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Abstract: We focus on window flow control as used in packet-switched communication networks. The approach consists in studying the stability of a system where each node on the path followed by the packets of the controlled connection is modeled by a FIFO (First-In-First-Out) queue of infinite capacity which receives in addition some cross traffic represented by an exogenous flow. Under general stochastic assumptions, namely for stationary and ergodic input processes, we show the existence of a maximum throughput allowed by the flow control. Then we establish bounds on the value of this maximum throughput. These bounds which do not coincide in general, are reached by time-space scalings of the exogenous flows. Therefore, the performance of window flow control depends not only on the traffic intensity of the cross flows, but also on fine statistical characteristics such as the burstiness of these flows. These results are illustrated by several examples, including the case of a non–monotone, non–convex and fractal stability region.

Key-words: Window flow control, TCP, stability, multiclass networks, stationary ergodic point processes, (max,+)–linear system.

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Contrôle de Flux à Fenêtre dans les Réseaux FIFO en présence de Trafic Transverse

Résumé : Nous nous intéressons au contrôle de flux à fenêtre dans les réseaux à commutation de paquets. L'approche consiste à étudier la stabilité d'un système où chaque noeud du chemin suivi par les paquets de la connexion contrôlée est modélisé par une file d'attente FIFO (First-In-First-Out) de capacité infinie qui reçoit de plus un trafic transverse représenté par un flux exogène. Nous montrons sous des hypothèses stochastiques très générales, à savoir pour des processus d'entrée stationnaires et ergodiques, l'existence d'un débit maximum permis par le contrôle de flux. Puis nous établissons des bornes sur la valeur de ce débit maximum. Ces bornes ne coïncident pas en général, et sont atteintes par des changements d'échelle en temps et en espace des flux exogènes. Ainsi, les performances du contrôle de flux à fenêtre ne dépendent pas uniquement de l'intensité de trafic des flux transverses, mais aussi de caractéristiques statistiques plus fines telles que la variabilité de ces flux. Ces résultats sont illustrés par plusieurs exemples, parmi lesquels celui d'une région de stabilité non monotone, non convexe et fractale.

Mots-clés : Contrôle de flux à fenêtre, TCP, stabilité, réseaux multi-classes, processus ponctuels stationnaires ergodiques, système (max,+)–linéaire.

1 Introduction

Flow control mechanisms are used in packet-switched communication networks to prevent routers from congestion, by regulating the input traffic generated by the users. The most widely used mechanism is the window flow control, like that of TCP (Transmission Control Protocol) over the Internet [7]. This mechanism consists in limiting the number of packets in transit in the network to a given value called the *window*. The focus of this paper is not on the window dynamics of TCP, which was studied in [10, 14, 15] in cases where the connection consists of a single bottleneck link, but rather on the network dynamics. In particular, we assume that the window either stabilizes as in TCP Vegas [6] or varies very slowly compared to the time–scale of a packet, so that it can be considered as *static*, and we evaluate in this case the effect of *cross traffic* on the performance of the flow control, in terms of utilization of the network resources.

The main difficulty which arises in the analysis of current communication networks is that the traffic may exhibit periodicity [8] and long-range dependence [16] which are not captured by traditional Markovian models [11, 13]. In addition, it turns out that in the particular case of window flow control, the throughput of the controlled connection depends in a crucial way on fine statistical characteristics of the cross flows, and not only on their traffic intensity. In [1], bounds on the performance of window flow control are obtained in cases where the interaction with cross traffic may be modeled by so–called *service curves*. Here we model the cross traffic at each node of the network by an exogenous flow with general statistical assumptions, namely by a stationary and ergodic marked point process. The reference model on which the analysis is based consists of a series of FIFO queues in tandem. More general network topologies including multicast connections and/or propagation delays are also considered along the same lines.

The paper is organized as follows. The model, its dynamics and its basic monotonicity properties are described in Section 2. In Section 3, we present the stochastic framework of the analysis and investigate the stability region of the model, that is the conditions under which the system admits a finite stationary regime. In particular, we show that there exists a maximum arrival rate of the packets below which the system is stable, and we refer to this value as the *maximum throughput* of the connection. In Section 4, we establish bounds on the maximum throughput, and show that these bounds which do not coincide in general, are reached by time-space scalings of the exogenous flows. The results obtained are illustrated by several examples in Section 5, including the case of a non–monotone, non–convex and fractal stability region. Section 6 concludes the paper.

2 Model

2.1 Window flow control

The connection involves two users, referred to as the source and the destination. The source sends data packets to the destination, which sends back an acknowledgment to the source each time it receives a packet. The window flow control consists in limiting the number of packets the source can consecutively send without waiting for any acknowledgment, to a given value K called the *window*. Incoming packets that exceed the capacity of the window are buffered at the source in the *input queue* until a new acknowledgment is received.

In the reference model, the path followed by the packets and the acknowledgments of the controlled connection consists of N stations in tandem. The first stations represent the *forward* path followed by the packets sent from the source to the destination, and the last stations represent the *reverse* path followed by the acknowledgments sent back from the destination to the source. No difference will be made between the forward and the reverse path in the following, and we will simply refer to a packet of the controlled connection and its acknowledgment as a *controlled customer*. Note that the number of controlled customers in the network (excluding the input queue) is always limited by the window K. Each station receives in addition a flow of *cross customers* which leave the network after the completion of their service. At each station, all customers are served at unit rate by a single server under the global FIFO discipline. All queues are of infinite capacity.

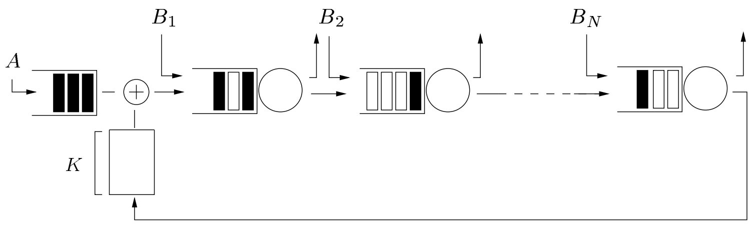

Figure 1: Reference model

The reference model with a window of $K = 4$ packets is shown in Figure 1 (in all figures, controlled customers are in black and cross customers in white). More general models including multicast connections and/or propagation delays are also considered along the same lines (see Appendix B).

2.2 Dynamics

Let $A = \{T_n\}_n$ be the arrival point process of the controlled customers, and $B_i = \{T_n^i\}_n$ the arrival point process of the cross customers at station i, with the usual conventions $T_0 \leq 0 < T_1$ and $T_0^i \leq 0 < T_1^i$.

We denote by α_n^i and U_n^i the service time and the departure time of the n–th controlled customer at (from) station i, and by β_n^i and V_n^i the service time and the departure time of the n–th cross customer at (from) station i. Denoting by U_n^0 the departure time of the n–th controlled customer from the input queue (or equivalently its arrival time in the network), it follows from the window flow control mechanism that

$$\forall n \in \mathbb{Z}, \quad U_n^0 = \max(T_n, U_{n-K}^N). \tag{1}$$

In addition, by the workload equation of a FIFO queue, we get for all $i = 1, \ldots, N$,

$$\forall n \in \mathbb{Z}, \quad U_n^i = \sup_{u \leq U_n^{i-1}} \left\{ u + \sum_{m:\ u \leq U_m^{i-1} \leq U_n^{i-1}} \alpha_m^i + \sum_{m:\ u \leq T_m^i < U_n^{i-1}} \beta_m^i \right\}, \tag{2}$$

and

$$\forall n \in \mathbb{Z}, \quad V_n^i = \sup_{u \leq T_n^i} \left\{ u + \sum_{m:\ u \leq U_m^{i-1} \leq T_n^i} \alpha_m^i + \sum_{m:\ u \leq T_m^i \leq T_n^i} \beta_m^i \right\}, \tag{3}$$

where we use the convention that, when a cross customer and a controlled customer arrive at a given station at the same time, the controlled customer is served first. We denote by $D_0 = \{U_n^0\}_n$ and $D_i = \{U_n^i\}_n$ the departure point processes of the controlled customers from the input queue and from station i, respectively.

