Dispersive and dissipative errors in the DPG method with scaled norms for Helmholtz equation

Jay Gopalakrishnan [1] Ignacio Muga [2] Nicole Olivares [1]

[1]Portland State University, Portland, Oregon

[2]Universidad de Valparaíso, Chile

17 July 2013

This work was partially supported by the NSF and by the AFOSR.

Portland State UNIVERSITY

Overview

- Helmholtz equation on $\Omega \subset \mathbb{R}^n$

$$-\Delta\phi - \omega^2\phi = \hat{\imath}\omega f \qquad \rightsquigarrow \qquad A(\vec{u}, \phi) = \begin{pmatrix} \hat{\imath}\omega\vec{u} + \vec{\nabla}\phi \\ \hat{\imath}\omega\phi + \vec{\nabla}\cdot\vec{u} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ f \end{pmatrix}$$

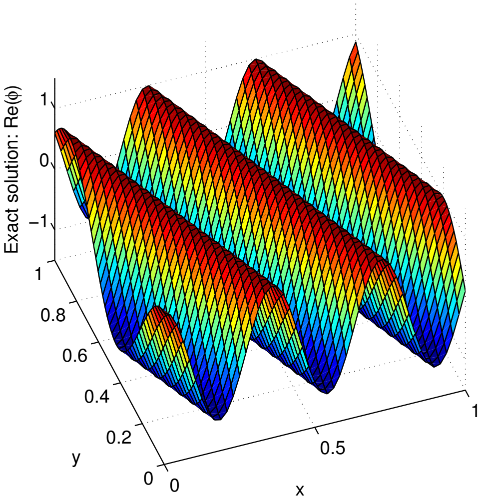
  Assume that the wavenumber ω is not a resonant frequency.

- When ϕ is a plane wave, the DPG method's approximation $(\vec{u}_h, \phi_h)$ satisfies

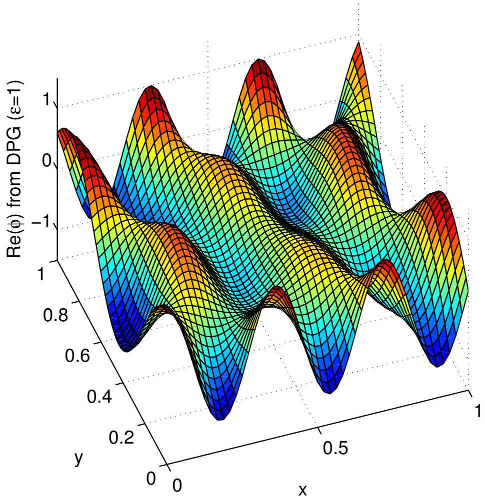
$$\|\vec{u} - \vec{u}_h\| + \|\phi - \phi_h\| \leq C\omega^2 h,$$

  (Demkowicz, Gopalakrishnan, Muga, Zitelli).

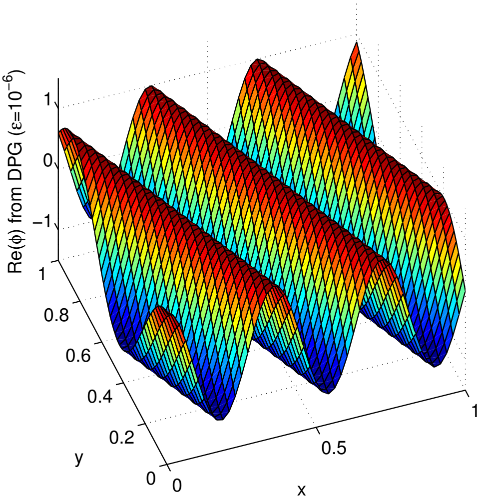
The ε-DPG method



The ε-DPG method



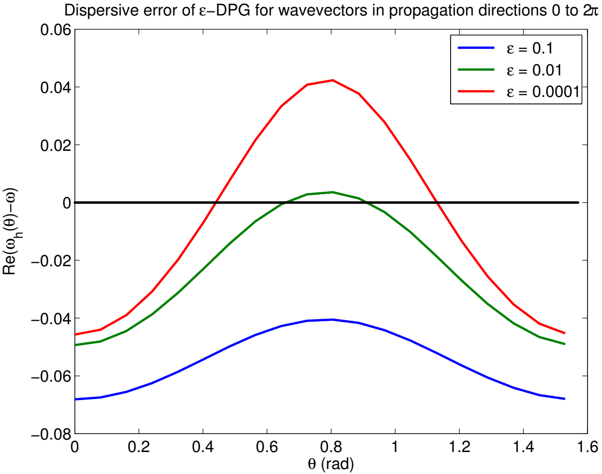
The ε-DPG method



Effect of ε

Compare wavevectors $\vec{k}$ and $\vec{k}_h$ in propagation direction θ,

$$\vec{k} = \omega \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \text{ and } \vec{k}_h = \omega_h \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$



