Stability of Mixed Jackson Networks

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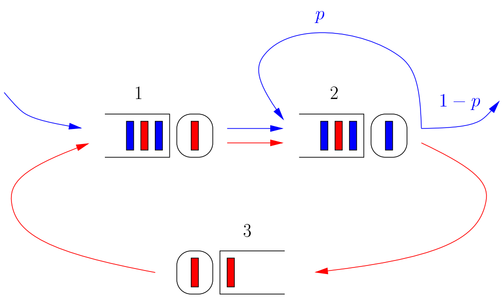
OUTLINE

- Model and its Markovian Structure
- Fluid Model
- Stability Result

Example



Model

- Network N FIFO stations
- Exogenous customers

  Interarrival times $\{\tau_i(n)\}_{n\geq 1}$ at queue i

  Service times $\{\sigma_i(n)\}_{n\geq 1}$ at queue i

  Routing matrix R **transient**, i.e.

$$I + R + R^2 + \ldots < \infty$$

- Permanent customers

  Service times $\{\delta_i(n)\}_{n\geq 1}$ at queue i

  A fixed number of M customers

  Routing matrix S **stochastic**, i.e.

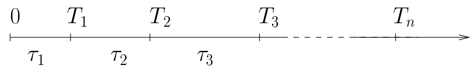
$$Se = e, \quad \text{with} \quad e = (1, \ldots, 1)',$$

  with K **irreducible** communicating classes.

Stochastic Assumptions

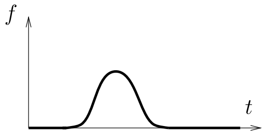
- The random variables $\{\tau_i(n)\}$, $\{\sigma_i(n)\}$ and $\{\delta_i(n)\}$ are **i.i.d.**, **mutually independent** and **integrable**.
- The interarrival times $\{\tau_i(n)\}$ of the exogenous customers are **unbounded** and **spread out**.
- For each communicating class k, the service times $\{\delta_{i_k}(n)\}$ of the permanent customers at **one** of the stations in this class are **unbounded** and **strongly spread out**.

Strong Spread Out Condition

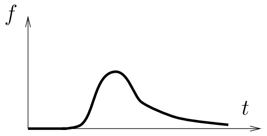


- The random variables $\{\tau(n)\}_{n\geq 1}$ are **spread out** if for some integer m,

$$\mathbb{P}(T(m) \in A) \geq \int_A f(t)dt, \quad \text{with} \quad \int_0^\infty f(t)dt > 0.$$



- The random variables $\{\tau(n)\}_{n\geq 1}$ are **strongly spread out** if in addition, f is **continuous** and **strictly positive** outside a compact set.



Traffic Intensities

- The **external arrival rate** and the mean service time of the exogenous customers at queue i are respectively denoted by

$$\nu_i = \frac{1}{\mathbb{E}(\tau_i(1))} \quad \text{and} \quad \sigma_i = \mathbb{E}(\sigma_i(1)).$$

- The **effective arrival rates** λ_i of the exogenous customers satisfy the traffic equation

$$\lambda = R'\lambda + \nu.$$

As the matrix R is transient, we get

$$\lambda = (I - R')^{-1}\nu.$$

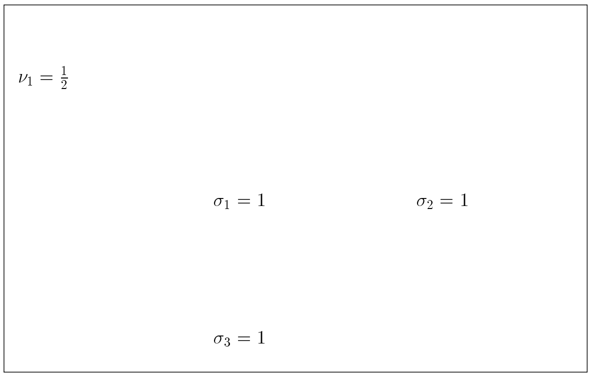
- The **traffic intensity** of the exogenous streams at queue i is then given by

$$\rho_i = \sigma_i \lambda_i,$$

that is with $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_N)$,

$$\boxed{\rho = \Sigma(I - R')^{-1}\nu}$$

Example



$$\rho = \Sigma(I - R')^{-1}\nu = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{1-p} & \frac{1}{1-p} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2(1-p)} \\ 0 \end{pmatrix}$$