2.3 Monotonicity

For all $k \leq l \in \mathbb{Z}$, let $A^{[k,l]}$ be the $[k,l]$–restriction of A, that is the arrival point process $\{T'_n\}_n$ defined by $T'_n = -\infty$ for all $n < k$, $T'_n = +\infty$ for all $n > l$, and $T'_n = T_n$ otherwise. We denote by $D_0^{[k,l]}, \ldots, D_N^{[k,l]}$ the corresponding departure point processes. When $k < l$, we also define

$$A^{[k,l)} = A^{[k,l-1]} \quad \text{and} \quad A^{(k,l]} = A^{[k-1,l]}.$$

The same notations are used for the corresponding departure point processes. The following monotonicity property is crucial for the rest of the analysis.

Lemma 1 *Let $\tilde{A}$ be any other arrival point process of controlled customers, and let $\tilde{D}_0, \ldots, \tilde{D}_N$ be the corresponding departure point processes. We have for all $k \leq l \in \mathbb{Z}$,*

$$A^{[k,l]} \prec \tilde{A}^{[k,l]} \implies \forall i = 0, \ldots, N, \quad D_i^{[k,l]} \prec \tilde{D}_i^{[k,l]},$$

where we use the notation $\{T_n\}_n \prec \{\tilde{T}_n\}_n$ to mean $T_n \leq \tilde{T}_n$ for all $n \in \mathbb{Z}$.

Proof. In view of (1), we have

$$D_0^{[k,k+K)} = A^{[k,k+K)} \quad \text{and} \quad \tilde{D}_0^{[k,k+K)} = \tilde{A}^{[k,k+K)},$$

so that

$$D_0^{[k,k+K)} \prec \tilde{D}_0^{[k,k+K)}.$$

It follows then from (2) that

$$\forall i = 1, \ldots, N, \quad D_i^{[k,k+K)} \prec \tilde{D}_i^{[k,k+K)}.$$

From (1), we get

$$D_0^{[k,k+2K)} \prec \tilde{D}_0^{[k,k+2K)},$$

and the proof follows by induction. □

Remark 1 *This monotonicity property does not hold with respect to the cross traffic flows. Consider the case $N = 2$, $K = 1$, and $\alpha_n^i = \beta_n^i = 1$ for all $n \in \mathbb{Z}$, $i = 1, 2$. Let $A = \{1, 2\}$ and $B_2 = \emptyset$. When $B_1 = \{0, 3\}$, the cross customers leave the network at times $\{1, 5\}$, and when $B_1 = \{\varepsilon, 3\}$, with $0 < \varepsilon < 1$, the cross customers leave the network at times $\{1 + \varepsilon, 4\}$.*

In the rest of the paper, we denote by $X_0(t)$ and $X_i(t)$ the number of controlled customers in the input queue and in station i at time t, respectively. We also denote by $Y_i(t)$ the number of cross customers in station i at time t. By convention, these processes are taken right–continuous with left–hand limits. Let $Z_i(t)$ be the cumulative number of controlled customers in all stations up to i at time t, that is

$$\forall i = 0, \ldots, N, \quad Z_i(t) = \sum_{j=0}^{i} X_j(t).$$

At any time t, we have

$$Z_0(t) \leq Z_1(t) \leq \ldots \leq Z_N(t) \leq Z_0(t) + K. \tag{4}$$

Define $Z(t) = (Z_0(t), \ldots, Z_N(t))$, and for all $k \leq l \in \mathbb{Z}$, denote by $Z^{[k,l]}(t)$ the value of $Z(t)$ when the arrival point process of the controlled customers is the restricted point process $A^{[k,l]}$. Lemma 1 admits the following key corollaries.

Corollary 1 *For all $k \leq l \in \mathbb{Z}$, we have component–wise,*

$$\forall t \in \mathbb{R}, \quad Z^{[k,l]}(t) \leq Z^{[k-1,l]}(t).$$

Proof. The proof follows from the fact that $A^{[k,l]} \prec A^{[k-1,l]}$ and from Lemma 1. $\square$

From Corollary 1, we can then define a function $\{\mathcal{Z}(t)\}$ by

$$\forall t \in \mathbb{R}, \quad \mathcal{Z}(t) = \lim_{k \to -\infty} Z^{[k,+\infty)}(t). \tag{5}$$

For the following corollaries, we assume that there exists $\lambda > 0$ such that

$$\lim_{n \to \pm\infty} \frac{n}{T_n} = \lambda.$$

Corollary 2 *Let $\tilde{A} = \{\tilde{T}_n\}_n$ be another arrival point process such that the associated function $\{\tilde{\mathcal{Z}}(t)\}$ is finite. Let*

$$\tilde{\lambda} = \liminf_{n \to -\infty} \frac{n}{\tilde{T}_n}.$$

If $\lambda < \tilde{\lambda}$, then the function $\{\mathcal{Z}(t)\}$ is also finite.

Proof. For any $t \in \mathbb{R}$, let m be the index of the last controlled customer arrived before time t, namely

$$m = \sup\{n \in \mathbb{Z},\ T_n \leq t < T_{n+1}\}.$$

Since $\lambda < \tilde{\lambda}$, there exists $l \leq m$ such that for all $k \leq l$,

$$A^{[k,l]} \prec \tilde{A}^{[k,l]},$$

and it follows from Lemma 1 that

$$\forall i = 0, \ldots, N, \quad D_i^{[k,l]} \prec \tilde{D}_i^{[k,l]}.$$

Hence,

$$\forall i = 0, \ldots, N, \quad \mathcal{Z}_i(t) \leq \tilde{\mathcal{Z}}_i(t) + (m - l),$$

so that the finiteness of $\{\tilde{\mathcal{Z}}(t)\}$ implies that of $\{\mathcal{Z}(t)\}$. $\square$

In the rest of the paper, we say that the system is *saturated* when the arrival point process of the controlled customers is $\bar{A} = \{\bar{T}_n\}_n$, where $\bar{T}_n = -\infty$ for all $n \leq 0$ and $\bar{T}_n = 0$ for all $n > 0$.

Corollary 3 *Denote by U_n the departure time of the n–th controlled customer from the input queue when the system is saturated. Let*

$$\nu = \limsup_{n \to +\infty} \frac{n}{U_n}.$$

If $\lambda > \nu$, then component–wise,

$$\lim_{t \to +\infty} \mathcal{Z}(t) = +\infty.$$

Proof. Denote by $\bar{D}_0, \ldots, \bar{D}_N$ the departure point processes associated with $\bar{A}$. It follows from the definition of $\bar{A}$ that for all $k \leq l \in \mathbb{Z}$,

$$\bar{A}^{[k,l]} \prec A^{[k,l]},$$

so that from Lemma 1,

$$\forall i = 0, \ldots, N, \quad \bar{D}_i^{[k,l]} \prec D_i^{[k,l]}.$$

In particular,

$$\nu \geq \limsup_{n \to +\infty} \frac{n}{U_n^0},$$

where $\{U_n^0\}_{n \geq 1}$ denotes the sequence of points of $D_0^{[1,+\infty)}$. Since $\lambda > \nu$,

$$\lim_{t \to +\infty} Z_0^{[1,+\infty)}(t) = +\infty,$$

and the result follows from (4) and Corollary 1. □

The following lemmas show that the departure point processes $D_0, \ldots, D_N$ are also monotone in the service times of the controlled customers and in the service times of the cross customers, as well as in the window K. As immediate corollaries, the function $\{\mathcal{Z}(t)\}$ is also monotone in the same quantities.

