Euler, Ritz, Galerkin, Courant: On the Road to the Finite Element Method

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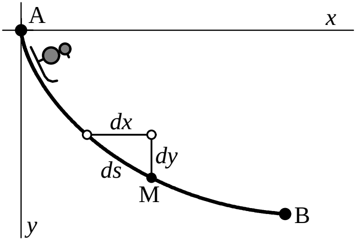
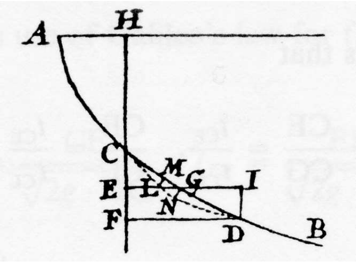
Bernoulli

($\beta\rho\alpha\chi\upsilon\varsigma$ =short, $\chi\rho o\nu o\varsigma$ =time)

Johann Bernoulli (1696), challenge to his brother Jacob:

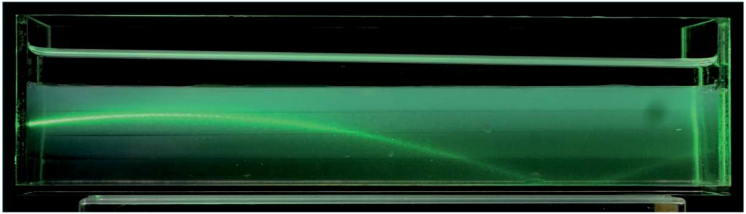
> *"Datis in plano verticali duobus punctis A & B, assignare Mobili M viam AMB, per quam gravitate sua descendens, & moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B."*





See already Galilei (1638)

Real Experiment Following Jacob's Idea



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Mathematical Formulation

Letter of de l'Hôpital to Joh. Bernoulli, June 15th, 1696:

> *Ce probleme me paroist des plus curieux et des plus jolis que l'on ait encore proposé et je serois bien aise de m'y appliquer ; mais pour cela il seroit necessaire que vous me l'envoyassiez reduit à la mathématique pure, car le phisique m'embarasse ...*

Time for passing through a small arc length ds: $dJ = \frac{ds}{v}$.
Speed (Galilei): $v = \sqrt{2gy}$
Need to find $y(x)$ with $y(a) = A$, $y(b) = B$ such that

$$J = \int_a^b \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \int_a^b \frac{\sqrt{1+p^2}}{\sqrt{y}}\, dx = \min \quad (p = \frac{dy}{dx})$$

Euler's Treatment

Euler (1744): general variational problem

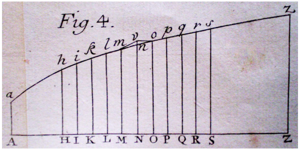
$$J = \int_a^b Z(x, y, p)\, dx = \min \quad (p = \frac{dy}{dx})$$

Theorem (Euler 1744)

The optimal solution satisfies the differential equation

$$\boxed{N - \frac{d}{dx}P = 0} \qquad \textit{where} \quad N = \frac{\partial Z}{\partial y}, \quad P = \frac{\partial Z}{\partial p}$$

Proof.



$$Z\,dx + Z'\,dx + Z''\,dx + Z'''\,dx + \&c.$$

$$(P + N'dx - P')$$

$$N - \frac{dP}{dx} = 0$$

Joseph Louis de Lagrange

August 12th, 1755: Ludovico de la Grange Tournier (19 years old) writes to *Vir amplissime atque celeberrime L. Euler*
September 6th, 1755: Euler replies to *Vir praestantissime atque excellentissime Lagrange* with an enthusiastic letter
Idea of Lagrange: suppose $y(x)$ is solution, and add an arbitrary *variation* $\varepsilon\delta y(x)$. Then

$$J(\varepsilon) = \int_a^b Z(x, y + \varepsilon\delta y, p + \varepsilon\delta p)\, dx$$

must *increase in all directions*, i.e. its derivative with respect to ϵ must be zero for $\epsilon = 0$:

$$\frac{\partial J(\varepsilon)}{\partial\varepsilon}\,|_{\varepsilon=0} = \int_a^b (N \cdot \delta y + P \cdot \delta p)\, dx = 0.$$

Since δp is the derivative of δy , we integrate by parts:

$$\int_a^b (N - \frac{d}{dx}P) \cdot \delta y \cdot dx = 0$$

Central Highway of Variational Calculus

Since δy is arbitrary, we conclude from

$$\int_a^b (N - \frac{d}{dx}P) \cdot \delta y \cdot dx = 0$$

that for all x

$$N - \frac{d}{dx}P = 0$$

Central Highway of Variational Calculus:

1. $J(y) \longrightarrow \min$
2. $\frac{dJ(y+\epsilon v)}{d\varepsilon}|_{\varepsilon=0} \stackrel{!}{=} 0$: **weak form**
3. Integration by parts, arbitrary variation: **strong form**

Connects the *Lagrangian* of a mechanical system (difference of potential and kinetic energy) to the differential equations of its motion. This later led to Hamiltonian mechanics.

Gravitation of a Complicated Body

Newton (Principia 1687): inverse square law for celestial bodies, $\mathbf{f} = (f_1, f_2, f_3)$ with (see also Euler 1749)

$$f_1 \approx \frac{x-\xi}{r^3}, \quad r := \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

Laplace (1785): What if the celestial body is not a point ?

$$f_1 = \iiint \rho(\xi,\eta,\zeta)\frac{x-\xi}{r^3}\, d\xi\, d\eta\, d\zeta$$

Idea of Laplace: Introduce the potential function

$$u = \iiint \rho(\xi,\eta,\zeta)\frac{1}{r}\, d\xi\, d\eta\, d\zeta$$

Taking a derivative with respect x, we obtain

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{x-\xi}{r^3} \quad \Longrightarrow \quad \mathbf{f} = -\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

Laplace's Equation

Differentiating once more, we obtain $\frac{\partial}{\partial x}\frac{x-\xi}{r^3} = \frac{r^3-3(x-\xi)^2 r}{r^6}$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0!$$

Laplace's equation was also discovered independantly in:

- theory of *stationary heat transfer* (Fourier 1822);
- theory of *magnetism* (Gauss and Weber in Göttingen 1839);
- theory of *electric fields* (W. Thomson, later Lord Kelvin 1847, Liouville 1847);
- *conformal mappings* (Gauss 1825);
- irrotational *fluid motion* in 2D (Helmholtz 1858)
- in *complex analysis* (Cauchy 1825, Riemann Thesis 1851);

Riemann Mapping Theorem

Riemann (Thesis 1851 in Göttingen):

"Eine vollkommen in sich abgeschlossene mathematische Theorie, welche ... fortschreitet, ohne zu scheiden, ob es sich um die Schwerkraft, oder die Electricität, oder den Magnetismus, oder das Gleichgewicht der Wärme handelt."