The boundary value problem

Helmholtz wave operator

$$A : H(\mathrm{div}, \Omega) \times H^1(\Omega) \to L^2(\Omega)^N \times L^2(\Omega)$$

$$A(\vec{v}, \eta) = (\hat{\imath}\omega\vec{v} + \vec{\nabla}\eta, \hat{\imath}\omega\eta + \vec{\nabla}\cdot\vec{v})$$

Let $R = H(\mathrm{div}, \Omega) \times H^1_0(\Omega)$ and consider the BVP:

Find $(\vec{u}, \phi) \in R$ satisfying $A(\vec{u}, \phi) = \underline{f}$

for a given $\underline{f} \in L^2(\Omega)^N \times L^2(\Omega)$.

The "broken" space

For a disjoint partition $\overline{\Omega} = \cup_{K \in \Omega_h} \overline{K}$ with ∂K Lipschitz, let

$$V = H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h),$$

where

$$H(\mathrm{div}, \Omega_h) = \{\vec{\tau} : \ \vec{\tau}|_K \in H(\mathrm{div}, K), \ \forall K \in \Omega_h\},$$
$$H^1(\Omega_h) = \{v : \ v|_K \in H^1(K), \ \forall K \in \Omega_h\}.$$

Define $A_h : V \to L^2(\Omega)^N \times L^2(\Omega)$ by

$$A_h(\vec{v}, \eta)|_K = (\hat{\imath}\omega\vec{v}|_K + \vec{\nabla}\eta|_K, \hat{\imath}\omega\eta|_K + \vec{\nabla} \cdot \vec{v}|_K).$$

Derivation of an ultraweak formulation

The equation $A(\vec{u}, \phi) = \underline{f}$ of the BVP can be expressed as

$$-\langle(\vec{u}, \phi), A_h(\vec{v}, \eta)\rangle_h + \langle\!\langle \mathrm{tr}_h(\vec{u}, \phi), (\vec{v}, \eta)\rangle\!\rangle_h = \langle \underline{f}, (\vec{v}, \eta)\rangle_h \quad , \forall (\vec{v}, \eta) \in V.$$

Notation:

$$\langle(\vec{w}, \psi), (\vec{v}, \eta)\rangle_h = \sum_{K \in \Omega_h} \int_K \vec{w} \cdot \overline{\vec{v}} + \psi \overline{\eta},$$

$$\langle\!\langle(\vec{w}, \psi), (\vec{v}, \eta)\rangle\!\rangle_h = \sum_{K \in \Omega_h} \int_{\partial K} (\vec{w} \cdot \vec{n}) \overline{\eta} + \int_{\partial K} \psi \overline{(\vec{v} \cdot \vec{n})}.$$

$$\mathrm{tr}_h : H(\mathrm{div}, \Omega) \times H^1(\Omega) \to \prod_K H^{-1/2}(\partial K)\vec{n} \times H^{1/2}(\partial K)$$

$$\mathrm{tr}_h(\vec{w}, \psi)|_{\partial K} = ((\vec{w} \cdot \vec{n})\vec{n}|_{\partial K}, \psi|_{\partial K}) \in H^{-1/2}(\partial K)\vec{n} \times H^{1/2}(\partial K).$$

Derivation of an ultraweak formulation

The equation $A(\vec{u}, \phi) = \underline{f}$ of the BVP can be expressed as

$$-\langle(\vec{u}, \phi), A_h(\vec{v}, \eta)\rangle_h + \langle\!\langle \underbrace{\mathrm{tr}_h(\vec{u}, \phi)}, (\vec{v}, \eta)\rangle\!\rangle_h = \langle \underline{f}, (\vec{v}, \eta)\rangle_h \quad, \forall (\vec{v}, \eta) \in V.$$

Replace with an independent unknown $(\hat{u}, \hat{\phi}) \in Q = \mathrm{tr}_h(R)$.