Lemma 2 *Let $\tilde{D}_0, \ldots, \tilde{D}_N$ be the departure point processes associated with the system where the service times of the controlled customers $\{\alpha_n^1\}_n, \ldots, \{\alpha_n^N\}_n$ are changed into the sequences $\{\tilde{\alpha}_n^1\}_n, \ldots, \{\tilde{\alpha}_n^N\}_n$. We have for all $k \leq l \in \mathbb{Z}$,*

$$\forall i = 1, \ldots, N, \quad \{\alpha_n^i\}_n \prec \{\tilde{\alpha}_n^i\}_n \quad \Longrightarrow \quad \forall i = 0, \ldots, N, \quad D_i^{[k,l]} \prec \tilde{D}_i^{[k,l]}.$$

Proof. In view of (1), we have

$$D_0^{[k,k+K)} = \tilde{D}_0^{[k,k+K)}.$$

It follows then from (2) that

$$\forall i = 1,\dots,N, \quad D_i^{[k,k+K)} \prec \tilde{D}_i^{[k,k+K)},$$

and from (1),

$$D_0^{[k,k+2K)} \prec \tilde{D}_0^{[k,k+2K)}.$$

The result follows by induction. □

Lemma 3 *Let $\tilde{D}_0,\dots,\tilde{D}_N$ be the departure point processes associated with the system where the service times of the cross customers $\{\beta_n^1\}_n,\dots,\{\beta_n^N\}_n$ are changed into the sequences $\{\tilde{\beta}_n^1\}_n,\dots,\{\tilde{\beta}_n^N\}_n$, and let $\{\tilde{\mathcal{Z}}(t)\}$ denote the corresponding queue size function. We have, for all $k \le l \in \mathbb{Z}$,*

$$\forall i = 1,\dots,N, \quad \{\beta_n^i\}_n \prec \{\tilde{\beta}_n^i\}_n \quad \Longrightarrow \quad \forall i = 0,\dots,N, \quad D_i^{[k,l]} \prec \tilde{D}_i^{[k,l]}.$$

Similarly

$$\forall i = 1,\dots,N, \quad \{\beta_n^i\}_n \prec \{\tilde{\beta}_n^i\}_n \quad \Longrightarrow \quad \forall t \in \mathbb{R}, \quad \mathcal{Z}(t) \le \tilde{\mathcal{Z}}(t).$$

Proof. The proof is exactly the same as that of Lemma 2. □

Lemma 4 *Let $\tilde{D}_0,\dots,\tilde{D}_N$ be the departure point processes associated with the system where the window is $\tilde{K}$ instead of K, and let $\{\tilde{\mathcal{Z}}(t)\}$ be the associated queue size function. We have, for all $k \le l \in \mathbb{Z}$,*

$$K \le \tilde{K} \quad \Longrightarrow \quad \forall i = 0,\dots,N, \quad \tilde{D}_i^{[k,l]} \prec D_i^{[k,l]}.$$

In addition

$$K \le \tilde{K} \quad \Longrightarrow \quad \forall t \in \mathbb{R}, \quad \tilde{\mathcal{Z}}(t) \le \mathcal{Z}(t).$$

Proof. In view of (1), we have

$$\tilde{D}_0^{[k,k+\tilde{K})} = A[k,k+\tilde{K}) \prec D_0^{[k,k+\tilde{K})}.$$

It follows then from (2) that

$$\forall i = 1,\dots,N, \quad \tilde{D}_i^{[k,k+\tilde{K})} \prec D_i^{[k,k+\tilde{K})},$$

and from (1),

$$\tilde{D}_0^{[k,k+2\tilde{K})} \prec D_0^{[k,k+2\tilde{K})},$$

The result follows by induction □

3 Stability and maximum throughput

In this section, we first present the stochastic framework of the analysis. The reader is referred to Appendix A for the main definitions and for the properties of stationary and ergodic point processes to be used in the following. By means of a Loynes' scheme [12], we then use the monotonicity property established in Section 2 to construct a stationary regime for the queueing process $\{Q(t) = (X(t), Y(t))\}$, $t \in \mathbb{R}$, where

$$X(t) = (X_0(t), \dots, X_N(t)) \quad \text{and} \quad Y(t) = (Y_1(t), \dots, Y_N(t)).$$

Finally, we prove the existence of a *maximum throughput* allowed by the flow control, that is a maximum arrival rate of the controlled customers below which the system is stable.

3.1 Stochastic framework

The arrival times and the service times of the controlled customers are defined on a probability space $(\Omega_A, \mathcal{F}_A, \mathbb{P}_A)$, equipped with an ergodic, measure–preserving flow $\{\theta_A(t)\}_{t\in\mathbb{R}}$. We assume that the point process A is simple, $\theta_A(t)$–compatible, and admits $\{\alpha_n\}_n = \{(\alpha_n^1, \dots, \alpha_n^N)\}_n$ as sequence of marks. We denote by λ its finite and non-null intensity, and by (A, α) the corresponding stationary and ergodic marked point process. Let $\mathbb{P}_A^0$ be the Palm probability associated with A. We assume that the mean service time of the controlled customers at station i, defined by $\alpha_i = \mathbb{E}_A^0(\alpha_0^i)$, is finite and non-null. We denote by $\mu_i = \alpha_i^{-1}$ the *service rate* of the controlled customers at station i.

The arrival times and the service times of the cross customers are defined on another probability space $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$, also equipped with an ergodic, measure–preserving flow $\{\theta_B(t)\}_{t\in\mathbb{R}}$. We assume that the point processes $B_1, \dots, B_N$ are simple, $\theta_B(t)$–compatible, and admit respectively $\{\beta_n^1\}_n, \dots, \{\beta_n^N\}_n$ as sequences of marks. Thus the corresponding marked point processes $(B_1, \beta_1), \dots, (B_N, \beta_N)$ are assumed to be *jointly* stationary and ergodic. When the finite intensity λ_i of B_i is non-null, we denote by $\mathbb{P}_{B_i}^0$ the associated Palm probability and assume that the mean service time of the cross customers at station i, defined by $\beta_i = \mathbb{E}_{B_i}^0(\beta_0^i)$, is finite and non-null. The *traffic intensity* of the cross flow at station i is then defined by $\rho_i = \lambda_i \beta_i$ if $\lambda_i > 0$, and $\rho_i = 0$ otherwise. We assume that

$$\forall i = 1, \dots, N, \quad \rho_i < 1.$$

The controlled flow is assumed to be *independent* of the cross flows. In particular, the probability space considered in the following is the product space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \Omega_A \times \Omega_B, \quad \mathcal{F} = \sigma(\mathcal{F}_A \times \mathcal{F}_B) \quad \text{and} \quad \mathbb{P} = \mathbb{P}_A \mathbb{P}_B.$$

We define a flow $\{\theta(t)\}_{t\in\mathbb{R}}$ on this space by $\theta(t) = \theta_A(t) \times \theta_B(t)$ for all $t \in \mathbb{R}$. This flow is measure–preserving but not necessarily ergodic. In the following, we say that a stochastic process is *stationary* if it is compatible with the flow $\{\theta(t)\}$.

3.2 Construction of the minimum stationary regime

Let $\{\mathcal{Z}(t)\}$ be the stochastic process defined by (5). By construction, this process is compatible with the flow $\{\theta(t)\}$.

