$$\sqrt{1+X} = e^{i\alpha/2}\sqrt{(1+X)e^{-i\alpha}} \approx R_{N_p}^{\alpha}(X) = C_0(\alpha) + \sum_{\ell=1}^{N_p} \frac{A_\ell(\alpha)X}{1+B_\ell(\alpha)X}.$$

Localization of the nonlocal operator $\mathcal{S}^{\mathrm{sq},\varepsilon}u = -\imath k\sqrt{1+\frac{\Delta_\Sigma}{k_\varepsilon^2}}\; u$

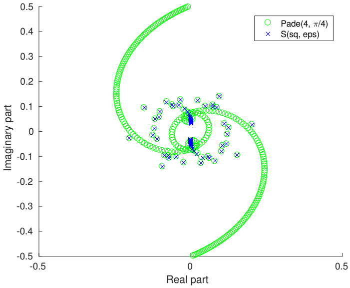
$$\mathcal{S}^{\mathsf{GIBC}(N_p,\,\alpha,\,\varepsilon)}u = -ikC_0(\alpha)u - ik\sum_{\ell=1}^{N_p} A_\ell(\alpha)\mathsf{div}_\Sigma\left(\frac{1}{k_\varepsilon^2}\nabla_\Sigma\right)\left(\mathcal{I} + B_\ell(\alpha)\mathsf{div}_\Sigma\left(\frac{1}{k_\varepsilon^2}\nabla_\Sigma\right)\right)^{-1} u.$$

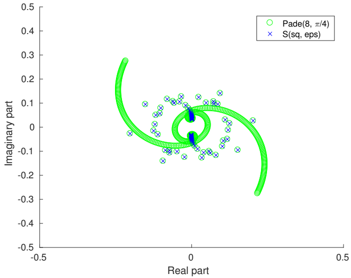
Transmission Operators for Helmholtz equation

Modal decomposition for different number of N_p and $\alpha = \pi/4$



Transmission Operators for Helmholtz equation

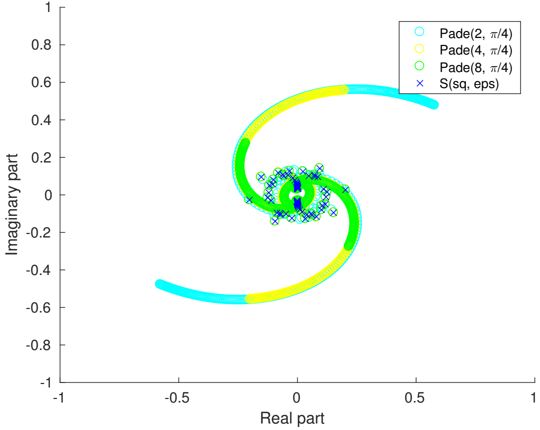




Eigenvalue distribution in the complex plane for the exact and Padé-localized square-root transmission operator of order 4 (left) and 8 (right).

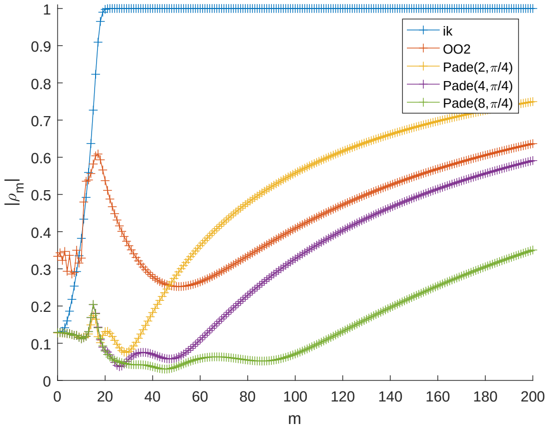
Transmission Operators for Helmholtz equation

Eigenvalues distribution with respect to the number of Padé approximants



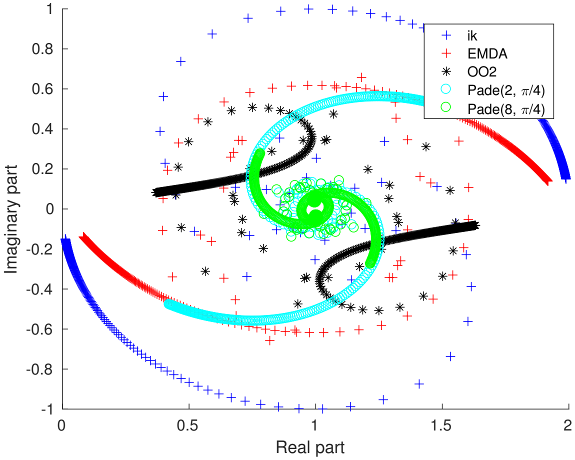
Comparison of the transmission operators

Convergence factor



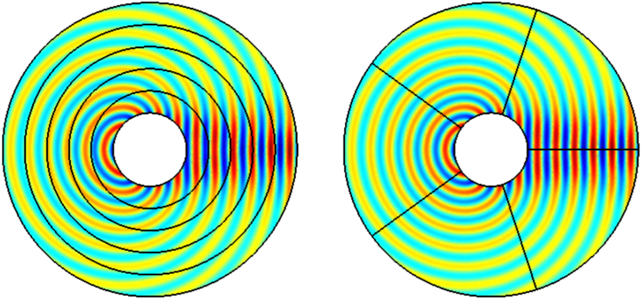
Comparison of the transmission operators