Derivation of an ultraweak formulation

The equation $A(\vec{u}, \phi) = \underline{f}$ of the BVP can be expressed as

$$-\langle(\vec{u}, \phi), A_h(\vec{v}, \eta)\rangle_h + \langle\!\langle \underbrace{\mathrm{tr}_h(\vec{u}, \phi)}, (\vec{v}, \eta)\rangle\!\rangle_h = \langle \underline{f}, (\vec{v}, \eta)\rangle_h \quad , \forall (\vec{v}, \eta) \in V.$$

Replace with an independent unknown $(\hat{u}, \hat{\phi}) \in Q = \mathrm{tr}_h(R)$.

Bilinear form:

$$b((\vec{u}, \phi, \hat{u}, \hat{\phi}), (\vec{v}, \eta)) = -\langle(\vec{u}, \phi), A_h(\vec{v}, \eta)\rangle_h + \langle\!\langle(\hat{u}, \hat{\phi}), (\vec{v}, \eta)\rangle\!\rangle_h.$$

Ultraweak formulation: Find $\underline{u} = (\vec{u}, \phi, \hat{u}, \hat{\phi})$ in

$$U = L^2(\Omega)^N \times L^2(\Omega) \times Q$$

satisfying

$$b(\underline{u}, \underline{v}) = \langle \underline{f}, \underline{v}\rangle_h \qquad , \forall \underline{v} = (\vec{v}, \eta) \in V.$$

The ε-DPG method

Let $U_h \subset U$ be finite dimensional. Find $\underline{u}_h \in U_h$ satisfying

$$b(\underline{u}_h, \underline{v}_h) = \langle \underline{f}, \underline{v}_h \rangle_h,$$

for all $\underline{v}_h$ in the space

$$V_h = T\, U_h,$$

where $T : U \to V$ is defined by

$$\langle T\, \underline{w}, \underline{v} \rangle_V = b(\underline{w}, \underline{v}), \qquad \forall \underline{v} \in V,$$

and the V-inner product $\langle \cdot, \cdot \rangle_V$ is generated by the norm

$$\|\underline{v}\|_V^2 = \|A_h \underline{v}\|^2 + \varepsilon^2 \|\underline{v}\|^2.$$

Define U-norm

$$\|(w, \psi, \hat{w}, \hat{\psi})\|_U^2 = \|(w, \psi)\|^2 + \|(\hat{w}, \hat{\psi})\|_Q^2.$$

The ε-DPG method

Let $U_h \subset U$ be finite dimensional. Find ${\underline{u}_h}^r \in U_h$ satisfying

$$b({\underline{u}_h}^r, {\underline{v}_h}^r) = \langle \underline{f}, {\underline{v}_h}^r \rangle_h,$$

for all ${\underline{v}_h}^r$ in the space

$${V_h}^r = T^r U_h,$$

where $T^r : U \to V^r \subset V$ is defined by

$$\langle T^r \underline{w}, \underline{v} \rangle_V = b(\underline{w}, \underline{v}), \qquad \forall \underline{v} \in V^r,$$

and the V-inner product $\langle \cdot, \cdot \rangle_V$ is generated by the norm

$$\| \underline{v} \|_V^2 = \| A_h \underline{v} \|^2 + \varepsilon^2 \| \underline{v} \|^2.$$

Define U-norm

$$\| (w, \psi, \hat{w}, \hat{\psi}) \|_U^2 = \| (w, \psi) \|^2 + \| (\hat{w}, \hat{\psi}) \|_Q^2.$$

Theorem

Suppose there exists $C(\omega)$ such that

$$\|(\vec{r}, \psi)\| \le C(\omega) \|A(\vec{r}, \psi)\|, \qquad \forall (\vec{r}, \psi) \in R.$$

Then the DPG solution admits the quasioptimal error estimate

$$\frac{\|\underline{u} - \underline{u}_h\|_U}{\inf_{\underline{w} \in U_h} \|\underline{u} - \underline{w}\|_U} \le 1 + c\,\varepsilon,$$

with $c = C(\omega)\left(C(\omega)\varepsilon/2 + \sqrt{1 + C(\omega)^2\varepsilon^2/4}\right)$.