Lemma 5 *The process $\{\mathcal{Z}(t)\}$ is component–wise a.s. finite or a.s. infinite.*

Proof. From (4), the $\theta(t)$–compatible process $\{\mathcal{Z}(t)\}$ is a.s. finite if and only if the random variable $\mathcal{Z} = \mathcal{Z}_0(0)$ is a.s. finite. From the monotonicity property of Lemma 1, we have

$$\forall \omega = (\omega_A, \omega_B) \in \Omega, \quad \forall t \geq 0, \quad \mathcal{Z}(\theta_A(t)\omega_A, \omega_B) \leq \mathcal{Z}(\omega_A, \omega_B) + A(\omega_A, t),$$

where $A(\omega_A, t)$ is the number of points of A between 0 and t. Hence, defining for any fixed $\omega_B \in \Omega_B$ the event

$$F_A = \{\omega_A \in \Omega_A, \quad \mathcal{Z}(\omega_A, \omega_B) < \infty\},$$

we get $\theta_A(t)F_A \subset F_A$ for all $t \geq 0$. It follows then from the ergodicity of $(\mathbb{P}_A, \theta_A(t))$ that $\mathbb{P}_A(F_A) \in \{0, 1\}$. The mapping

$$\begin{array}{rccc} \chi : & \Omega_B & \longrightarrow & \{0,1\} \\ & \omega_B & \longmapsto & \mathbb{P}_A(F_A) \end{array}$$

defines a random variable on $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$. But by the same argument as above,

$$\forall \omega = (\omega_A, \omega_B) \in \Omega, \quad \forall t \leq 0, \quad \mathcal{Z}(\omega_A, \theta_B(t)\omega_B) \leq \mathcal{Z}(\omega_A, \omega_B) + A(\omega_A, t),$$

so that $\chi \circ \theta_B(t) \geq \chi$ for all $t \leq 0$. It follows then from the ergodicity of $(\mathbb{P}_B, \theta_B(t))$ that χ is $\mathbb{P}_B$–a.s. constant, and

$$\mathbb{E}_B(\chi) \in \{0, 1\}.$$

The result follows then from Fubini Theorem since

$$\begin{aligned} \mathbb{P}(\mathcal{Z} < \infty) &= \int_{\Omega_A} \int_{\Omega_B} \mathbb{I}_{\{\mathcal{Z}(\omega_A,\omega_B)<\infty\}} \mathbb{P}_A(d\omega_A)\mathbb{P}_B(d\omega_B), \\ &= \int_{\Omega_B} \left[\int_{\Omega_A} \mathbb{I}_{\{\mathcal{Z}(\omega_A,\omega_B)<\infty\}} \mathbb{P}_A(d\omega_A) \right] \mathbb{P}_B(d\omega_B), \\ &= \int_{\Omega_B} \chi \, \mathbb{P}_B(d\omega_B). \end{aligned}$$

□

Corollary 4 *Since the discrete flow $\{\theta_A^n\}_n$ associated with the point process A is ergodic on the Palm probability space $(\Omega_A^0, \mathcal{F}_A^0, \mathbb{P}_A^0)$ (see Appendix A), the result of Lemma 5 also holds on the Palm probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ with respect to A, given by*

$$\Omega^0 = \Omega_A^0 \times \Omega_B, \quad \mathcal{F}^0 = \sigma(\mathcal{F}_A^0 \times \mathcal{F}_B) \quad \text{and} \quad \mathbb{P}^0 = \mathbb{P}_A^0 \mathbb{P}_B.$$

Lemma 6 *The process $\{\mathcal{Z}(t)\}$ is the minimum stationary regime of $\{Z(t)\}$.*

Proof. Assume that there exists a finite stationary regime $\{\tilde{\mathcal{Z}}(t)\}$ of $\{Z(t)\}$, and denote by $\tilde{D}_0, \dots, \tilde{D}_N$ the corresponding departure point processes. For any $k \in \mathbb{Z}$, we get from (1),

$$D_0^{[k,k+K)} = A^{[k,k+K)} \prec \tilde{D}_0^{(-\infty,k+K)},$$

so that from (2),

$$\forall i = 1, \dots, N, \quad D_i^{[k,k+K)} \prec \tilde{D}_i^{(-\infty,k+K)},$$

and by induction,

$$\forall i = 0, \dots, N, \quad D_i^{[k,+\infty)} \prec \tilde{D}_i.$$

Therefore,

$$\forall t \in \mathbb{R}, \quad Z^{[k,+\infty)}(t) \le \tilde{\mathcal{Z}}(t),$$

and the proof follows from the fact that this inequality holds for all $k \in \mathbb{Z}$. □

Lemma 7 *There exists a finite stationary regime for the queueing process $\{Q(t)\}$ if and only if the process $\{\mathcal{Z}(t)\}$ is a.s. finite.*

Proof. If there exists a finite stationary regime for the queueing process $\{Q(t)\}$, there exists also a finite stationary regime for $\{Z(t)\}$, and it follows from from Lemma 6 that $\{\mathcal{Z}(t)\}$ is a.s. finite. Conversely, assume that the process $\{\mathcal{Z}(t)\}$ is a.s. finite. We can then define a finite stationary queueing process $\{\mathcal{X}(t)\}$ for the controlled customers by $\mathcal{X}_0(t) = \mathcal{Z}_0(t)$ and

$$\forall i = 1, \dots, N, \quad \mathcal{X}_i(t) = \mathcal{Z}_i(t) - \mathcal{Z}_{i-1}(t).$$

Let $D_0, \dots, D_N$ be the associated $\theta(t)$–compatible departure point processes. In view of (3), we can then construct the associated $\theta(t)$–compatible departure point processes of the cross customers $\{V_n^1\}, \dots, \{V_n^N\}$, and define a finite stationary queueing process $\{\mathcal{Y}(t)\}$ by

$$\forall i = 1, \dots, N, \quad \mathcal{Y}_i(t) = \sum_{n \in \mathbb{Z}} \mathbb{I}_{\{T_n^i \le t < V_n^i\}}.$$

□

Remark 2 *It follows from the $\theta(t)$–invariance of the probability measure $\mathbb{P}$ that when the process $\{\mathcal{Z}(t)\}$ is a.s. infinite, the number of customers in the input queue tends in probability to infinity, that is component–wise,*

$$Z^{[1,+\infty)}(t) \xrightarrow{\mathbb{P}} +\infty \quad \textit{when} \quad t \to +\infty.$$

3.3 Maximum throughput

In the following, we say that the system is *stable* if there exists a finite stationary regime for the queueing process $\{Q(t)\}$. From Lemma 7, the finiteness of $\{\mathcal{Z}(t)\}$ provides a necessary and sufficient condition for the stability of the system. Note that since the process $\{\mathcal{Z}(t)\}$ is $\{\theta(t)\}$–compatible, its finiteness under the probability measures $\mathbb{P}$ and $\mathbb{P}_0$ are equivalent. In particular, the stability region of the system depends on the controlled flow only through its distribution under $\mathbb{P}^0$. Theorem 1 below which is the central result of the paper, shows that the stability region is actually *insensitive* to the distribution of the interarrival times of the controlled customers, and depends on the point process A only through its intensity λ.

Theorem 1 (Maximum throughput) *There exists a constant $\bar{\lambda}$, which only depends on the law of the cross flows under $\mathbb{P}_B$ and on the law of the service times of the controlled customers under $\mathbb{P}^0_A$, and such that the system is stable whenever $\lambda < \bar{\lambda}$ and unstable whenever $\lambda > \bar{\lambda}$.*

In this sense, $\bar{\lambda}$ is the maximum throughput of the controlled flow.