Eigenvalue distribution in the complex plane for $(I - \mathcal{A})$

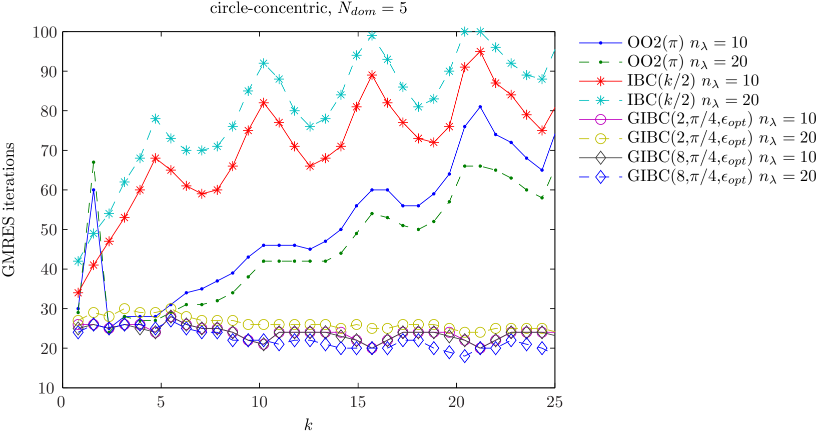


Numerical Example

“concentric-” and “pie-” decomposition

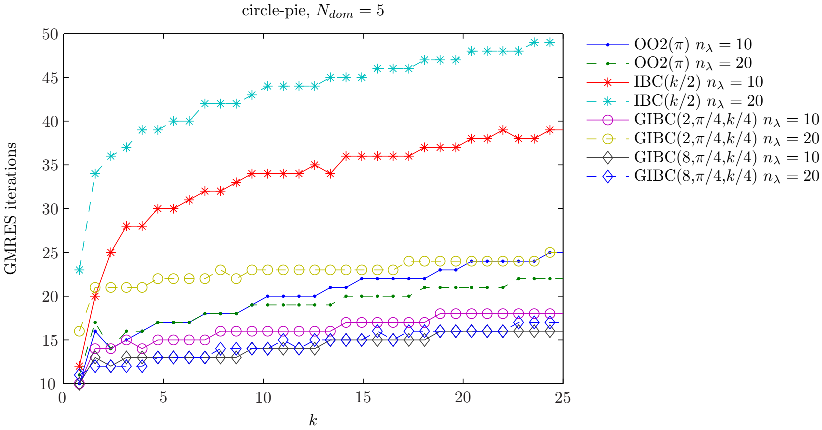


Numerical Example



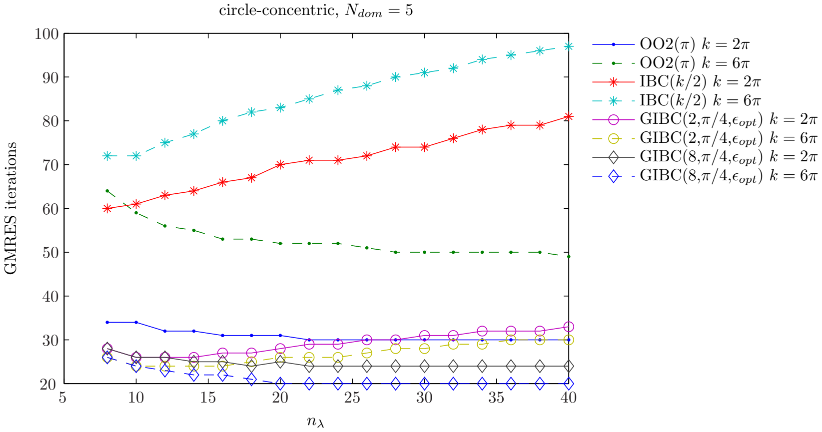
Convergence for the "circle-concentric" decomposition. Number of iterations vs. wavenumber

Numerical Example



Convergence for the "circle-pie" decomposition. Number of iterations vs. wavenumber.

Numerical Example



Convergence for the "circle-concentric" decomposition. Number of iterations vs. mesh density

Non-Overlapping DDM for Maxwell

P-L. Lions algorithm

From iteration n to $n+1$:

1. Solve, for $i = 1, \dots, N$:

$$\left\{\begin{array}{rcll} \mathbf{curl}\ \mathbf{curl}\ \mathbf{E}_i^{(n+1)} - k^2\ \mathbf{E}_i^{(n+1)} & = & \mathbf{0} & \text{in } \Omega_i, \\ \gamma_i^T(\mathbf{E}_i^{(n+1)}) & = & -\gamma_i^T(\mathbf{E}^{\text{inc}}) & \text{in } \Gamma_i^D, \\ -\frac{\imath}{k}\gamma_i^t(\mathbf{curl}\ \mathbf{E}_i^{(n+1)}) + \mathcal{S}(\gamma_i^T(\mathbf{E}_i^{(n+1)})) & = & \boldsymbol{g}_{ij}^{(n)} & \text{in } \Sigma_{ij}. \end{array}\right.$$