This follows from

$$C_1 \|\underline{v}\|_V \le \sup_{\underline{w} \in U} \frac{|b(\underline{w}, \underline{v})|}{\|\underline{w}\|_U} \le C_2 \|\underline{v}\|_V, \qquad \forall \underline{v} \in V.$$

Working out the ε-dependence of the norms, we conclude that the DPG errors for fluxes and traces admit a better bound for smaller ε.

Numerical experiment

The theorem gives

$$\frac{\|\underline{u} - \underline{u}_h\|_U}{\inf_{\underline{w} \in U_h} \|\underline{u} - \underline{w}\|_U} \leq 1 + c\,\varepsilon.$$

We compute the ratio

$$\left(\frac{\|\vec{u} - \vec{u}_h^r\|^2 + \|\phi - \phi_h^r\|^2}{\inf_{(\vec{w},\psi,0,0) \in U_h} \|\vec{u} - \vec{w}\|^2 + \|\phi - \psi\|^2} \right)^{1/2}$$

and expect it to be closer to 1 for smaller ε.

Numerical experiment

- For a range of wavenumbers ω, compute

$$\left(\frac{\|\vec{u} - \vec{u}_h^r\|^2 + \|\phi - \phi_h^r\|^2}{\inf_{(\vec{w},\psi,0,0)\in U_h} \|\vec{u} - \vec{w}\|^2 + \|\phi - \psi\|^2} \right)^{1/2}$$

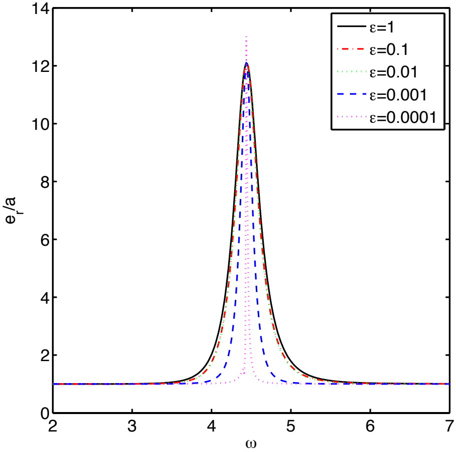
- Data $\underline{f} = (\vec{0}, f)$ such that $\phi = x(1-x)y(1-y)$ on the unit square.
- Near resonant frequencies, $C(\omega)$ blows up.

$\omega = \pi\sqrt{m^2 + n^2}$	Excited?
$\pi\sqrt{2} \approx 4.4$	yes
$\pi\sqrt{5} \approx 7.0$	no
$\pi\sqrt{8} \approx 8.9$	no
$\pi\sqrt{13} \approx 11.3$	no
$\pi\sqrt{18} \approx 13.3$	yes

- Compare ratio plots for various values of ε (with $h = 1/16$, $r = 3$ fixed).

Regularizing effect of ε-DPG

The ratio $\frac{e_r}{a} = \left(\frac{\|\vec{u} - \vec{u}_h^r\|^2 + \|\phi - \phi_h^r\|^2}{\inf_{(\vec{w},\psi,0,0)\in U_h} \|\vec{u} - \vec{w}\|^2 + \|\phi - \psi\|^2} \right)^{1/2}$ near a resonance.



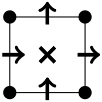
Computing lowest order method

Local matrix

$$B_{i,j} = b(e_j, T^r e_i),$$

where $\{e_i\}$ spans

$$\{(\vec{w}, \psi) : \vec{w} \text{ and } \psi \text{ constant functions on } K\}$$
$$\times\{(\hat{w}, \hat{\psi}) : \hat{w} \text{ constant on each edge of } \partial K, \hat{\psi} \text{ piece-wise linear and continuous on each edge of } \partial K\}$$



Computing lowest order method

Local matrix

$$B_{i,j} = b(e_j, T^r e_i),$$

where $\{e_i\}$ spans

$$\{(\vec{w}, \psi) : \vec{w} \text{ and } \psi \text{ constant functions on } K\}$$
$$\times\{(\hat{w}, \hat{\psi}) : \hat{w} \text{ constant on each edge of } \partial K, \hat{\psi} \text{ piece-wise linear and continuous on each edge of } \partial K\}$$

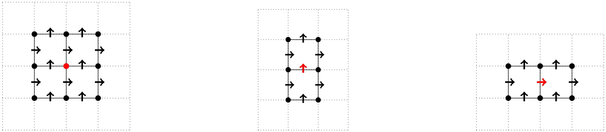
Eliminate interior variables to obtain 8×8 condensed matrix.