Proof. The proof is given in the Palm setting, that is on the probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ defined in Corollary 4. For any $\tau > 0$, denote by $\{\mathcal{Z}_\tau(t)\}$ the minimum stationary regime obtained when the arrival point process of the controlled customers is the periodic point process $A_\tau = \{n\tau\}_n$. We know from Corollary 4 that the process $\{\mathcal{Z}_\tau(t)\}$ is either a.s. finite or a.s. infinite. Define

$$\bar{\tau} = \inf\{\tau > 0,\ \{\mathcal{Z}_\tau(t)\} \text{ is a.s. finite}\},$$

and $\bar{\tau} = \infty$ if this set is empty. In view of Corollary 2, the process $\{\mathcal{Z}_\tau(t)\}$ is a.s. finite for all $\tau > \bar{\tau}$.

If $\lambda\bar{\tau} < 1$, then there exists $\tau > \bar{\tau}$ such that $\lambda\tau < 1$. From Corollary 2, the finiteness of the process $\{\mathcal{Z}_\tau(t)\}$ implies that of $\{\mathcal{Z}(t)\}$. On the other hand, if $\lambda\bar{\tau} > 1$, then there exists $\tau < \bar{\tau}$ such that $\lambda\tau > 1$. Assume that the process $\{\mathcal{Z}(t)\}$ is finite with strictly positive probability. Then from Corollary 2, the process $\{\mathcal{Z}_\tau(t)\}$ is also finite with strictly positive probability, so that it is finite with probability 1, and $\tau \geq \bar{\tau}$, a contradiction. □

Remark 3 *In view of the proof of Theorem 1, the result still holds if the controlled flow is not independent of the cross flows under* $\mathbb{P}$*, provided that the service times of the controlled customers are independent of the cross flows under* $\mathbb{P}^0$*.*

Remark 4 *When the measure–preserving flow* $\{\theta_B(t)\}$ *is not ergodic with respect to* $\mathbb{P}_B$*, Theorem 1 still applies, except that the maximum throughput* $\bar{\lambda}$ *is no more a constant but a* $\theta_B(t)$*–invariant random variable on* $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$*, that is the system is stable on the event* $\{\lambda < \bar{\lambda}(\omega_B)\}$ *and unstable on the event* $\{\lambda > \bar{\lambda}(\omega_B)\}$*. This case is illustrated by an example in Section 5.*

In the rest of the section, we assume that the service times of the controlled customers are *deterministic*. We show that in this case, the maximum throughput of the controlled flow is equal to the departure rate of the controlled customers from the input queue when the system is *saturated*, i.e. when the arrival point process of the controlled customers is given by $\bar{A}$. Thus the stability condition of the system is given by the so–called saturation rule, although the system does not enter the monotone–separable framework of [5].

Theorem 2 (Saturation rule) *Assume that the service times of the controlled customers are deterministic and denote by* U_n *the departure time of the* n*–th controlled customer from the input queue when the system is saturated. We have*

$$\lim_{n \to +\infty} \frac{n}{U_n} = \bar{\lambda} \quad \textit{a.s.}$$

where $\bar{\lambda}$ *is the maximum throughput defined in Theorem 1.*

Proof. Since the service times of the controlled customers are deterministic, $\{U_n\}_{n \geq 1}$ is a functional of the cross traffic only, so that the reference probability space is actually $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$. Denote by $D(t)$ the number of points of $\{U_n\}_{n \geq 1}$ between 0 and t. Using the fact that $U_1 = U_2 = \ldots = U_K = 0$, it follows from the monotonicity property of Lemma 2 that

$$\forall t, s \geq 0, \quad D(t+s) \leq D(t) + D(s) \circ \theta_B(t).$$

From Kingman's subadditive ergodic theorem [9], there exists a constant ν such that

$$\lim_{t \to +\infty} \frac{D(t)}{t} = \lim_{t \to +\infty} \frac{\mathbb{E}_B(D(t))}{t} = \nu \quad \mathbb{P}_B \text{ a.s.}$$

and we know from Corollary 3 that $\nu \geq \bar{\lambda}$.

Now using the same notations as in the proof of Theorem 1, let τ be any fixed constant such that $\tau < \bar{\tau} = \bar{\lambda}^{-1}$. In view of the definition of $\bar{\tau}$, the process $\{\mathcal{Z}_\tau(t)\}$ associated with A_τ is a.s. infinite. Define

$$H_\tau(t) = \sum_{n=0}^{[\frac{t}{\tau}]} \mathbb{I}_{\{X_\tau((n+1)\tau)=0\}}, \quad \forall t \geq 0,$$

where $[x]$ denotes the integer value of x and $X_\tau(t)$ is the number of controlled customers in the input queue at time t when the arrival point process is $\{n\tau\}_{n\geq 0}$. In view of Remark 2, we have

$$X_\tau(t) \xrightarrow{\mathbb{P}_B} +\infty \quad \text{when} \quad t \to +\infty,$$

so that

$$\lim_{t\to+\infty} \frac{\mathbb{E}_B(H_\tau(t))}{t} = 0.$$

Denote by $A_\tau(t)$ the number of points of A_τ between 0 and t and define another arrival point process by

$$\forall t \geq 0, \quad \tilde{A}_\tau(t) = A_\tau(t) + \left(K + \left[\frac{\tau}{\alpha_1}\right]\right) H_\tau(t),$$

where α_1 is the (non-null) service time of the controlled customers at station 1. By construction, the input queue is then never empty at any time $t \geq 0$, so that

$$\forall t \geq 0, \quad \tilde{A}_\tau(t) \geq D(t).$$

Therefore, using the fact that

$$\lim_{t\to+\infty} \frac{\mathbb{E}_B(\tilde{A}_\tau(t))}{t} = \lim_{t\to+\infty} \frac{A_\tau(t)}{t} = \tau^{-1},$$

we get $\tau^{-1} \geq \nu$. Since this inequality holds for any $\tau < \bar{\lambda}^{-1}$, we obtain $\bar{\lambda} \geq \nu$. $\square$

Corollary 5 *Assume that service times of the controlled customers are deterministic. If there exists a stationary regime for the saturated system, that is a $\theta_B(t)$–compatible arrival process of the controlled customers $\tilde{A} = \{\tilde{T}_n\}_n$ defined on the probability space $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$, such that the departure time of the n–th controlled customer is equal to $\tilde{T}_{n+K}$, for all $n \in \mathbb{Z}$, then the intensity of $\tilde{A}$ is equal to $\bar{\lambda}$.*

Proof. Denote by $\tilde{\lambda}$ the intensity of $\tilde{A}$. By the definition of $\bar{A}$, we have $\bar{A} \prec \tilde{A}$, so that by Lemma 1, the corresponding departure point processes from the input queue satisfy $\bar{D} \prec \tilde{D}$. But since the arrival time $\tilde{T}_n$ of the n–th controlled customer corresponds to the departure time of the $(n-K)$–th controlled customer, it follows from (1) that $\tilde{D} = \tilde{A}$. Using Theorem 2, we obtain $\tilde{\lambda} \leq \bar{\lambda}$.