2. Update surface quantities:

$$\begin{aligned} \boldsymbol{g}_{ji}^{(n+1)} &= \frac{\imath}{k}\gamma_i^t(\mathbf{curl}\ \mathbf{E}_i^{(n+1)}) + \mathcal{S}(\gamma_i^T(\mathbf{E}_i^{(n+1)})) \\ &= -\boldsymbol{g}_{ij}^{(n)} + 2\mathcal{S}(\gamma_i^T(\mathbf{E}_j^{(n+1)})), \quad \text{on } \Sigma_{ij}. \end{aligned}$$

where we have introduce the trace operators:

$$\gamma_i^t : \mathbf{v}_i \mapsto \mathbf{n}_i \times \mathbf{v}_{i|\partial\Omega_i} \quad \text{and} \quad \gamma_i^T : \mathbf{v}_i \mapsto \mathbf{n}_i \times (\mathbf{v}_{i|\partial\Omega_i} \times \mathbf{n}_i).$$

Non-Overlapping DDM for Maxwell

Krylov acceleration

As for the Helmholtz case, the whole algorithm can be recast into a linear system:

$$(\mathcal{I} - \mathcal{A})\boldsymbol{g} = \boldsymbol{b}.$$

Transmission operators

Again, the transmission operator $\mathcal{S}$ has a direct impact on the iteration operator $\mathcal{A}$.

Transmission Operators for Maxwell

Half-space problem ($\Omega := (-\infty, 0) \times \mathbb{R}^2$)

$$
\begin{cases}
\mathbf{curl}\ \mathbf{H} + \imath k \mathbf{E} = 0 & (\Omega) \\
\mathbf{curl}\ \mathbf{E} - \imath k \mathbf{H} = 0 & (\Omega) \\
\gamma^T(\mathbf{E}) = -\gamma^T(\mathbf{E}^{\text{inc}}) & (\Sigma) \\
\lim\limits_{\|\mathbf{x}\| \to \infty} \|\mathbf{x}\| \left(\mathbf{E} + \dfrac{\mathbf{x}}{\|\mathbf{x}\|} \times \mathbf{H} \right) = 0 &
\end{cases}
$$

Surface electric and magnetic currents

$$
\mathbf{J} = \mathbf{n} \times \mathbf{E}, \qquad \mathbf{M} = \mathbf{n} \times \mathbf{H}
$$

MtE

$$
\mathbf{M} + \Lambda^{\text{sq}}(\mathbf{n} \times \mathbf{J}) = 0,
$$

with

$$
\Lambda^{\text{sq}} = (\Lambda_1^{\text{sq}})^{-1} \Lambda_2^{\text{sq}},
$$

$$
\Lambda_1^{\text{sq}} = \left(\mathbf{I} + \nabla_\Sigma \frac{1}{k^2} \text{div}_\Sigma - \mathbf{curl}_\Sigma \frac{1}{k^2} \text{curl}_\Sigma \right)^{1/2}, \qquad \Lambda_2^{\text{sq}} = \left(\mathbf{I} - \frac{1}{k^2} \mathbf{curl}_\Sigma \text{curl}_\Sigma \right).
$$

Transmission Operators for Maxwell

$$\Lambda^{\mathrm{sq}} = (\Lambda_1^{\mathrm{sq}})^{-1}\Lambda_2^{\mathrm{sq}},$$

$$\Lambda_1^{\mathrm{sq}} = \left(\mathbf{I} + \nabla_\Sigma \frac{1}{k^2}\mathrm{div}_\Sigma - \mathbf{curl}_\Sigma \frac{1}{k^2}\mathrm{curl}_\Sigma\right)^{1/2}, \qquad \Lambda_2^{\mathrm{sq}} = \left(\mathbf{I} - \frac{1}{k^2}\mathbf{curl}_\Sigma\mathrm{curl}_\Sigma\right).$$

0th-order transmission condition IBC(0) [Després, 1992]

$$\mathcal{S}_{\mathsf{IBC(0)}}(\gamma^T(\mathbf{E})) = \gamma^T(\mathbf{E}).$$

Optimized impedance boundary condition GIBC(α) [Alonso-Rodriguez and Gerardo-Giorda, 2006]

$$\mathcal{S}_{\mathsf{GIBC}(\alpha)}(\gamma^T(\mathbf{E})) = \alpha\,\left(\mathbf{I} - \frac{1}{k^2}\mathbf{curl}_\Sigma\mathrm{curl}_\Sigma\right)\,\gamma^T(\mathbf{E}),$$

where α is chosen thanks to an optimization process.

Transmission Operators for Maxwell

$$\Lambda^{\rm sq} = (\Lambda_1^{\rm sq})^{-1}\Lambda_2^{\rm sq},$$

$$\Lambda_1^{\rm sq} = \left(\mathbf{I} + \nabla_\Sigma \frac{1}{k^2}\mathrm{div}_\Sigma - \mathbf{curl}_\Sigma \frac{1}{k^2}\mathrm{curl}_\Sigma\right)^{1/2}, \qquad \Lambda_2^{\rm sq} = \left(\mathbf{I} - \frac{1}{k^2}\mathbf{curl}_\Sigma \mathrm{curl}_\Sigma\right).$$

Optimized second-order GIBC(α,β) [Rawat and Lee, 2010]

$$\mathcal{S}_{\mathsf{GIBC}(a,\,b)}(\gamma^T(\mathbf{E})) = \left(\mathbf{I} + \frac{a}{k^2}\nabla_\Sigma \mathrm{div}_\Sigma\right)^{-1}\left(\mathbf{I} - \frac{b}{k^2}\mathbf{curl}_\Sigma \mathrm{curl}_\Sigma\right)\gamma^T(\mathbf{E}),$$

where a and b are chosen so that an optimal convergence rate is obtained for the (TE) and (TM) modes.