Theorem

If $f(z) = u(x,y) + iv(x,y)$ is holomorph in Ω, then

$$\Delta u = u_{xx} + u_{yy} = 0 \qquad \textit{and} \qquad \Delta v = v_{xx} + v_{yy} = 0.$$

Proof:

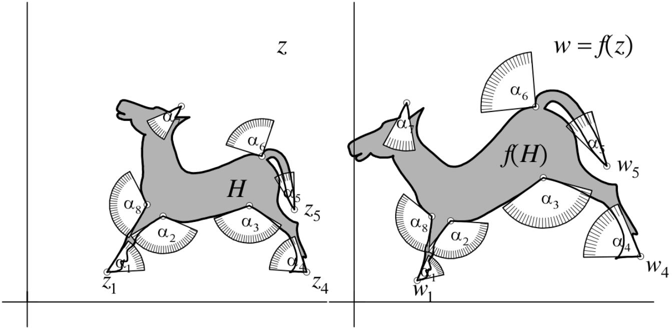
$$\frac{df(z)}{dz} = \begin{cases} \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{\partial f}{\partial x} \\ \lim_{h\to 0} \frac{f(z+ih)-f(z)}{ih} = -i\frac{\partial f}{\partial y} \end{cases} \Longrightarrow u_x + iv_x = -iu_y + v_y$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Conformal maps

The Jacobian of such a function satisfies

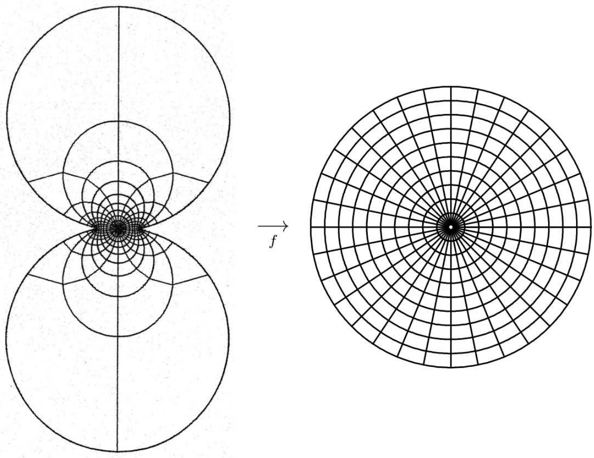
$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = ||(u_x, u_y)|| \cdot \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$



" ... und ihre entsprechenden kleinsten Theile ähnlich sind;" (Thesis §21)

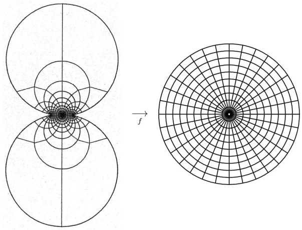
Riemann Mapping Theorem

"Zwei gegebene einfach zusammenhängende Flächen können stets so aufeinander bezogen werden, dass jedem Punkte der einen ein mit ihm stetig fortrückender Punkt entspricht...;"



(drawing M. Gutknecht 18.12.1975)

Idea of Riemann's Proof



Find f which maps Ω to the unit disk and z_0 to 0: set

$$f(z) = (z - z_0)e^{g(z)}, \; g = u + iv \implies z_0 \text{ only zero}$$

In order to arrive on the boundary of the disk

$$|f(z)| = 1, \; z \in \partial\Omega \implies u(z) = -log|z - z0|, \; z \in \partial\Omega.$$

Once harmonic u with this boundary condition is found, construct v with the Cauchy-Riemann equations.

Question: Does such a u exist ???

Riemann's Audacious "Proof"

Riemann 1857, ***Werke*** **p. 97:**

> *"Hierzu kann in vielen Fällen . . . ein Princip dienen, welches Dirichlet zur Lösung dieser Aufgabe für eine der Laplace'schen Differentialgleichung genügende Function . . . in seinen Vorlesungen . . . seit einer Reihe von Jahren zu geben pflegt."*

Idea: For all functions defined on a given domain Ω with the prescribed boundary values, the integral

$$J(u) = \iint_\Omega \frac{1}{2}\left(u_x^2 + u_y^2\right)\,dx\,dy \quad \text{is always} > 0.$$

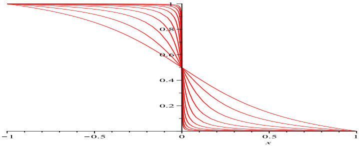
Choose among these functions the one for which this integral is minimal !
(see citation; from here originates the name "Dirichlet Principle" and "Dirichlet boundary conditions").

But is the Dirichlet Principle Correct?

Weierstrass's Critique (1869, Werke 2, p. 49):

$$\int_{-1}^{1} (x \cdot y')^2 \, dx \to \min \quad y(-1) = a, \; y(1) = b.$$

$$\Longrightarrow y = \frac{a+b}{2} + \frac{b-a}{2} \frac{\arctan \frac{x}{\epsilon}}{\arctan \frac{1}{\epsilon}}$$



"Die Dirichlet'sche Schlussweise führt also in dem betrachteten Falle offenbar zu einem falschen Resultat."

Riemann's Answer to Weierstrass: "... meine Existenztheoreme sind trotzdem richtig". (see F. Klein)

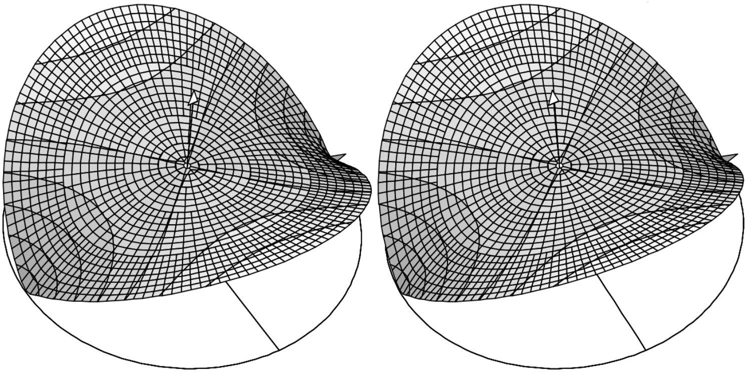
Helmholtz: "Für uns Physiker bleibt das Dirichletsche Prinzip ein Beweis"

International Challenge

Find harmonic functions $\Delta u = 0$ on any domain Ω with prescribed boundary conditions $u = g$ for $(x, y) \in \partial\Omega$.

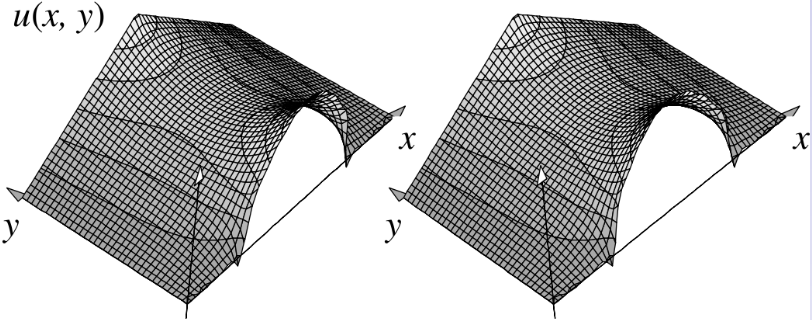
Solution easy for circular domain (Poisson 1815) ...