Now define for all $n \in \mathbb{Z}$,

$$\tilde{S}_n = \max_{1 \leq i \leq N} \inf\{t \geq \tilde{T}_{n+K},\ \tilde{Y}_i^{(-\infty,n]}(t) = 0\},$$

where $\{\tilde{Y}^{(-\infty,n]}(t)\}$ is the queueing process of the cross customers obtained when the arrival point process $\tilde{A}$ is restricted to $(-\infty, n]$. Since

$$\forall n \in \mathbb{Z}, \quad \tilde{S}_n - \tilde{T}_n = (\tilde{S}_0 - \tilde{T}_0) \circ \theta(\tilde{T}_n),$$

there exists a.s. $m \leq 0$ such that $\tilde{S}_m \leq 0$. Let $\hat{A} = \{\hat{T}_n\}_n$ be the point process defined by $\hat{T}_n = \tilde{T}_n$, for all $n \leq m$, and $\hat{T}_n = 0$ otherwise, and denote by $\hat{D} = \{\hat{U}_n\}_n$ the corresponding departure point process from the input queue. By construction, we know from Theorem 2 that

$$\lim_{n \to +\infty} \frac{n}{\hat{U}_n} = \bar{\lambda}.$$

From the monotonicity property of Lemma 1, we get $\tilde{A} \prec \hat{A}$ and $\tilde{D} \prec \hat{D}$, so that $\tilde{\lambda} \geq \bar{\lambda}$. □

4 Bounds on the maximum throughput

In this section, we establish an upper bound and a lower bound on the maximum throughput of the controlled flow, as defined in Section 3. Then we show that these bounds which coincide only when there is a single *bottleneck* in the network, are tight in the sense that both are reached by time-space scaling of the cross flows.

4.1 Upper bound

The upper bound is obtained by comparing the model with systems where the controlled customers leave each station earlier. We first use the monotonicity of the departure times of the controlled customers in the *service times* of the cross customers, then the monotonicity of these departure times in the *window* K.

System without cross flows

The system without cross flows, obtained by reducing the service times of the cross customers to zero, can be represented by a Petri network, as shown in Figure 2. This Petri network is an event graph, the dynamics of which can be represented by linear equations in the (max,+)–algebra (see [4]). Since this system depends on the marked point process (A, α) only, which is compatible with the ergodic shift $\{\theta_A(t)\}$ on the probability space $(\Omega_A, \mathcal{F}_A, \mathbb{P}_A)$, a necessary condition for the stability of this (max,+)–linear system is that

$$\lambda \leq \mu(K),$$

where $\mu(K)$ denotes the inverse of the associated *Lyapunov exponent*.

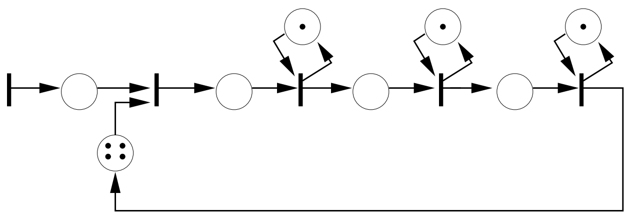

Figure 2: System without cross flows ($N = 3$, $K = 4$)

Remark 5 *By definition, $\mu(K)$ is given by the departure rate of the customers when the input of the system is saturated [2]. In the case of deterministic service times of the controlled customers, we get*

$$\mu(K) = \min\left(\frac{K}{\sum_{i=1}^{N}\frac{1}{\mu_i}}, \min_{1\leq i\leq N}\mu_i\right).$$

From Lemma 3, a necessary condition for the stability of the original system is given by

$$\bar{\lambda} \leq \mu(K). \tag{6}$$

System without flow control

The system without flow control, obtained by letting the window K tend to infinity, is shown in Figure 3. A necessary condition for the stability of this system is given by the usual traffic conditions, namely

$$\forall i = 1, \dots, N, \quad \frac{\lambda}{\mu_i} + \rho_i \leq 1.$$

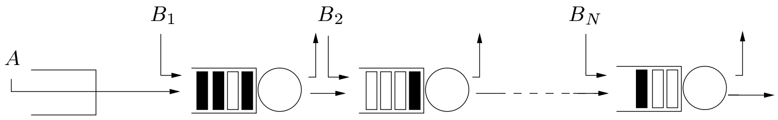

Figure 3: System without flow control

From Lemma 4, this provides a necessary condition for the stability of the original system, so that

$$\bar{\lambda} \leq \min_{1 \leq i \leq N} \mu_i(1 - \rho_i). \tag{7}$$

Remark 6 *The value of this upper bound can be viewed as the so-called available bandwidth of the controlled connection. It is then natural to define the utilization of the network resources as the ratio* $U = \bar{\lambda}/\mu$*, where*

$$\mu = \min_{1 \leq i \leq N} \mu_i(1 - \rho_i).$$

4.2 Lower bound

In the following, we denote by $\{\mathcal{W}(t)\} = \{(\mathcal{W}_1(t), \ldots, \mathcal{W}_N(t))\}$ the stationary workload process of the system in the absence of controlled customers, and define

$$\gamma = \mathbb{P}(\mathcal{W}(0) \neq 0).$$

Note that, since the input processes of the cross flows are not necessarily mutually independent, we can have $\gamma = 1$, which corresponds to cases where the system is never empty.

The lower bound is obtained by comparing the system with that obtained when the cross customers have a *global preemptive priority* (GPP) over controlled customers. That is, whenever a cross customer arrives in any station of the network, the services of *all* controlled customers present in the network are preempted, and can only restart when there are no more cross customers in the network.

System under the GPP service discipline

It is easy to check that Lemma 1, Corollary 1 and Lemmas 5-6 hold under this service discipline, so that we can define the corresponding minimum stationary regime $\{\tilde{\mathcal{Z}}(t)\}$.

Lemma 8 *We have* $\mathcal{Z}(t) \leq \tilde{\mathcal{Z}}(t)$*, for all* $t \in \mathbb{R}$.

Proof. Denote by $\tilde{D}_0 = \{\tilde{U}_n^0\}, \ldots, \tilde{D}_N = \{\tilde{U}_n^N\}$, the departure point processes of the controlled customers under the GPP service discipline. As under this service discipline, a controlled customer can be served at station i only if all cross customers and all controlled customers arrived before itself in station i have been served, we get

$$\forall n \in \mathbb{Z},\ \tilde{U}_n^i \geq \sup_{u \leq \tilde{U}_n^{i-1}} \left\{ u + \sum_{m:\ u \leq \tilde{U}_m^{i-1} \leq \tilde{U}_n^{i-1}} \alpha_m^i + \sum_{m:\ u \leq T_m^i < \tilde{U}_n^{i-1}} \beta_m^i \right\}.$$

Hence if for some $p \in \mathbb{Z}$, we have $\tilde{U}_n^{i-1} \geq U_n^{i-1}$, for all $n \leq p$, we obtain

$$\begin{aligned} \forall n \leq p, \quad \tilde{U}_n^i \quad &\geq \quad \sup_{u \leq \tilde{U}_n^{i-1}} \left\{ u + \sum_{m:\ u \leq \tilde{U}_m^{i-1} \leq \tilde{U}_n^{i-1}} \alpha_m^i + \sum_{m:\ u \leq T_m^i < U_n^{i-1}} \beta_m^i \right\} \\ &\quad + \sum_{m:\ U_n^{i-1} \leq T_m^i < \tilde{U}_n^{i-1}} \beta_m^i, \end{aligned}$$

and it follows from (2) that $\tilde{U}_n^i \geq U_n^i$, for all $n \leq p$.

Now let $k \leq l \in \mathbb{Z}$ be fixed. From (1),

$$D_0^{[k,k+K)} = \tilde{D}_0^{[k,k+K)},$$

so that from the above property,

$$\forall i = 1, \ldots, N, \quad D_i^{[k,k+K)} \prec \tilde{D}_i^{[k,k+K)},$$

and by induction,

$$\forall i = 0, \ldots, N, \quad D_i^{[k,l]} \prec \tilde{D}_i^{[k,l]}.$$

The result follows from the definition of the processes $\{\mathcal{Z}(t)\}$ and $\{\tilde{\mathcal{Z}}(t)\}$. □

Lemma 9 *Assume that $\gamma < 1$. If $\lambda < \mu(K)(1-\gamma)$, the system under the GPP service discipline is stable, that is there exists a finite stationary regime for the associated queueing process $\{\tilde{Q}(t)\}$.*

Proof. Since under the GPP service discipline, the cross customers have preemptive priority over the controlled customers, there exists a unique $\theta(t)$–compatible process $\{\tilde{\mathcal{Y}}(t)\}$ for the number of cross customers in each station, and it is enough to show the finiteness of the process $\{\tilde{\mathcal{Z}}(t)\}$.