This condition has been generalized in [Dolean, Gander, Lanteri, Lee and Peng, 2015].

Transmission Operators for Maxwell

Modified square root operator

$$\Lambda^{\mathrm{sq},\varepsilon} = (\Lambda_1^{\mathrm{sq},\varepsilon})^{-1}\Lambda_2^{\mathrm{sq},\varepsilon},$$

$$\Lambda_1^{\mathrm{sq},\varepsilon} = \left(\mathbf{I} + \nabla_\Sigma \frac{1}{k_\varepsilon^2}\mathrm{div}_\Sigma - \mathbf{curl}_\Sigma \frac{1}{k_\varepsilon^2}\mathrm{curl}_\Sigma\right)^{1/2}, \qquad \Lambda_2^{\mathrm{sq},\varepsilon} = \left(\mathbf{I} - \frac{1}{k_\varepsilon^2}\mathbf{curl}_\Sigma\mathrm{curl}_\Sigma\right).$$

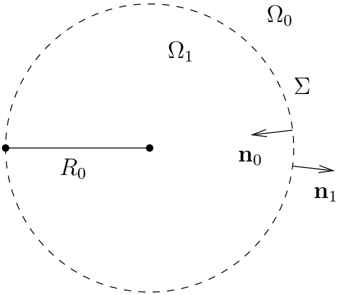
Padé-localized square-root transmission condition GIBC(N_p, α, ε) [El Bouajaji, Antoine, Geuzaine, Thierry, 2014]

$$\mathcal{S}_{\mathsf{GIBC}(N_p,\,\alpha,\,\varepsilon)}(\gamma^T(\mathbf{E})) = \left(C_0 + \sum_{\ell=1}^{N_p} A_\ell X\left(\mathcal{I} + B_\ell X\right)^{-1}\right)^{-1} \left(\mathcal{I} - \mathbf{curl}_\Sigma \frac{1}{k_\varepsilon^2}\mathrm{curl}_\Sigma\right)\gamma^T(\mathbf{E}),$$

with $X := \nabla_\Sigma \frac{1}{k_\varepsilon^2}\mathrm{div}_\Sigma - \mathbf{curl}_\Sigma \frac{1}{k_\varepsilon^2}\mathrm{curl}_\Sigma$, and where k_ε, C_0, A_ℓ and B_ℓ are defined as in the Helmholtz case.

Transmission Operators for Maxwell

Convergence Analysis for a Model Problem



Model problem with two subdomains and a spherical interface:

$$\Omega_0 = \{\mathbf{x} \in \mathbb{R}^3, ||\mathbf{x}|| > R_0\}, \quad \Omega_1 = \{\mathbf{x} \in \mathbb{R}^3, ||\mathbf{x}|| < R_0\}.$$

Convergence Analysis for a Model Problem

Let

$$\begin{cases} A_{m,1} = \imath\mu_{m,\varepsilon}^{-\frac{1}{2}}\xi_m^{(1)'}(kR_0) - \xi_m^{(1)}(kR_0), & B_{m,1} = \imath\mu_{m,\varepsilon}^{-\frac{1}{2}}\psi_m'(kR_0) - \psi_m(kR_0), \\ A_{m,2} = \imath\xi_m^{(1)'}(kR_0) - \mu_{m,\varepsilon}^{\frac{1}{2}}\xi_m^{(1)}(kR_0), & B_{m,2} = \imath\psi_m'(kR_0) - \mu_{m,\varepsilon}^{\frac{1}{2}}\psi_m(kR_0), \\ A_{m,3} = \imath\mu_{m,\varepsilon}^{-\frac{1}{2}}\psi_m'(kR_0) + \psi_m(kR_0), & B_{m,3} = \imath\mu_{m,\varepsilon}^{-\frac{1}{2}}\xi_m^{(1)'}(kR_0) + \xi_m^{(1)}(kR_0), \\ A_{m,4} = \imath\psi_m'(kR_0) + \mu_{m,\varepsilon}^{\frac{1}{2}}\psi_m(kR_0), & B_{m,4} = \imath\xi_m^{(1)'}(kR_0) + \mu_{m,\varepsilon}^{\frac{1}{2}}\xi_m^{(1)}(kR_0), \end{cases}$$

where

- $\mu_{m,\varepsilon} = 1 - \frac{m(m+1)}{k_\varepsilon^2 R^2}$
- ψ_m and ζ_m are respectively the first- and second-kind Ricatti-Bessel functions of order m
- $\xi_m^{(1)} = \psi_m + \imath\zeta_m$ is the first-kind spherical Hankel's function of order m

Convergence Analysis for a Model Problem

We can show that:

$$\mathbf{g}^{(n+1),m} = \begin{pmatrix} (\mathbf{g}_{12}^{(n+1),m})_1 \\ (\mathbf{g}_{12}^{(n+1),m})_2 \\ (\mathbf{g}_{21}^{(n+1),m})_1 \\ (\mathbf{g}_{21}^{(n+1),m})_2 \end{pmatrix} = \mathbb{A}_m \mathbf{g}^{(n),m} := \begin{pmatrix} 0 & 0 & \frac{B_{m,1}}{A_{m,3}} & 0 \\ 0 & 0 & 0 & \frac{B_{m,2}}{A_{m,4}} \\ \frac{B_{m,3}}{A_{m,1}} & 0 & 0 & 0 \\ 0 & \frac{B_{m,4}}{A_{m,2}} & 0 & 0 \end{pmatrix} \mathbf{g}^{(n),m}$$

with $\mathbb{A}_m$ the iteration matrix for a mode $m \geq 1$, with eigenvalues

$$\lambda_{m,1} = \sqrt{\frac{B_{m,1}\ B_{m,3}}{A_{m,1}\ A_{m,3}}} = -\lambda_{m,2}, \quad \lambda_{m,3} = \sqrt{\frac{B_{m,2}\ B_{m,4}}{A_{m,2}\ A_{m,4}}} = -\lambda_{m,4}.$$