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\phi-\psi)+r^2}\, f(\psi)\, d\psi \ .$$



International Challenge

... and for rectangular domains (Fourier 1807):

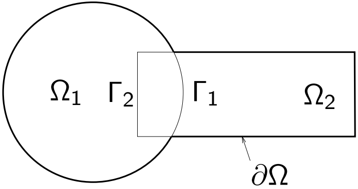


But existence of solutions of Laplace equation on arbitrary domains appears hopeless !

Proof without Dirichlet Principle

H.A. Schwarz (1870, Crelle 74, 1872)

Solve alternatingly in subdomains Ω_1 and Ω_2 which are rectangles or discs...



D. Hilbert (Göttingen 1901, Annalen 1904, Crelle J. 1905): Rehabilitation of Dirichlet Principle:

"... eine besondere Kraftleistung beweisender Mathematik, ..." (F. Klein *Entw. Math. 19. Jahrh.*, p. 266).

"Mittlerweile war das verachtete und scheintote *Dirichlet*sche Prinzip durch Hilbert wieder zum Leben erweckt worden;..." (Hurwitz-Courant *Funktionentheorie*, Springer Grundlehren 3, p. 392)

Birth of Numerical Methods for PDE's

Finite difference methods:

Über eine Methode die partielle Differentialgleichung
$\Delta u =$ Constans numerisch zu integrieren.

Von C. Runge in Göttingen.

Variational methods ⇒ Ritz-Galerkin ⇒ FE methods:

Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik.

Von Herrn *Walter Ritz* in Göttingen.

Both in 1908; both in Göttingen !

Ritz: Vaillant Prize 1907

Preisaufgaben der Académie des Sciences de Paris aus der angewandten Mathematik und Physik.

Für 1907. [1]

Prix Vaillant (4000 fr.): Perfectionner en un point important le problème d'Analyse relatif à l'équilibre des plaques élastiques encastrées, c'est-à-dire le problème de l'intégration de l'équation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x, y)$$

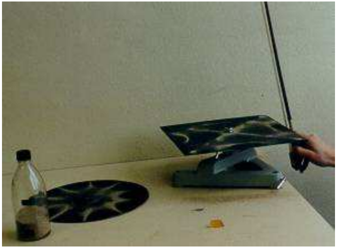
avec les conditions que la fonction u et sa dérivée suivant la normale au contour de la plaque soient nulles. Examiner plus spécialement le cas d'un contour rectangulaire. — Les Mémoires devront être envoyés au Secrétariat avant le 1er janvier 1907.

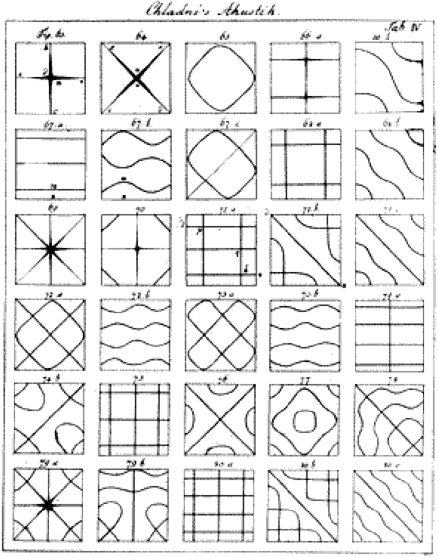
Ritz had worked with many such problems in his thesis, where he tried to explain the Balmer series in spectroscopy (1902); it therefore appeared to him that he had good chances to succeed in this competition.

But: Hadamard will win the price...

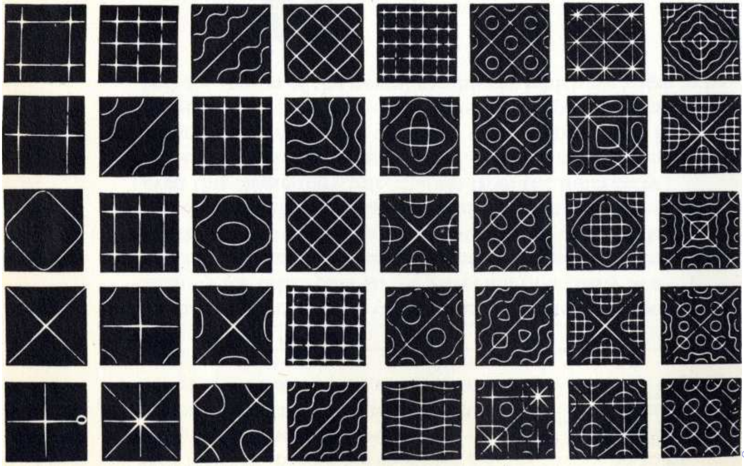
Ritz: Chladni Figures 1909

Ernst Florens Friedrich Chladni (1787): Entdeckung über die Theorie des Klangs, Leipzig.





Modern Experiments (U. San Diego, Munich)



Mathematical Model

Walther Ritz (1909): Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern

> *"Die Differentialgleichungen und Randbedingungen für die transversalen Schwingungen ebener, elastischer Platten mit freien Rändern sind bekanntlich zuerst in teilweise unrichtiger Form von Sophie Germain und Poisson, in definitiver Gestalt aber von Kirchhoff im Jahre 1850 gegeben worden."*

Chladni figures correspond to eigenpairs of the bi-harmonic operator

$$\Delta^2 w = \lambda w \qquad \text{in } \Omega := (-1,1)^2$$

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + (2-\mu)\frac{\partial^2 w}{\partial y^2}\right) &= 0, \quad \frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2} = 0, \; x = \{-1,1\} \\ \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2} + (2-\mu)\frac{\partial^2 w}{\partial x^2}\right) &= 0, \quad \frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2} = 0, \; y = \{-1,1\} \end{aligned}$$

Here, μ is the elasticity constant.

Solution Attempts before Ritz

Wheatstone (1833): Approximation by cosine and sine functions (Ritz: "..., dass es sich hier nur um einen in besonderen Fällen anwendbaren Kunstgriff handelt").

Kirchhoff (1850): Solution for circular plate.

R. König (1864): Careless experimental results lead to the conclusion that Chladni figures can only contain straight lines.

S. Tanaka (1887): Integration starting from straight lines in order to obtain solutions (Ritz: " ... Tanaka glaubt, allgemeinere und strengere Formeln zu erhalten. Dies ist aber schon deswegen nicht der Fall, weil übersehen ist, dass *eine* Randbedingung die Lösung gar nicht bestimmt.")