State Description

At any time t, the state of the system is described by

$$X(t) = (\alpha(t), \beta(t), \gamma(t)),$$

where

- $\alpha(t)$ gives the remaining **interarrival time** at each queue,
- $\beta(t)$ gives the remaining **service time** at each queue,
- $\gamma(t)$ gives the **position of the customers** in the network, namely

$$\gamma(t) = (Q(t), Q^P(t), P(t))$$

Number of exogenous customers in each queue

Number of permanent customers in each queue

Position of the permanent customers

Example

$$Q(t) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \quad Q^P(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad P(t) = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Markov Process

- The process $X = \{X(t),\ t \in \mathbb{R}_+\}$ has the **strong** Markov property. The state space is

$$E \subset \mathbb{R}_+^N \times \mathbb{R}_+^N \times (\mathbb{N}^N \times \mathbb{N}^N \times \mathbb{N}^M).$$

- Lemma 1 The process X is φ-irreducible. In addition, the compact sets of E are **petite**.

Proof. X is a T-process and the state $x = 0$ where there are no exogenous customers in the system and all permanent customers are in queues $i_1, \dots, i_K$ is reachable. ♢

- In the following, we study the stability of the system in terms of **positive Harris recurrence** vs **transience** of the Markov process X.

A Criterion for Stability

- For any element $x = (\alpha, \beta, \gamma)$ of the state space, define

$$\|x\| = |\alpha| + |\beta| + |q|.$$

The sets $\{x \in E,\ \|x\| \le u\}_{u>0}$ are **compact**.

- Theorem (Dai 1995) A sufficient condition for stability is the existence of some $t > 0$, such that

$$\lim_{\|x\|\to\infty} \mathbb{E}_x \left\| \frac{X(\|x\|t)}{\|x\|} \right\| = 0.$$

- Remark 1 Since the initial residual times α and β only introduce delay in the evolution of the system, it is sufficient to consider limits such that

$$\|x\| \sim |q| \quad \text{when} \quad \|x\| \to \infty.$$

- Remark 2 Since the interarrival times and the service times are integrable, we have for all $t > 0$,

$$\lim_{\|x\|\to\infty} \mathbb{E}_x \left| \frac{\alpha(\|x\|t)}{\|x\|} \right| = 0 \text{ and } \lim_{\|x\|\to\infty} \mathbb{E}_x \left| \frac{\beta(\|x\|t)}{\|x\|} \right| = 0.$$

Fluid Limit

- The evolution of the number of exogenous customers in the system is given by

$$Q_i(t) = Q_i(0) + \underbrace{A_i(t)}_{\substack{\text{Arrivals}\\\text{from outside}}} + \underbrace{\textstyle\sum_{j=1}^N \Phi_{ji}(D_j(T_j(t))}_{\substack{\text{Arrivals}\\\text{from the network}}} - \underbrace{D_i(T_i(t))}_{\text{Departures}}$$

where $T_i(t)$ is the total time spent by server i on **exogenous** customers up to time t.

- Theorem (Dai 1995) For any sequence $\{x_n\}$ with $\|x_n\| \to \infty$ and almost all sample paths, there is a subsequence $\{x_{n_k}\}$ such that uniformly on compact sets,

$$\frac{Q(\|x_{n_k}\|t)}{\|x_{n_k}\|} \to \bar{Q}(t) \quad \text{and} \quad \frac{T(\|x_{n_k}\|t)}{\|x_{n_k}\|} \to \bar{T}(t).$$

The **fluid limit** $(\bar{Q}, \bar{T})$ is continuous and satisfies

$$\bar{Q}(t) = \bar{Q}(0) + \nu t + (R' - I)\Sigma^{-1}\bar{T}(t).$$

Fluid Model

The collection of all fluid limits is called the **fluid model**.