One can prove that $\mathcal{A} = \mathrm{diag}((\mathbb{A}_m)_{m\geq 1})$. Therefore, studying the global convergence of the DDM for $\mathcal{A}$ requires the spectral study of the modal iteration matrices $\mathbb{A}_m$, for $m \geq 1$.

Convergence Analysis for a Model Problem

Quasi-optimality of GIBC(sq, ε)

One can prove that $\rho(\mathbb{A}_m) < 1, \forall m \geq 1$, and that

$$\lim_{m\to\infty} \lambda_{m,(1,3)} = -\lim_{m\to\infty} \lambda_{m,(2,4)} = \frac{\imath\varepsilon}{2k+\imath\varepsilon},$$

i.e., we have two opposite accumulation points in the complex plane for the evanescent modes.

Optimal parameters for GIBC(α) and GIBC(α, β)

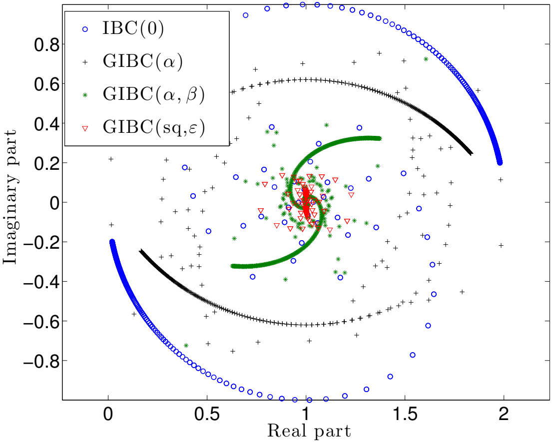
In what follows, the optimal parameters α and β were computed numerically by solving the min-max problem

$$\min_{(\alpha,\beta)\in\mathbb{C}^2} \max_{m\geq 1} \rho(\mathbb{A}_m)$$

with the Matlab function `fminsearch`.

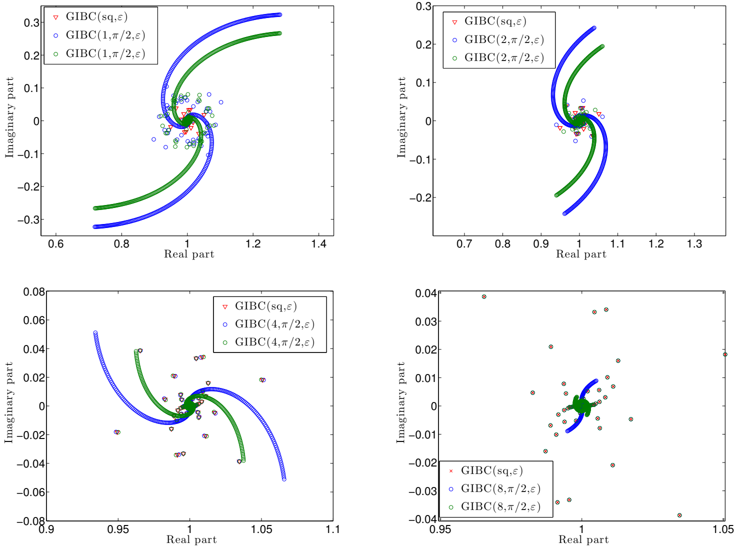
Convergence Analysis for a Model Problem

Eigenvalue distribution in the complex plane for $(I - \mathcal{A})$ and different transmission operators.



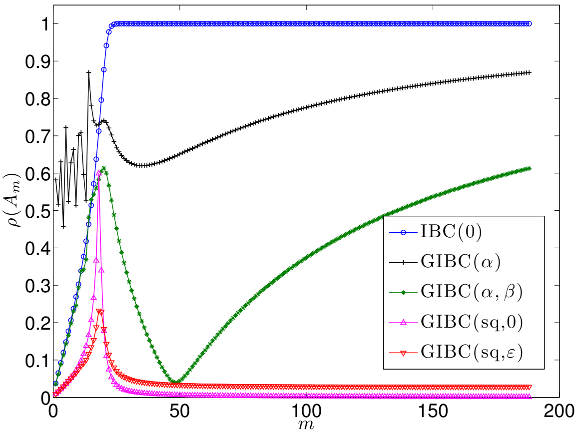
Convergence Analysis for a Model Problem

Influence of the Padé approximation on the eigenvalue distribution.



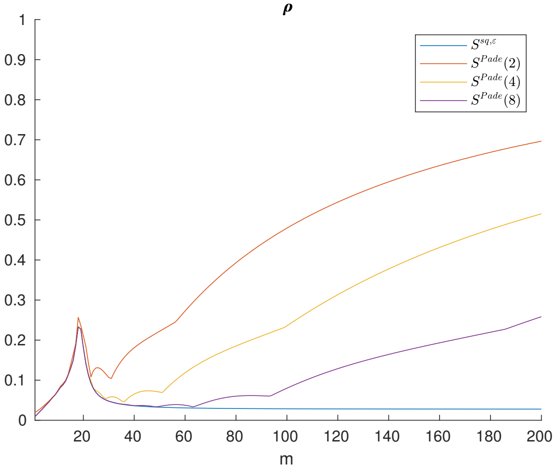
Convergence Analysis for a Model Problem

Spectral radius of $\mathbb{A}_m$ for different transmission operators.