W. Voigt (1893): Solution for rectangular plate with two or four clamped boundaries by elementary integration.

John William Strutt, Baron Rayleigh (1894): "The Problem of a rectangular plate, whose edges are free, is one of great difficulty, and has for the most part resisted attack"

The Ritz Method

Walther Ritz (1909): "Das wesentliche der neuen Methode besteht darin, dass nicht von den Differentialgleichungen und Randbedingungen des Problems, sondern *direkt vom Prinzip der kleinsten Wirkung* ausgegangen wird, aus welchem ja durch Variation jene Gleichungen und Bedingungen gewonnen werden können."

$$J(w) := \int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]$$

Minimization principle: solution w is a minimum of

$$J(w) \to \min, \quad \int_{-1}^{1}\int_{-1}^{1} w^2 dxdy = const.$$

This leads to the differential equation form with the Bi-Laplacian (Lagrange parameter λ, integration by parts, Hamilton principle).

Ritz' Invention

Walther Ritz (1909): "Im folgenden entwickle ich am Beispiel der quadratischen Platten mit freien Rändern eine neue Integrationsmethode, die ohne wesentliche Änderungen auch auf rechteckige Platten angewandt werden kann, sei es mit freien, sei es auch mit teilweise oder ganz eingespannten oder gestützten Rändern. Theoretisch ist die Lösung in ähnlicher Weise sogar für eine beliebige Gestalt der Platte möglich; eine genaue Berechnung einer grösseren Anzahl von Klangfiguren, wie sie im folgenden für den klassischen Fall der quadratischen Scheibe durchgeführt ist, wird aber nur bei geeigneter Wahl der Grundfunktionen, nach welchen entwickelt wird, praktisch ausführbar."

$$w_s = \sum_{m=0}^{s} A_m u_m(x, y)$$

"Für den Grundton, wofern grosse Genauigkeit nicht gefordert wird, führt das Verfahren für die meisten Platten durch den Ansatz von Polynomen zum Ziel."

Ritz' Choice of 'Coordinate Functions'

Walther Ritz (1909): "Sämtliche Eigentöne der Platte lassen sich bis auf einige Prozent darstellen durch die Formeln:"

$$\begin{aligned} w_{mn} &= u_m(x)u_n(y) + u_m(y)u_n(x) \\ w'_{mn} &= u_m(x)u_n(y) - u_m(y)u_n(x) \end{aligned}$$

where $u_m(x)$ are the known eigenfunctions of a free one dimensional bar (see Lord Rayleigh, The Theory of Sound)

$$\frac{d^4 u_m}{dx^4} = k_m^4 u_m, \quad \text{with } \frac{d^2 u_m}{dx^2} = 0,\ \frac{d^3 u_m}{dx^3} = 0 \text{ at } x = \{-1, 1\},$$

which are

$$u_m = \begin{cases} \frac{\cosh k_m \cos k_m x + \cos k_m \cosh k_m x}{\sqrt{\cosh^2 k_m + \cos^2 k_m}}, & \tan k_m + \tanh k_m = 0,\ m \text{ even} \\ \frac{\sinh k_m \sin k_m x + \sin k_m \sinh k_m x}{\sqrt{\sinh^2 k_m - \sin^2 k_m}}, & \tan k_m - \tanh k_m = 0,\ m \text{ odd} \end{cases}$$

Hence the key idea of Ritz: approximate w by

$$w_s := \sum_{m=0}^{s} \sum_{n=0}^{s} A_{mn} u_m(x) u_n(y)$$

Approximate Minimization

How to determine A_{mn} ?

Ritz: "Es liegt nahe, als Massstab des Gesamtfehlers die Abweichung der potentiellen Energie von ihrem exakten Wert beim wirklichen Vorgang zu wählen; dies kommt auf die Forderung hinaus: *es sind die A_{mn} so zu wählen, dass der Ausdruck* (this is the functional $J(w)$!)

$$\int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{\partial^2 w_s}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_s}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_s}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w_s}{\partial x\partial y}\right)^2\right]$$

unter der Bedingung

$$U(w_s) := \int_{-1}^{1}\int_{-1}^{1} w_s^2 dxdy = const$$

möglichst klein werde."

Calculating . . .

To evaluate $J(w_s)$, we thus have to evaluate

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial^2 w_s}{\partial x^2}\right)^2 = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial^2 \sum_{m,n} A_{mn}u_m(x)u_n(y)}{\partial x^2}\right)^2 dxdy$$

$$= \sum_{m,n}\sum_{p,q} A_{mn}A_{pq} \underbrace{\int_{-1}^{1}\int_{-1}^{1} \frac{\partial^2 u_m(x)}{\partial x^2} u_n(y) \frac{\partial^2 u_p(x)}{\partial x^2} u_q(y) dxdy}_{c^1_{mnpq}:=}.$$

Now c^1_{mnpq} can be computed, since u_n is known! Similarly

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial^2 w_s}{\partial y^2}\right)^2 dxdy &= \sum_{m,n}\sum_{p,q} A_{mn}A_{pq}c^2_{mnpq} \\
\int_{-1}^{1}\int_{-1}^{1} 2\mu \frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_s}{\partial y^2} dxdy &= \sum_{m,n}\sum_{p,q} A_{mn}A_{pq}c^3_{mnpq} \\
\int_{-1}^{1}\int_{-1}^{1}(1-\mu)\left(\frac{\partial^2 w_s}{\partial x \partial y}\right)^2 dxdy &= \sum_{m,n}\sum_{p,q} A_{mn}A_{pq}c^4_{mnpq} \\
\int_{-1}^{1}\int_{-1}^{1} w_s^2 dxdy &= \sum_{m,n} A_{mn}^2 \quad \text{(orthogonality!)}
\end{aligned}$$

and Calculating ...

Using a Lagrange multiplier λ, we need to minimize

$$J(w_s) - \lambda U(w_s) \longrightarrow \min$$

which is equivalent to minimize

$$J_s(\mathbf{a}) := \mathbf{a}^T \widetilde{K} \mathbf{a} - \lambda \mathbf{a}^T \mathbf{a} \longrightarrow \min$$

with respect to **a**, where we defined the vector

$$\mathbf{a} := [A_{00}, A_{01}, A_{10}, \ldots]$$

and the matrix

$$\widetilde{K} := \begin{bmatrix} \alpha_{00}^{00} & \alpha_{01}^{00} & \alpha_{10}^{00} & \cdots \\ \alpha_{00}^{01} & \alpha_{01}^{01} & \alpha_{10}^{01} & \cdots \\ \alpha_{00}^{10} & \alpha_{01}^{10} & \alpha_{10}^{10} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with $\alpha_{mn}^{pq} := c_{mnpq}^1 + c_{mnpq}^2 + c_{mnpq}^3 + c_{mnpq}^4$.

and Calculating ...