Dispersion analysis

- Goal is to compute the numerical wave vector for a discrete approximation to a plane wave propagating over an infinite lattice.
- The discrete method may propagate faster or slower than the true wave speed. We compare

$$\vec{k} = \omega \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \text{and} \quad \vec{k}_h = \omega_h \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

- The real and imaginary parts of $\omega_h - \omega$ measure numerical dispersion and dissipation, respectively.
- Approach adapted from Deraemaeker, Babuška, and Brouillard.



For $t = 1, 2, 3$:

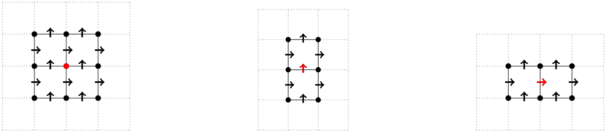
- Consider the t^{th} stencil centered at the origin.
- Denote stencil nodes as $\bigcup_{s=1}^{3}\{\vec{j}h : \vec{j} \in J_s\}$, where $J_s \subset (\mathbb{Z}/2)^2$ locates nodes of type s within the stencil.
- Apply the stencil to the solution values

$$\psi_{1,\vec{j}} = \hat{\phi}_h(\vec{x}_{\vec{j}}) \qquad \forall \vec{x}_{\vec{j}} \in (h\mathbb{Z})^2,$$

$$\psi_{2,\vec{j}} = \hat{u}_h(\vec{x}_{\vec{j}}) \qquad \forall \vec{x}_{\vec{j}} \in (h\mathbb{Z} + h/2) \times h\mathbb{Z},$$

$$\psi_{3,\vec{j}} = \hat{u}_h(\vec{x}_{\vec{j}}) \qquad \forall \vec{x}_{\vec{j}} \in h\mathbb{Z} \times (h\mathbb{Z} + h/2).$$

$$\sum_{s=1}^{3} \sum_{\vec{j}\in J_s} D_{t,s,\vec{j}}\, \psi_{s,\vec{j}}$$



For $t = 1, 2, 3$:

- Consider the t^{th} stencil centered at the origin.
- Denote stencil nodes as $\bigcup_{s=1}^{3}\{\vec{\jmath}h : \vec{\jmath} \in J_s\}$, where $J_s \subset (\mathbb{Z}/2)^2$ locates nodes of type s within the stencil.
- Apply the stencil to the solution values

$$\psi_{1,\vec{\jmath}} = \hat{\phi}_h(\vec{x}_{\vec{\jmath}}) = a_1 e^{\hat{\imath}\vec{k}_h\cdot\vec{x}_{\vec{\jmath}}} \qquad \forall \vec{x}_{\vec{\jmath}} \in (h\mathbb{Z})^2,$$

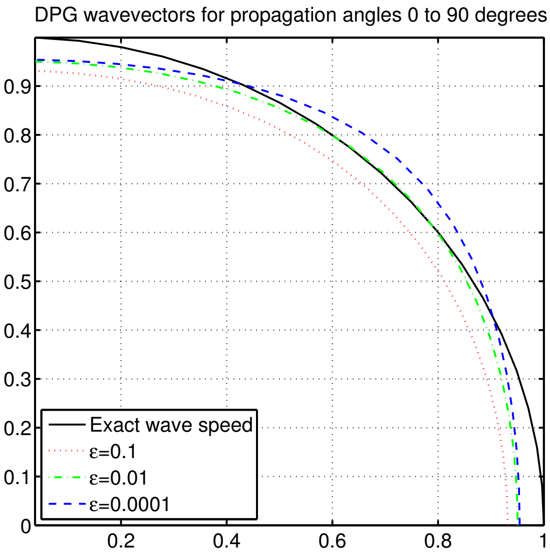
$$\psi_{2,\vec{\jmath}} = \hat{u}_h(\vec{x}_{\vec{\jmath}}) = a_2 e^{\hat{\imath}\vec{k}_h\cdot\vec{x}_{\vec{\jmath}}} \qquad \forall \vec{x}_{\vec{\jmath}} \in (h\mathbb{Z} + h/2) \times h\mathbb{Z},$$

$$\psi_{3,\vec{\jmath}} = \hat{u}_h(\vec{x}_{\vec{\jmath}}) = a_3 e^{\hat{\imath}\vec{k}_h\cdot\vec{x}_{\vec{\jmath}}} \qquad \forall \vec{x}_{\vec{\jmath}} \in h\mathbb{Z} \times (h\mathbb{Z} + h/2).$$