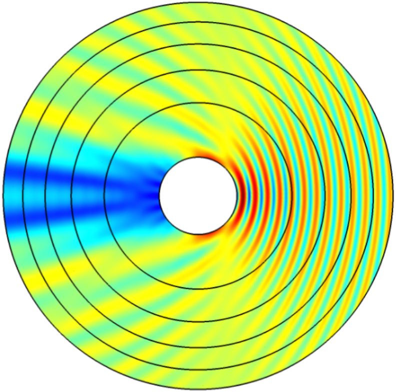
Convergence Analysis for a Model Problem

Spectral radius of $\mathbb{A}_m$ for different Padé transmission operators.



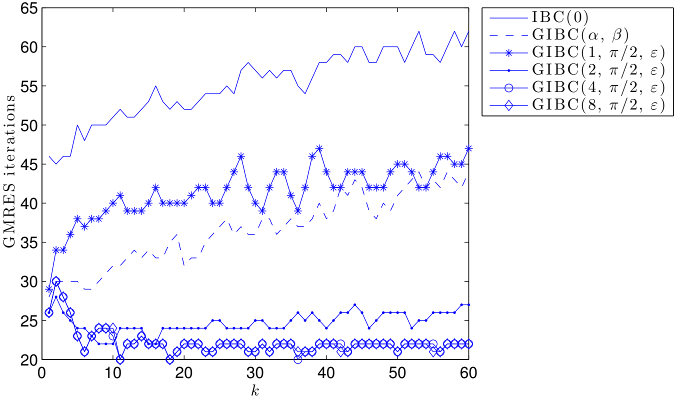
Numerical example

Concentric cylinder decomposition: Number of GMRES iterations vs. wavenumber ($N_{\mathrm{dom}} = 5$, $n_\lambda = 20$).



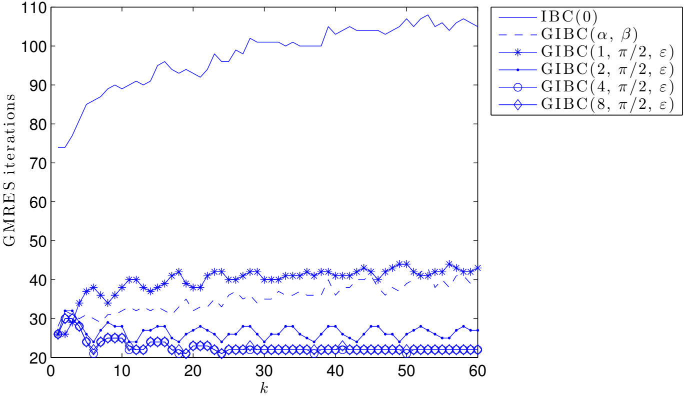
Numerical example

Concentric cylinder decomposition (TE case): Number of GMRES iterations vs. wavenumber ($N_{\mathrm{dom}} = 5$, $n_\lambda = 20$).



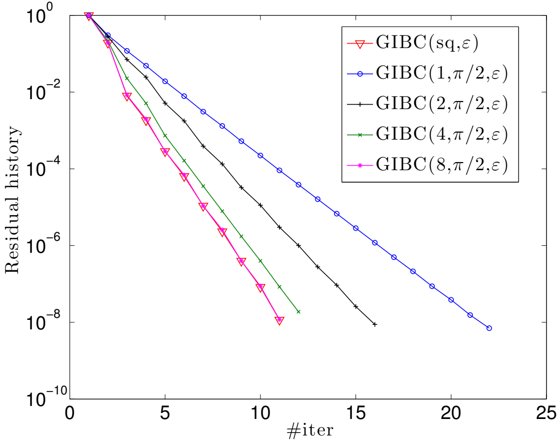
Numerical example

Concentric cylinder decomposition (TM case): Number of GMRES iterations vs. wavenumber ($N_{\text{dom}} = 5$, $n_\lambda = 20$).



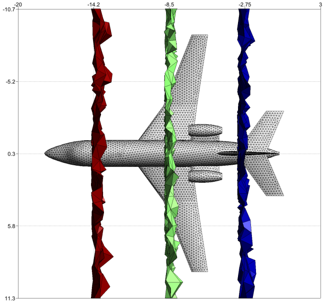
Numerical example

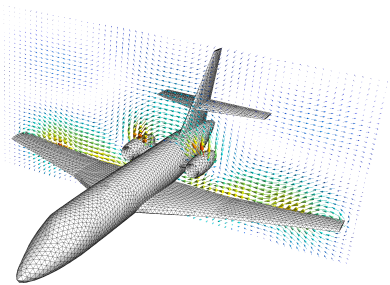
GMRES convergence history for different Padé orders.