In order to minimize

$$J_s(\mathbf{a}) := \mathbf{a}^T \widetilde{K}\mathbf{a} - \lambda \mathbf{a}^T \mathbf{a} \longrightarrow \min$$

we compute the gradient with respect to **a** and set it to zero, to obtain with $K := \frac{1}{2}(\widetilde{K} + \widetilde{K}^T)$

$$K\mathbf{a} = \lambda \mathbf{a}$$

a discrete eigenvalue problem. For each eigenvalue λ^ℓ, we get an eigenvector $\mathbf{a}^\ell = [A^\ell_{00}, A^\ell_{01}, \ldots]$, and the corresponding eigenfunction

$$w_s^\ell = \sum_{m=0}^{s} \sum_{n=0}^{s} A^\ell_{mn} u_m(x) u_n(y)$$

Note the similarity with the underlying continuous eigenvalue problem

$$\Delta^2 w = \lambda w.$$

Problems at the Time of Ritz

1) How to compute c^j_{mnpq} ?

Ritz (1909):

- Die Wurzeln von $\tan k_m + \tanh k_m = 0$ unterscheiden sich nur wenig von $m\pi/2 - \pi/4\ldots$
- Für $m > 2$ ist auf vier Stellen genau

$$u_m = \cos(\frac{m}{2} - \frac{1}{4})\pi x + \frac{(-1)^{\frac{m}{2}} \cosh(\frac{m}{2} - \frac{1}{4})\pi x}{\sqrt{2}\cosh(\frac{m}{2} - \frac{1}{4})\pi}$$

  für gerade m, ...
- Begnügt man sich mit vier genauen Ziffern, ...
- Mit einer Genauigkeit von mindestens 2 Prozent...
- Zur Vereinfachung wird man die aus der Symmetrie der Lösung sich ergebenden Beziehungen zwischen den A_{mn} sogleich einführen: ...

The Eigenvalue Problem of Ritz

The eigenvalue problem found by Ritz

$$
(53)\quad \begin{cases}
0 = (a_{00}^{(00)} - \lambda_s) A_{00} + a_{01}^{(00)} A_{01} + a_{10}^{(00)} A_{10} + \ldots + a_{ss}^{(00)} A_{ss}, \\
0 = a_{00}^{(01)} A_{00} + (a_{01}^{(01)} - \lambda_s) A_{01} + a_{10}^{(01)} A_{10} + \ldots + a_{ss}^{(01)} A_{ss}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
0 = a_{00}^{(ss)} A_{00} + a_{01}^{(ss)} A_{01} + a_{10}^{(ss)} A_{10} + \ldots + (a_{ss}^{(ss)} - \lambda_s) A_{ss}.
\end{cases}
$$

and the actual one solved using simplified formulas and:

$u_m(y) = v_m$ gesetzt und

$$
w = A_0 u_1 v_1 + A_1(u_1 v_3 + v_1 u_3) + A_2 u_3 v_3 + A_3(u_1 v_5 + u_5 v_1) + A_4(u_3 v_5 + u_5 v_3) + A_5 u_5 v_5.
$$

Das System (53) wird hier

$$
(54)\quad \begin{cases}
0 = (13{,}95 - \lambda) A_0 - 32{,}08 A_1 + 18{,}60 A_2 + 32{,}08 A_3 - 37{,}20 A_4 + 18{,}60 A_5, \\
0 = -16{,}04 A_0 + (411{,}8 - \lambda) A_1 - 120{,}0 A_2 - 133{,}6 A_3 + 166{,}8 A_4 + 140 A_5, \\
0 = +18{,}60 A_0 - 240{,}0 A_1 + (1686 - \lambda) A_2 - 218{,}0 A_3 - 1134 A_4 + 330 A_5, \\
0 = +16{,}04 A_0 - 133{,}6 A_1 + 109{,}0 A_2 + (2945 - \lambda) A_3 - 424 A_4 + 179 A_5, \\
0 = -18{,}6 A_0 + 166{,}8 A_1 - 567 A_2 - 424 A_3 + (6303 - \lambda) A_4 - 1437 A_5, \\
0 = +18{,}6 A_0 + 280 A_1 - 330 A_2 + 358 A_3 - 2874 A_4 + (13674 - \lambda) A_5.
\end{cases}
$$

Nowadays, the symbolic calculator Maple can easily perform all these operations!

High Accuracy of the Results of Ritz

$$
\begin{aligned}
0 &= (13{,}95-\lambda)A_0 - \overset{22}{32{,}08}\,A_1 + 18{,}60\,A_2 + \overset{22}{32{,}08}\,A_3 - 37{,}20\,A_4 + 18{,}60\,A_5\\
0 &= -\overset{11}{16{,}04}\,A_0 + (\overset{05{,}9\ 6.1}{411{,}8}-\lambda)A_1 - \overset{19{,}8\ 9}{120{,}0}\,A_2 - \overset{4\ 5}{133{,}6}\,A_3 + \overset{71{,}9\ 2.1}{166{,}8}\,A_4 + \overset{-83.6}{140}\,A_5\\
0 &= +18{,}60\,A_0 - \overset{39{,}6\ 9}{240{,}0}\,A_1 + (\overset{1\ 4}{1686}-\lambda)A_2 - \overset{+217{,}8\ 8.0}{218{,}0}\,A_3 - \overset{7}{1134}\,A_4 + \overset{29\ 30}{330}\,A_5,\\
0 &= +\overset{11}{16{,}04}\,A_0 - \overset{4\ 5}{133{,}6}\,A_1 + \overset{8{,}9\ 9.0}{109{,}0}\,A_2 + (2945-\lambda)A_3 - \overset{7}{424}\,A_4 + 179\,A_5,\\
0 &= -18{,}6\,A_0 + \overset{71{,}9\ 2.1}{166{,}8}\,A_1 - \overset{8}{567}\,A_2 - \overset{7}{424}\,A_3 + (\overset{21\ 5}{6303}-\lambda)A_4 - \overset{40\ 2}{1437}\,A_5,\\
0 &= +18{,}6\,A_0 + \overset{-167}{280}\,A_1 - \overset{+329\ 9.5}{330}\,A_2 + \overset{80\ 83}{358}\,A_3 - \overset{2}{2874}\,A_4 + (13674-\lambda)A_5.
\end{aligned}
$$

The results of Ritz were extremely accurate:

- In red the digits in Ritz' results that need to be modified when computing with full accuracy, using the approximations Ritz used for the functions
- In green the exact result obtained using no approximations

Second Problem at the Time of Ritz

2) How to solve the eigenvalue problem $K\mathbf{a} = \lambda\mathbf{a}$?

- **Ritz (1909):** …setzen wir $A_0 = 1$, und in erster Annäherung $\lambda_0 = 13.95$. Dann ergeben die fünf letzten Gleichungen die übrigen A_i.
- Wir berechnen für die A_i eine erste Approximation, indem wir alle Glieder rechts vernachlässigen neben den Diagonalgliedern …
- Ein oder zwei sukzessive Korrektionen genügen meist, um die vierte Stelle bis auf wenige Einheiten festzustellen.