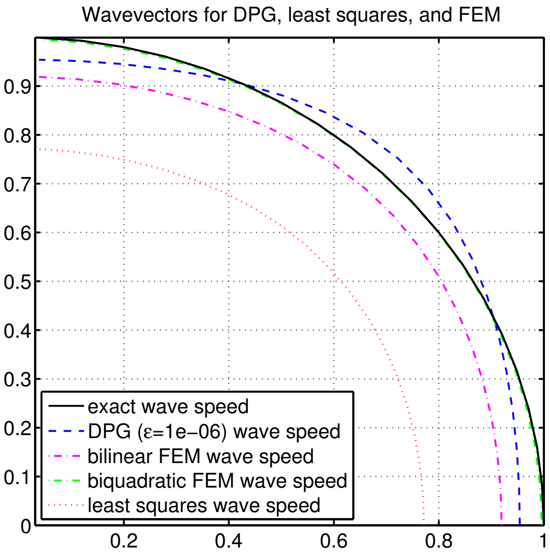
- Suppose the DPG solution interpolates a plane wave

$$\sum_{s=1}^{3}\sum_{\vec{\jmath}\in J_s} D_{t,s,\vec{\jmath}}\,\psi_{s,\vec{\jmath}} = 0.$$

Dependence on θ

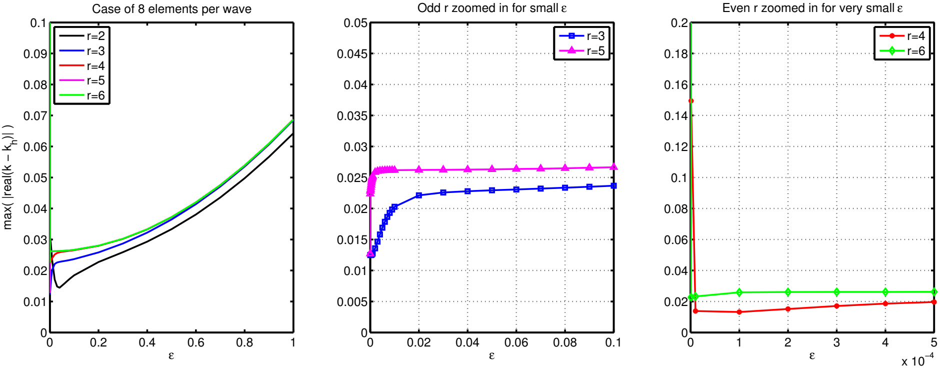


Dependence on θ



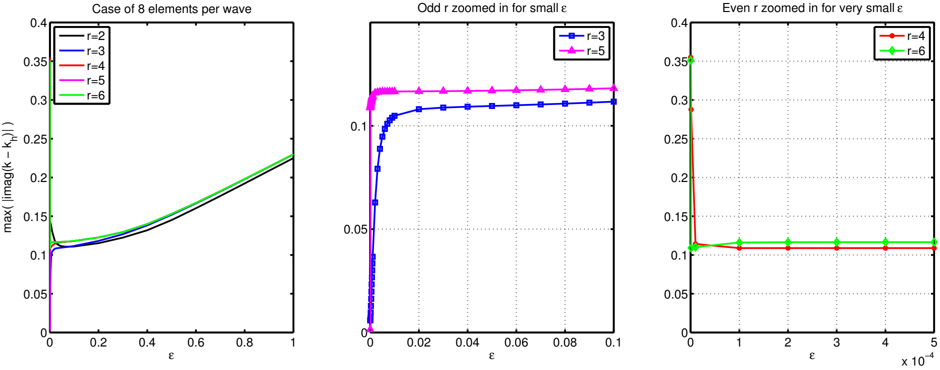
Dependence on ε and r

Plots of $\max_\theta |\mathrm{Re}(\omega_h(\theta)) - \omega|$

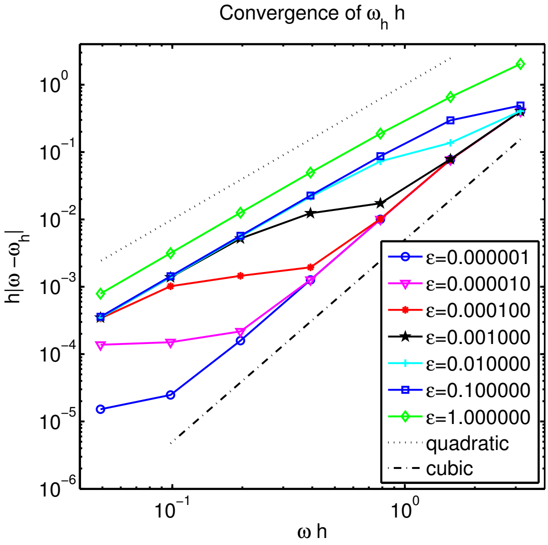


Dependence on ε and r

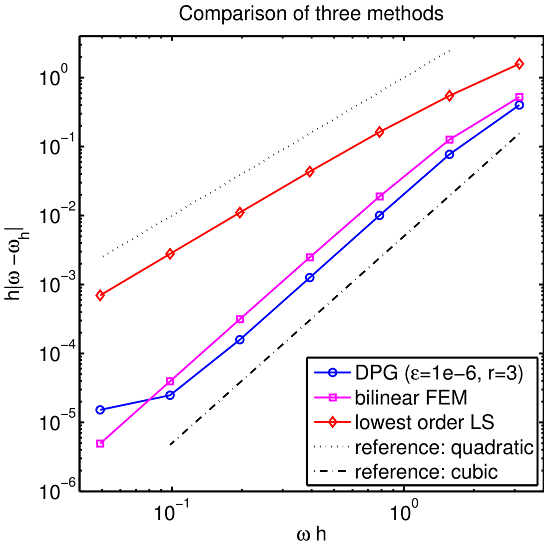
Plots of $\eta = \max_\theta |\mathrm{Im}(\omega_h(\theta))|$

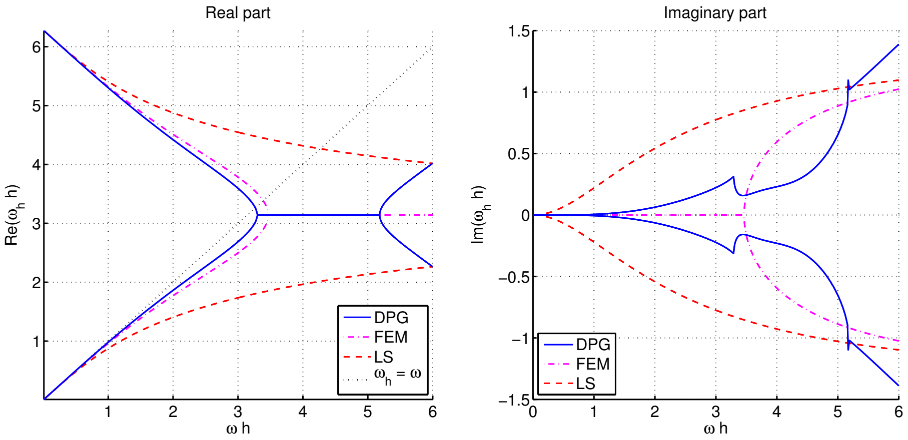


Dependence on ω



Dependence on ω




Real part
Re(ω_h h)
ω h
DPG
FEM
LS
$\omega_h = \omega$
Imaginary part
Im(ω_h h)
ω h
DPG
FEM
LS

Conclusions

For the lowest order DPG method:

- Both dispersive and dissipative errors exist.
- Solutions have higher accuracy than an L^2-based least-squares method with a stencil of identical size.
- Errors do not compare favorably with a standard (higher order) finite element method having a stencil of the same size.
- There is theoretical justification for considering the ε-modified DPG method.

Topics for further study include:

- A theoretical explanation of the discrete effects that cause the errors to continually decrease as $\varepsilon \to 0$ only for the case of odd enrichment degree r.