Denote by $\{A(t)\}$ the counting process associated with A such that $A(0) = 0$, and let $\tilde{A}$ be the point process associated with the counting process $\{\tilde{A}(t)\}$, defined by

$$\forall t \in \mathbb{R}, \quad \tilde{A}(t) = A(\varphi(t)),$$

where

$$\varphi(t) = \sup \left\{ u \in \mathbb{R}, \quad t = \int_0^u \mathbb{I}_{\{\mathcal{W}(0)=0\}} \circ \theta_B(s)\, ds \right\}.$$

Note that this point process is not necessarily simple. The stochastic process defined by (5) for the system without cross flows, and with the arrival point process $\tilde{A}$ is given by $\{\tilde{\mathcal{Z}}(\varphi(t))\}$. Applying the saturation rule of [5], a sufficient condition for this process to be finite is that the intensity of $\tilde{A}$ is smaller than the departure rate of the customers in the associated saturated system, namely $\mu(K)$. But since $\gamma < 1$, we have

$$\varphi(t) \xrightarrow{a.s.} \pm\infty \quad \text{when} \quad t \to \pm\infty,$$

and noting that

$$t = \int_0^{\varphi(t)} \mathbb{I}_{\{\mathcal{W}(0)=0\}} \circ \theta_B(s)\, ds,$$

if follows from the ergodicity of $\{\theta_B(t)\}$ that

$$\frac{t}{\varphi(t)} \xrightarrow{a.s.} \mathbb{P}(\mathcal{W}(0) = 0) \quad \text{when} \quad t \to \pm\infty.$$

In particular,

$$\frac{\tilde{A}(t)}{t} = \frac{A(\varphi(t))}{\varphi(t)} \frac{\varphi(t)}{t} \xrightarrow{a.s.} \lambda(1-\gamma)^{-1} \quad \text{when} \quad t \to \pm\infty.$$

Thus if $\lambda < \mu(K)(1-\gamma)$, the process $\{\tilde{\mathcal{Z}}(t)\}$ is a.s. finite. □

From Lemmas 8 and 9, a sufficient condition for the stability of the original system (under the FIFO service discipline), is that $\lambda < \mu(K)(1-\gamma)$. Therefore,

$$\bar{\lambda} \geq \mu(K)(1-\gamma). \tag{8}$$

4.3 Tightness of the bounds

From (6)-(7) and (8), the constants

$$\lambda_{\max} = \min\left(\mu(K), \min_{1\le i\le N} \mu_i(1-\rho_i)\right) \quad \text{and} \quad \lambda_{\min} = \mu(K)(1-\gamma) \tag{9}$$

respectively provide an upper and a lower bound on the maximum throughput of the controlled connection. It is not difficult to see that these bounds coincide if and only if $\rho_i > 0$ for at most one i, say j, and $\mu(K) = \mu_j$. In particular, $\mu_j = \min_{1\le i\le N}\mu_i$, so that the station j is the *bottleneck* of the connection, in the following strong sense:

- In the absence of cross customers, the maximum throughput of the connection is given by the service rate μ_j at station j. In particular, increasing the window K has no effect on this maximum throughput;
- No station except station j receives a cross flow.

In this case, the maximum throughput of the controlled connection is equal to the *available bandwidth* as defined in Remark 6, that is

$$\bar{\lambda} = \mu_j(1-\rho_j).$$

Remark 7 *The single station case $N = 1$ is a particular case where the bounds always coincide. This is not surprising since the model can then be seen as a single G/G/1 queue with two types of customers, the controlled customers and the cross customers, and a conservative service discipline which depends on the type of customers present in the queue. Such a system is known to be stable when $\rho < 1$ and unstable when $\rho > 1$, where $\rho = \lambda/\mu_1 + \rho_1$ is the total traffic intensity of both flows.*

In the general case, the bounds do not coincide. In fact, we will show that in the case of deterministic service times of the controlled customers, the upper bound on the maximum throughput is reached when the cross flows are scaled in time and space by a factor which tends to zero, so as to get *fluid* flows. Then we show that the lower bound is reached when this scaling factor tends to infinity, so as to get *bursty* flows.

For any $c > 0$, we denote by $\{\mathcal{W}(c,t)\}$ the stationary workload process of the system without controlled customers, and where the cross flows are scaled by a factor c, namely the arrivals on station i take place at times $\{cT_n^i\}_n$ and the corresponding service times are $\{c\beta_n^i\}_n$. Note that the scaled cross flows are compatible with the measure–preserving, ergodic flow $\{\theta_B(c,t)\} = \{\theta_B\left(\frac{t}{c}\right)\}$, and that

$$\forall t \in \mathbb{R}, \quad \mathcal{W}(c,t) = c\,\mathcal{W}\left(\frac{t}{c}\right).$$

System with asymptotically fluid cross flows

When the scaling factor c tends to zero, a direct application of the pointwise ergodic theorem shows that the cross traffic workload arriving to station i in the interval $(0,t]$, namely

$$\sum_{n\in\mathbb{Z}} \mathbb{I}_{\{0<cT_n^i\leq t\}}\, c\beta_n^i = t\,\frac{c}{t}\sum_{n\in\mathbb{Z}} \mathbb{I}_{\{0<T_n^i\leq \frac{t}{c}\}}\beta_n^i,$$

tends a.s. to $t\rho_i$, so that the cross flows tend to be *fluid*. Theorem 3 below states that, provided that the service times of the controlled customers are deterministic, the maximum throughput of the controlled connection will then tend to the upper bound $\lambda_{\max}$.

For sake of completeness, but also to facilitate the reading of the proof of Theorem 3, we first study the limiting case where the cross flows are fluid. The dynamics of the system is then given by (1) and

$$\forall n\in\mathbb{Z},\quad U_n^i = U_n^{i-1} + \sup_{v\leq 0}\left\{(1-\rho_i)v + \sum_{m:\ U_n^{i-1}+v\leq U_m^{i-1}\leq U_n^{i-1}} \alpha_m^i\right\}. \quad (2')$$

The monotonicity properties of §2.3 still hold. Using this, we show the existence of a maximum throughput as in Theorem 1, and also that this maximum throughput is smaller than $\lambda_{\max}$, as in §4.1.

Lemma 10 *If the service times of the controlled customers are deterministic, the maximum throughput of the controlled connection with fluid cross flows is equal to* $\lambda_{\max}$.