In today's terms: $K\mathbf{a} = \lambda\mathbf{a} \iff \mathbf{f}(\lambda, A_1, \ldots, A_n) = 0$.
Starting with $\lambda^0 = 13.95$ and $A_1 \ldots A_n = 0$, for $k = 0, 1, \ldots$

$$f_j(\lambda^k, A_1^k, \ldots, A_{j-1}^k, A_j^{k+1}, A_{j+1}^k, \ldots, A_n^k) = 0 \quad j = 1, 2, \ldots, n$$

and then solve for λ^{k+1}

$$f_0(\lambda^{k+1}, A_1^{k+1}, \ldots, \ldots, A_n^{k+1}) = 0.$$

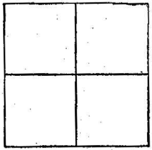
Chladni Figures Computed by Ritz

A. *Lösungen, die in x und y ungerade und symmetrisch sind.*

I. *Grundton.* $\underline{\lambda = 12{,}43 - 18{,}0\,\delta\mu.}$

$$w = \mathbf{u}_1\mathbf{v}_1 + 0{,}0394(u_1v_3 + v_1u_3) - 0{,}0040\,u_3v_3 - 0{,}0034(u_1v_5 + u_5v_1) + 0{,}0011(u_3v_5 + u_5v_3) - 0{,}0019\,u_5v_5.$$

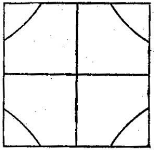
Fig. 1.



II. $\underline{\lambda = 378 - 57\,\delta\mu.}$

$$w = -0{,}075\,u_1v_1 + (\mathbf{u}_1\mathbf{v}_3 + \mathbf{u}_3\mathbf{v}_1) + 0{,}173\,u_3v_3 + 0{,}045(u_1v_5 + u_5v_1) - 0{,}015(u_3v_5 + u_5v_3) - 0{,}029\,u_5v_5.$$

Fig. 2.



Es ist

y beob. :	0,530	0,578	0,630	0,690	0,752	0,819	0,893
x beob. :	$0{,}937_5$	0,8750	$0{,}812_5$	0,7500	$0{,}687_5$	0,6250	$0{,}562_5$
x ber. — x beob. :	—0,003	—0,002	0,000	—0,001	$-0{,}000_5$	0,000	0,000

Chladni Figures Computed by Ritz

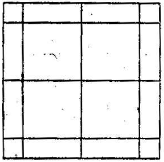
III. $\underline{\lambda = 1554.}$

$$w = 0{,}009\,u_1 v_1 - 0{,}075(u_1 v_3 + v_1 u_3) + u_3 v_3 - 0{,}057(u_1 v_5 + u_5 v_1) + 0{,}121(u_3 v_5 + u_5 v_3) - 0{,}007\,u_5 v_5.$$

Messungen fehlen.

R.

Fig. 3.



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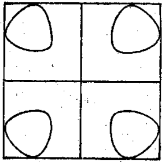
IV. $\underline{\lambda = 2945.}$

$$w = u_1 v_5 + u_5 v_1.$$

Diese Figur fehlt bei Chladni.

Fig. 4.



Chladni Figures Computed by Ritz

V. $\lambda = 6303.$

$$w = u_3 v_5 + u_5 v_3.$$

VI. $\lambda = 13674.$

$$w = u_5 v_5.$$

Fig. 5.

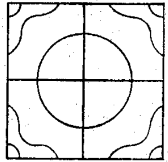
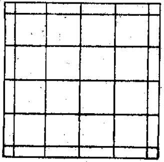


Fig. 6.



Frequency Table Computed by Ritz

Tabelle der Tonhöhen ($\mu = 0{,}225$).

HAUPTGLIEDER	λ	BER.	BEOB.	HAUPTGLIEDER	λ	BER.	BEOB.
$u_1 v_1$	12,43	G*	G	$u_3 v_4 \pm u_4 v_3$	3240	g_3 +	fis_3
$u_0 v_2 - v_0 u_2$	26,40	d*	d	$u_5 v_2 \pm u_2 v_5$	3927	a_3 +	gis_3 +
$u_0 v_2 + v_0 u_2$	35,73	e*	e	$u_4 v_4$	5480	ais_3 +	ais_3
$u_1 v_2 \pm u_2 v_1$	80,8	h*	h	$u_0 v_6 - u_6 v_0$	5500	c_4 —	— (2)
$u_0 v_3 \pm u_3 v_0$	237,1	gis_1* +	gis_1 +	$u_3 v_5 - u_5 v_3$	5570	c_4 —	ais_3 —
$u_1 v_1$	266,0	ais_1* —	ais_1* —	$u_0 v_6 + u_6 v_0$	5640	c_4 —	— (2)
$u_1 v_3 - u_3 v_1$	316,1	h_1*	h_1	$u_1 v_6 \pm v_1 u_6$	6036	c_4 +	c_4 —
$u_1 v_3 + u_3 v_1$	378	cis_2*	cis_2	$u_3 v_5 + u_5 v_3$	6303	cis_4	c_4 —
$u_2 v_3 \pm u_3 v_2$	746	fis_2* +	fis_2	$u_2 v_6 - u_6 v_2$	7310	d_4 +	cis_4 +
$u_0 v_4 - v_0 u_4$	886	gis_2	gis_2	$u_2 v_6 + u_6 v_2$	7840	dis_4 —	d_4 —
$u_0 v_4 + v_0 u_4$	941	gis_2 +	gis_2 +	$u_5 v_4 \pm u_4 v_5$	9030	e_4	dis_4
$u_1 v_4 \pm u_4 v_1$	1131	ais_2	ais_2 —	$u_6 v_3 \pm u_3 v_6$	10380	f_4	e_4
$u_3 v_3$	1554	c_3 +	c_3	$u_5 v_5$	13670	g_4 +	fis_4 +
$u_2 v_4 - u_4 v_2$	1702	d_3 +	cis_3	$u_6 v_4 - u_4 v_6$	13840	g_4 +	g_4 +
$u_2 v_4 + u_4 v_2$	2020	dis_3	d_3	$u_6 v_4 + u_4 v_6$	15120	gis_4 +	g_4 +
$u_0 v_5 \pm v_0 u_5$	2500	f_3 —	f_3 —	$u_6 v_5 \pm u_5 v_6$	20400	h_4	ais_4 —
$u_1 v_5 - v_1 u_5$	2713	fis_3	fis_3 —	$u_6 v_6$	28740	d_5	— (2)
$u_1 v_5 + v_1 u_5$	2945	fis_3 +	fis_3 (1)				

S.P. Timoshenko (1878–1972)

Timoshenko was the first to realize the importance of Ritz' invention for applications (1913):