- The fluid model is **stable** if there exists $t > 0$, such that for every fluid limit $(\bar{Q}, \bar{T})$,

$$\bar{Q}(t + \cdot) \equiv 0.$$

- Theorem (Dai 1995) If the fluid model is stable, the Markov process X is **positive Harris recurrent**. In particular, there exists a unique stationary distribution.
- The fluid model is **unstable** if there exists $a > 0$ and $b \geq 0$, such that for every fluid limit $(\bar{Q}, \bar{T})$,

$$|\bar{Q}(t)| \geq at - b.$$

- Theorem (Meyn 1995) If the fluid limit model is unstable, then the Markov process X is **transient**.

Stability Result

- Lemma 2 The fluid model of the mixed Jackson network, is completely **characterized** by the equations

$$\bar{Q}(t) = \bar{Q}(0) + \nu t - (I - R')\Sigma^{-1}\bar{T}(t),$$

$$\bar{Q}(t).\left(\frac{d\bar{T}(t)}{dt} - e\right) = 0.$$

Proof. If there is some fluid in queue i at time t, there is actually an **infinite** number of 'exogenous particles'. Since the number of 'permanent customers' is **bounded** by M and the service discipline is FIFO, these particles are negligible and the server spends all its time to serve the exogenous customers. $\diamondsuit$

- Theorem The stability condition of the **mixed** Jackson network is the same as that of the corresponding **open** network, i.e. **without** the permanent customers.

  If $\rho_i < 1$ for all i, the fluid model is stable.

  $\hookrightarrow X$ is **positive Harris recurrent**.

  If $\rho_i > 1$ for some i, the fluid model is unstable.

  $\hookrightarrow X$ is **transient**.

Proof of Stability

- Define the **workload** of the network as

$$\bar{W}(t) = \Sigma(I - R')^{-1}\bar{Q}(t).$$

The evolution equation becomes

$$\boxed{\bar{W}(t) = \bar{W}(0) + \rho t - \bar{T}(t)}$$

so that

$$\frac{d\bar{W}(t)}{dt} = \rho - \frac{d\bar{T}(t)}{dt}.$$

- Conservation property If $\rho < e$, we have for all t,

$$\frac{d\bar{T}(t)}{dt} \geq \rho.$$

- If there is some fluid in queue i at time t, it follows from the **non-idling property** that

$$\frac{d\bar{T}_i(t)}{dt} = 1$$

so that from the **conservation property**,

$$\frac{d|\bar{W}(t)|}{dt} \leq \rho_i - 1.$$

Hence from time $t = |\bar{W}(0)|/(1 - \max_i \rho_i)$, there is no more fluid in the system.

Proof of Instability

- Let i be such that $\rho_i > 1$. From the evolution equation,
$$\bar{W}_i(t) \geq (\rho_i - 1)t.$$
It follows then from
$$\boxed{\bar{Q}(t) = (I - R')\Sigma^{-1}\bar{W}(t)}$$
that
$$|\bar{Q}(t)| \geq \frac{1 - |R'_i|}{\sigma_i}(\rho_i - 1)t.$$
- If $|R'_i| < 1$, i.e. if the exogenous customers leave the network with **positive probability** after the completion of their service at station i, the proof is concluded.
- Otherwise, for all stations j such that $R_{ij} > 0$, we have
$$\bar{W}_j(t) \geq R_{ij}\frac{\sigma_j}{\sigma_i}\bar{W}_i(t).$$
The result follows from the fact that the exogenous customers **eventually** leave the network.

Conclusion

- We have shown that the stability condition of the **mixed** Jackson network is the same as that of the corresponding **open** network.
- In this respect, the key results are

  the 'good' **structural properties** of the Markov process , i.e. φ-irreducibility and compact sets being petite;

  the extension of the usual **non-idling property** to the mixed network.
- The fluid approach is very **intuitive** and **efficient** in this case.