Numerical example

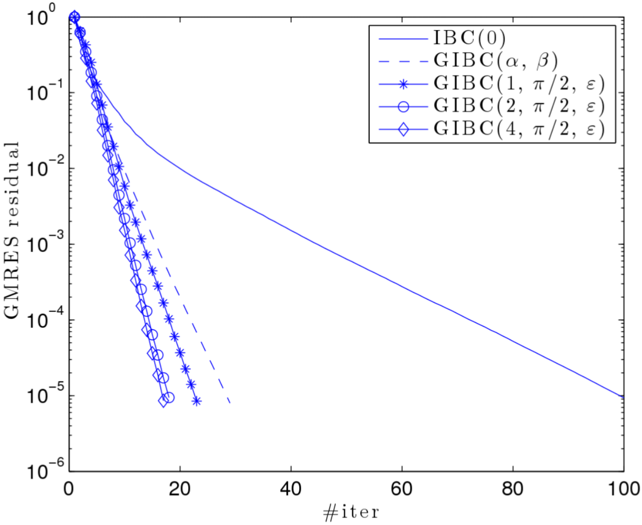
Falcon jet ($N_{dom} = 4, \lambda = 10, n_\lambda = 10$)





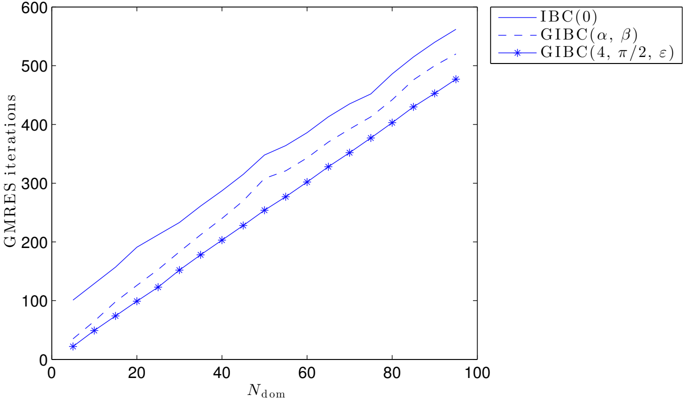
Numerical example

Falcon jet ($N_{dom} = 4, \lambda = 10, n_\lambda = 10$)



Numerical example: scalability issue

Concentric cylinder decomposition (TM case): of iterations vs. number of subdomains ($k = 30$, $n_\lambda = 20$).



Integral Equation based transmission operator

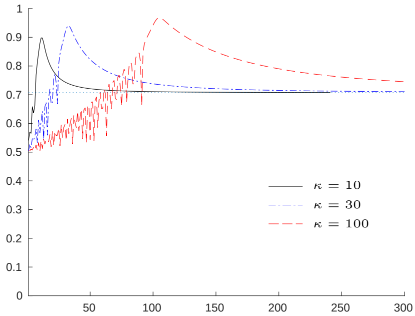
Joined work with X. Claeys and F. Collino

Dissipative Electric Field Integral Equation

$$\int_\Sigma \boldsymbol{u}\,\mathcal{S}(\overline{\boldsymbol{v}})\,d\sigma := \int_{\Sigma\times\Sigma} \mathcal{G}_\alpha(\mathbf{x}-\boldsymbol{y})[\ \alpha^{-1}\mathrm{div}_\Sigma\boldsymbol{u}(\mathbf{x})\mathrm{div}_\Sigma\boldsymbol{v}(\boldsymbol{y}) + \alpha\,\boldsymbol{u}(\mathbf{x})\cdot\boldsymbol{v}(\boldsymbol{y})\ ]d\sigma(\mathbf{x},\boldsymbol{y})$$

with $\mathcal{G}_\alpha(\mathbf{x}) := \exp(-\alpha|\mathbf{x}|)/(2\pi|\mathbf{x}|)$ satisfies $-\Delta\mathcal{G}_\alpha + \alpha^2\mathcal{G}_\alpha = 2\delta_0$.
We have $(\alpha = k)$:

$$\forall m, \quad \rho_m < 1, \qquad \text{and} \qquad \lim_{m\to\infty} \rho_m = \frac{1}{\sqrt{2}}.$$

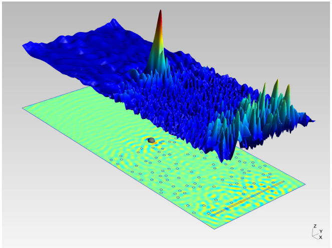


ONELAB

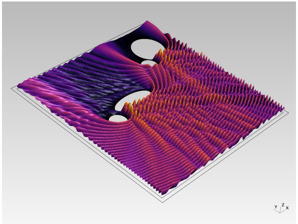
Open Numerical Engineering LABoratory

Provides ready-to-use finite element codes for different community.

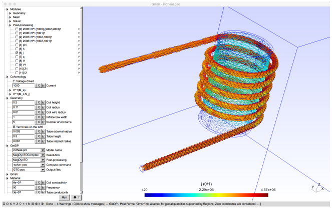
- Magnetostatic
- Acoustic time reversal
- 2D Acoustic scattering
- GetDDM
- ...



Time reversal



2D acoustic scattering



Magnetodynamic

http://onelab.info. Available on Android and iOS markets

GetDDM

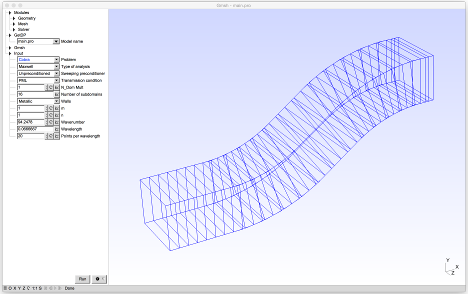
A simple, flexible and ready-to-use environment

- Direct link between discrete and continuous weak-formulations

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\Omega \qquad \forall v \longleftrightarrow$$

```
Galerkin { [ Grad Dof{u}, {Grad u} ];
  In Omega; Jacobian JVol; Integration I1;
}
```

- Parallelism made simple
- Click & run: GUI, full examples and scripts, numerous geometries, . . .