"Nous ne nous arrêterons plus sur le côté mathématique de cette question: un ouvrage remarquable du savant suisse, M. Walter Ritz, a été consacré à ce sujet. En ramenant l'intégration des équations à la recherche des intégrales, M. W. Ritz a montré que pour une classe très vaste de problèmes, en augmentant le nombre de paramètres a_1, a_2, a_3,. . . , on arrive à la solution exacte du problème. Pour le cycle de problèmes dont nous nous occuperons dans la suite, il n'existe pas de pareille démonstration, mais l'application de la méthode approximative aux problèmes pour lesquels on possède déjà des solutions exactes, montre que la méthode donne de très bons résultats et pratiquement on n'a pas besoin de chercher plus de deux approximations"

швейцарского ученого Вальтера Ритца

schweizarskogo utshenogo Waltera Ritza

Ivan Bubnov (1872-1919)

СТРОИТЕЛЬНАЯ МЕХАНИКА КОРАБЛЯ[1]

[РАЗДЕЛЫ, ОТНОСЯЩИЕСЯ К ТЕОРИИ ПЛАСТИН]

Structural Mechanics of Shipbuilding

[Part concerning the theory of shells]

- Bubnov was a Russian submarine engineer and constructor
- Worked at the Polytechnical Institute of St. Petersburg (with Galerkin, Krylov, Timoshenko)
- Work motivated by Timoshenko's application of Ritz' method to study the stability of plates and beams

Main Contribution of Bubnov

Bubnov (1913): Report on the works of Prof. Timoshenko which were awarded the Zhuranskii prize

"... extremely simple solutions can also be obtained in the usual way, i.e., without resorting to a consideration of the energy of the system... we simply substitute the expansion for w in the general differential expression for equilibrium, multiply the expression obtained by $\varphi_k dxdy$ and integrate over the entire volume of the body, then we obtain an equation relating the coefficient a_k with all others provided that the functions φ_k are chosen so that

$$\int\int \varphi_n \varphi_k dxdy = 0 \quad \text{for } n \neq k\text{"}$$

Substituting $w_s = a_1\varphi_1 + a_2\varphi_2 + \ldots$ into $\Delta^2 w = \lambda w$, multiplying by φ_k and integrating, we get

$$\int\int \Delta^2 w_s \varphi_k dxdy = \lambda \int\int w_s \varphi_k dxdy$$

Integration by Parts

Inserting the expansion $w_s = a_1\varphi_1 + a_2\varphi_2 + \dots$ into

$$\int\int \Delta^2 w_s \varphi_k dxdy = \lambda \int\int w_s \varphi_k dxdy$$

and integrating by parts, we obtain for $k = 1, 2, \dots$

$$a_1 \int\int \Delta\varphi_1 \Delta\varphi_k + a_2 \int\int \Delta\varphi_2 \Delta\varphi_k + \dots = \lambda a_k \int\int \varphi_k^2$$

the discrete eigenvalue problem

$$\tilde{K}\mathbf{a} = \lambda\mathbf{a}, \qquad \tilde{K}_{jk} = \frac{\int\int \Delta\varphi_j \Delta\varphi_k}{\int\int \varphi_k^2},$$

which is equivalent to the problem found by Ritz for the coefficients a_k. (Bubnov: "... and will be identical to those found by Prof. Timoshenko")

Remark: Bubnov used trigonometric functions for φ_k

Boris Grigoryevich Galerkin (1871-1945)

СТЕРЖНИ И ПЛАСТИНКИ

РЯДЫ В НЕКОТОРЫХ ВОПРОСАХ УПРУГОГО РАВНОВЕСИЯ
СТЕРЖНЕЙ И ПЛАСТИНОК*

(Петроград, 1915)

Beams and Plates: Series solution of some problems in elastic equilibrium of rods and plates (Petrograd, 1915)

- Studies in the Mechanics Department of St. Petersburg Technological Institute
- Worked for Russian Steam-Locomotive Union and China Far East Railway
- Arrested in 1905 for political activities, imprisoned for 1.5 years.
- Devote life to science in prison.
- Visited Switzerland (among other European countries) for scientific reasons in 1909.

Connections Between Galerkin and Ritz, Bubnov

Из приближенных методов решений широкое применение получил в последнее время метод Ритца [6, 7]. Этот метод сводится вкратце к следующему.

метод Ритца [6, 7]

ЛИТЕРАТУРА

...

3. Бубнов И. Г. Строительная механика корабля. Ч. 2, СПб., 1914.

...

6. Ritz W. Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik.— Journal f. reine u. angewandte Mathematik, 1908, Bd. 135, S. 1—61.
7. Ritz W. Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern.- Annalen der Physik, 1909, Bd. 28, N. 4, S. 737—786.
8. Тимошенко С. П. Об устойчивости упругих систем. Киев, 1910, 186 с.

Main Contribution of Galerkin

- The functions φ_k do not need to be orthogonal:

$$a_1 \int\int \Delta\varphi_1 \Delta\varphi_k + a_2 \int\int \Delta\varphi_2 \Delta\varphi_k + \dots$$
$$= \lambda(a_1 \int\int \varphi_1\varphi_k + a_2 \int\int \varphi_2\varphi_k + \dots)$$

  which then leads to the generalized eigenvalue problem

$$K\mathbf{a} = \lambda M\mathbf{a}.$$

- The method can also be applied to problems where there is no energy minimization principle

The method is now mostly called '**the Galerkin Method**' (Google on 8.9.09: 'Galerkin Method' 191000 hits, 'Ritz Method' 51200 hits)

How is this possible ?

Hurwitz and Courant: the Birth of FEM

While Russian scientists immediately used Ritz' method to solve many difficult problems, pure mathematicians had little interest:

Hurwitz and Courant (1922): Funktionentheorie

(footnote, which disappeared in the second edition (1925))

[1]) Die wirkliche, für den bloßen Existenzbeweis unerhebliche Konstruktion solcher Minimalfolgen macht keinerlei prinzipielle Schwierigkeiten. Ist z. B. G ein ganz im Endlichen gelegener Bereich, begrenzt von Kurven C ohne mehrfache Punkte, so denken wir uns denselben mit einem noch von dem Index j abhängigen Dreiecksnetz T_j überdeckt, dessen Maschen mit wachsendem j immer enger werden. Wir betrachten nun nur solche Funktionen φ bzw. $\Phi = \varphi - S$, wo die Differenz $\varphi - \frac{x}{x^2+y^2}$ in jedem Dreieck von T_j eine lineare Funktion wird. Unter Φ_j verstehen wir diejenige unter den so entstehenden zu T_j konstruierten Funktionen, für welche $D[\Phi]$ einen möglichst kleinen Wert erhält. Diese Forderung $D[\Phi] = \text{Min.}$ ist jetzt ein Problem eines Minimums einer Funktion von einer endlichen Anzahl von Variablen, nämlich des Integrals, aufgefaßt in seiner Abhängigkeit von den Werten von φ in den Eckpunkten der Dreieckseinteilung; dieses Problem ist gewiß lösbar, und zwar, wie leicht ersichtlich, mittels linearer Gleichungen. Daß die so entstehenden Funktionen Φ_j wirklich eine Minimalfolge bilden, folgt sofort aus der unschwer beweisbaren Tatsache, daß man jede zulässige Funktion Φ und deren Dirichletsches Integral mit Hilfe unserer Konstruktion bei hinreichend großem j beliebig genau approximieren kann.