GetDDM

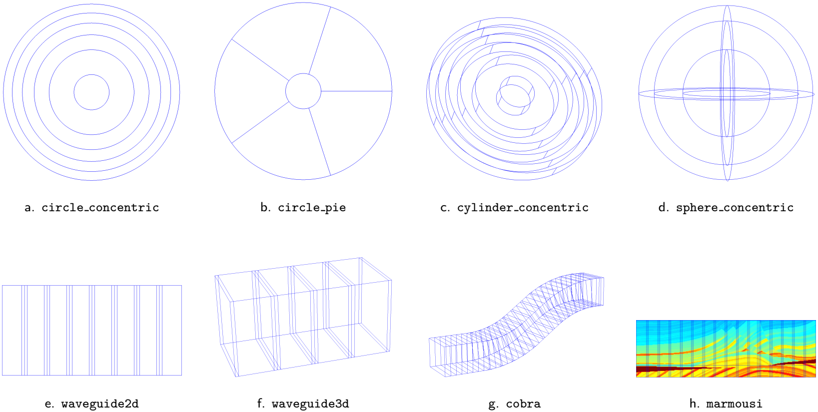
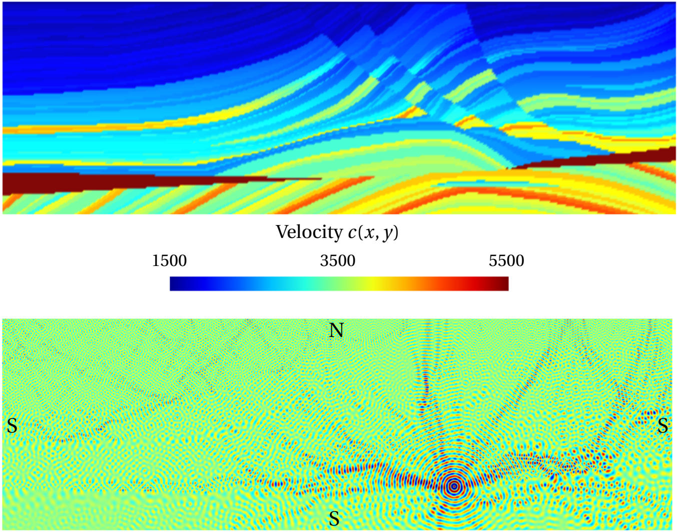


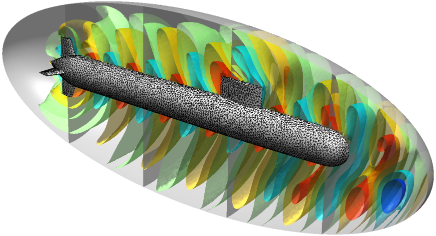
Figure: Sample models available online at `http://onelab.info/wiki/GetDDM`.

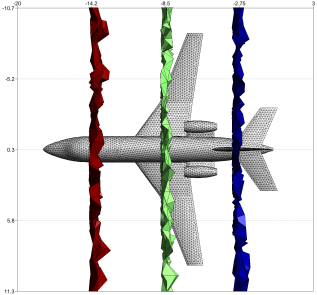
GetDDM

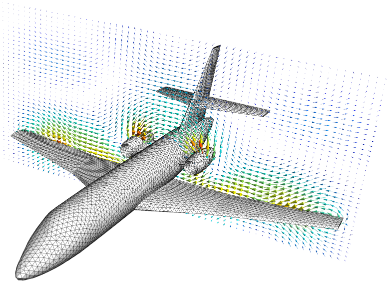


Velocity profile and pressure field. Dimensions: $9192m \times 2904m$. $700Hz$ ($\sim 4000\lambda$ in the domain) with $N = 358$ subdomains on 4296 CPUs: > 2.3 billions unknowns.

Remark: also works on non academic cases







Conclusion

Efficient transmission condition for Helmholtz and Maxwell

- Quasi-optimal in wavenumber and mesh refinement
- Suitable for FE framework

Open source implementation readily available for testing:

- Preprint, code and examples on `http://onelab.info/wiki/GetDDM`
- Work from laptops to massively parallel computer clusters:
  - `marmousi.pro` test-case (Helmholtz) at $700Hz$ (approx. 4000 wavelengths in the domain) with $N = 358$ subdomains on 4296 CPUs: > 2.3 billions unknowns.
  - `waveguide3d.pro` test-case (Maxwell) with $N = 3,500$ subdomains on 3,500 CPUs (cores): > 300 million unknowns.

Perspectives

Mathematics side

- "Padé" operator:
  - Fine analysis on the number of Pade approximants
  - Stability at high frequency regime
- Integral Equation operator:
  - Coupling with Padé for propagative modes
  - Kernel truncation
  - Other integral equation
- Optimization method on complexified square root operator

GetDDM

- Link with HPDDM library (P. Jolivet, P-H. Tournier, F. Nataf)
- Automatic partitioning (Scotch, Metis, . . .)
- "Production mode": real physical cases