Richard Courant (1888-1972)

Variational Methods for the Solution of Problems of Equilibrium and Vibrations (**Richard Courant**, address delivered before the meeting of the AMS, May 3rd, 1941)

As Henri Poincaré once remarked, "solution of a mathematical problem" is a phrase of indefinite meaning. Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal. Such one sided views seem to reflect human limitations rather than objective values.

Praise of Ritz' Work by Courant

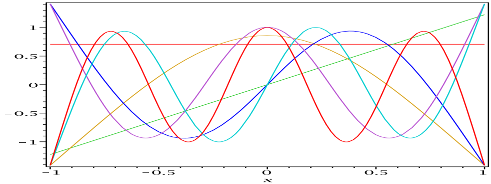
Courant (1941): "At first, the theoretical interest in existence proofs dominated and only much later were practical applications envisaged by two physicists, Lord Rayleigh and Walther Ritz. They independently conceived the idea of utilizing this equivalence for numerical calculation of the solutions, by substituting for the variational problems simpler approximating extremum problems in which but a finite number of parameters need be determined"

"But only the spectacular success of Walther Ritz and its tragic circumstances caught the general interest. In two publications of 1908 and 1909, Ritz, conscious of his imminent death from consumption, gave a masterly account of the theory, and at the same time applied his method to the calculation of the nodal lines of vibrating plates, a problem of classical physics that previously had not been satisfactorily treated."

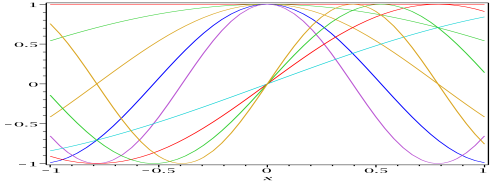
Courant's Main Contribution

"However, the difficulty that challenges the inventive skill of the applied mathematician is to find suitable coordinate functions"

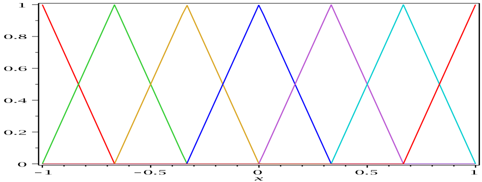
Ritz' choice:
Eigenfunctions of the 1d beam, or polynomials



Bubnov/Galerkin:
Use of trigonometric functions or polynomials

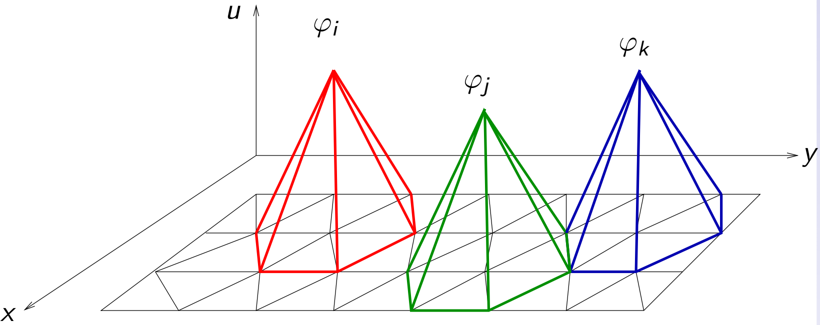


Courant's choice:
Use hat functions, or polynomials on elements

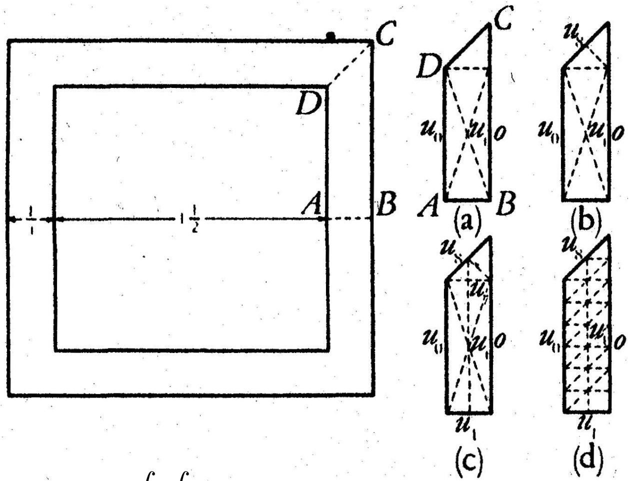


Where are the Finite Elements?

Courant (1941): "Instead of starting with a quadratic or rectangular net we may consider from the outset any polyhedral surfaces with edges over an *arbitrarily chosen* (preferably triangular) *net*.



First Finite Element Solution by Courant



$$\int\int (\nabla u)^2 + 2u \quad \longrightarrow \quad \min$$

with $u = 0$ on outer boundary, and $u = c$, unknown constant on the inner boundary.

Comparison for Different 'Coordinate' Functions

Polynomials: ("with negligible amount of numerical labor")

$$\varphi_1 := a(1-x) \quad S = 0.339,\ c = -0.11$$

$$\varphi_2 := a(1-x)[1+\alpha(x-\frac{3}{4})y] \quad S = 0.340,\ c = -0.109$$

Finite Elements: ("The results, easily obtainable, were")

$$\text{Case (a):} \quad S = 0.344,\ c = -0.11$$

$$\text{Case (b):} \quad S = 0.352,\ c = -0.11$$

$$\text{Case (c):} \quad S = 0.353,\ c = -0.11$$

$$\text{Case (d):} \quad S = 0.353,\ c = -0.11$$

Courant: "These results show in themselves and by comparison that the generalized method of triangular nets seems to have advantages."

The Name Finite Element Method

The term Finite Element Method was then coined by Ray Clough in:

Ray W. Clough: The finite element method in plane stress analysis, Proc ASCE Conf Electron Computat, Pittsburg, PA, 1960

Based on joint work with Jon Turner from Boeing on structural dynamics, and this work led to the first published description of the finite element method, without the name yet, in

N. J. Turner and R. W. Clough and H. C. Martin and L. J. Topp: Stiffness and Deflection analysis of complex structures, J. Aero. Sci., Vol. 23, pp. 805–23, 1956.

Summary

- **Euler (1744)** "invents" variational calculus **by piecewise linear discretization**.
- **Lagrange (1755)** puts it on a solid foundation.
- **Riemann (1851)** theory of analytic functions.
- **Ritz (1908)** proposes and analyzes approximate solutions based on linear combinations of simple functions, and solves two difficult problems of his time.
- **Timoshenko (1913), Bubnov (1913) and Galerkin (1915)** realize the tremendous potential of Ritz's method and solve many difficult problems.
- **Courant (1941)** proposes to use piecewise linear functions on triangular meshes.
- **Clough et al. (1960)** name the method the **Finite Element Method**.

The mathematical development of the finite element method was however just to begin...