Proof. First note that in view of (9) and Remark 5,

$$\lambda_{\max} = \min\left(\frac{K}{\sum_{1\leq i\leq N}\alpha_i},\ \min_{1\leq i\leq N}\frac{1}{\alpha_i}(1-\rho_i)\right).$$

Let $\tau = {\lambda_{\max}}^{-1}$ and denote by $\tilde{A}$ the arrival point process with points $\{n\tau\}_n$. In view of Corollary 2, it is sufficient to show that the (deterministic) process $\{\tilde{\mathcal{Z}}(t)\}$ associated with $\tilde{A}$ is finite to conclude the proof. We denote by $\tilde{D}_0,\ldots,\tilde{D}_N$ the departure point processes associated with $\tilde{A}$, and by σ_i the cumulative service time of the controlled customers up to station i, namely

$$\forall i=1,\ldots,N,\quad \sigma_i = \sum_{j=1}^{i}\alpha_j \quad\text{and}\quad \sigma_0 = 0.$$

In view of ($2'$), using the fact that $\tau(1-\rho_i) \geq \alpha_i$ for all i, we have for any $k \in \mathbb{Z}$,

$$\forall i = 0, \ldots, N, \quad \tilde{D}_i^{[k,k+K)} = \tilde{A}^{[k,k+K)} + \sigma_i.$$

But since $\tilde{T}_{n+K} = \tilde{T}_n + K\tau \geq \tilde{T}_n + \sigma_N$ for all $n \in \mathbb{Z}$, it follows from (1) that

$$\tilde{D}_0^{[k,k+2K)} = \tilde{A}^{[k,k+2K)}.$$

By induction, we get for all $k \leq l \in \mathbb{Z}$,

$$\forall i = 0, \ldots, N, \quad \tilde{D}_i^{[k,l]} = \tilde{A}^{[k,l]} + \sigma_i.$$

Thus using (4),

$$\forall t \in \mathbb{R}, \quad 0 = \tilde{Z}_0^{[k,l]}(t) \leq \tilde{Z}_1^{[k,l]}(t) \leq \ldots \leq \tilde{Z}_N^{[k,l]}(t) \leq K.$$

The result follows from the fact that these inequalities hold for all $k \leq l \in \mathbb{Z}$. □

Remark 8 *When the service times of the controlled customers are not deterministic, the maximum throughput of the controlled connection in the presence of fluid cross flows is not equal to $\lambda_{\max}$ in general.*

Now we consider the original system with (discrete) cross flows scaled in time and space by a factor c, and study the behavior of the corresponding maximum throughput $\bar{\lambda}(c)$ when c tends to zero.

Theorem 3 *If the service times of the controlled customers are deterministic, the maximum throughput of the controlled connection tends to its maximum value $\lambda_{\max}$ when the cross flows are scaled in time and space by a factor which tends to zero, that is*

$$\lim_{c \to 0} \bar{\lambda}(c) = \lambda_{\max}.$$

Proof. The proof is given on the probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$. Let τ and ε be fixed constants such that $\tau\lambda_{\max} > 1$ and $\varepsilon\sigma_N < \tau$. For any $c > 0$, define the event

$$F(c) = \bigcap_{i=1}^{N} \left\{ \{ \int_{\sigma_{i-1}(1+\varepsilon)}^{\sigma_{i-1}+\tau} \mathbb{I}_{\{\mathcal{W}_i(c,t)=0\}} \, dt \geq \alpha_i \} \cap \{ \mathcal{W}_i(c, \sigma_{i-1}(1+\varepsilon)) \leq \alpha_i \varepsilon \} \right\},$$

where $\sigma_0, \dots, \sigma_N$ are the cumulative service times of the controlled customers as defined above. Let $\tilde{A}$ be the point process defined by the associated counting process

$$\tilde{A}(t) = \sum_{n=0}^{[\frac{t}{\tau}]} \mathbb{I}_{F(c)} \circ \theta(n\tau).$$

We denote by $\{\tilde{\mathcal{Z}}(t)\}$ the associated process defined by (5) and by $\tilde{D}_0, \dots, \tilde{D}_N$ the corresponding departure point processes. By the same inductive argument as that used in Lemma 10, we get for all $k \leq l \in \mathbb{Z}$,

$$\forall i = 0, \dots, N, \quad \tilde{A}^{[k,l]} + \sigma_i \leq \tilde{D}_i^{[k,l]} \leq \tilde{A}^{[k,l]} + \sigma_i(1+\varepsilon).$$

By choosing ε small enough, so that

$$\frac{\mu(K)}{1+\varepsilon}\,\tau > 1,$$

the system without cross flows, with arrival point process $\tilde{A}$ and deterministic service times $\alpha_1(1+\varepsilon), \dots, \alpha_N(1+\varepsilon)$, is stable, and the corresponding departure point processes are given by $\tilde{A}, \tilde{A}+\sigma_1(1+\varepsilon), \dots, \tilde{A}+\sigma_N(1+\varepsilon)$. The number of customers in this system is bounded by K at any time t, so that

$$\forall i = 0, \dots, N, \quad \tilde{Z}_i^{[k,l]}(t) \leq K.$$

This inequality holds for all $k \leq l \in \mathbb{Z}$, which implies the finiteness of the process $\{\tilde{\mathcal{Z}}(t)\}$.

In addition, since the discrete flow $\{\theta(n\tau)\}_n$ is measure–preserving, there exists an random variable $\tilde{\lambda}(c)$ such that

$$\frac{\tilde{A}(t)}{t} \xrightarrow{a.s.} \tilde{\lambda}(c) \quad \text{when} \quad t \to \pm\infty,$$

and the expectation of $\tilde{\lambda}(c)$ which is the same under $\mathbb{P}$ and under $\mathbb{P}^0$, is given by

$$\mathbb{E}(\tilde{\lambda}(c)) = \lim_{t\to\pm\infty} \frac{\mathbb{E}(\tilde{A}(t))}{t} = \frac{1}{\tau}\,\mathbb{P}(F(c)).$$

From Corollary 2, we have $\tilde{\lambda}(c) \leq \bar{\lambda}(c)$ a.s. and in particular,

$$\forall c > 0, \quad \frac{1}{\tau}\,\mathbb{P}(F(c)) \leq \bar{\lambda}(c).$$

But since

$$\mathbb{P}(F(c)) = \mathbb{P}\left(\bigcap_{i=1}^{N}\left\{\{\frac{1}{c}\int_0^{c(\tau-\varepsilon\sigma_{i-1})}\mathbb{I}_{\{\mathcal{W}_i(0)=0\}}\circ\theta_B(t)\,dt \geq \alpha_i\}\cap\{\mathcal{W}_i(0)\leq c\alpha_i\varepsilon\}\right\}\right),$$

by fixing $\varepsilon > 0$ small enough such that

$$\forall i = 1,\ldots,N, \quad (\tau - \varepsilon\sigma_{i-1})(1-\rho_i) > \alpha_i,$$

it follows from the ergodicity of the flow $\{\theta_B(t)\}$ that

$$\lim_{c\to+\infty}\mathbb{P}(F(c)) = 1.$$

Therefore,

$$\frac{1}{\tau} \leq \liminf_{c\to+\infty}\bar{\lambda}(c) \leq \lambda_{\max}.$$

The result follows from the fact that these inequalities hold for all τ such that $\tau\lambda_{\max} > 1$. $\square$

System with asymptotically bursty cross flows

When the scaling factor c tends to infinity, the cross flows become more *bursty*, and we will show that in this case, the maximum throughput of the controlled connection tends to the lower bound $\lambda_{\min}$.

We first consider the limiting system. On the event $\{\mathcal{W}(0) \neq 0\}$, using the fact that a.s. no cross customer arrives in the system at time 0, we have for some i,

$$\forall t \in \mathbb{Z}, \quad \lim_{c\to+\infty}\mathcal{W}_i(c,t) = \lim_{c\to+\infty} c\,\mathcal{W}_i\left(\frac{t}{c}\right) = +\infty,$$

so that the maximum throughput is equal to 0. On the complementary event $\{\mathcal{W}(0) = 0\}$, using the fact that a.s. no cross customer leaves the system at time 0, we have for all i

$$\forall t \in \mathbb{Z}, \quad \lim_{c\to+\infty}\mathcal{W}_i(c,t) = \lim_{c\to+\infty} c\,\mathcal{W}_i\left(\frac{t}{c}\right) = 0,$$

so that the maximum throughput is that of the (max,+)–linear system of §4.1, that is $\mu(K)$. So the limiting system is intrinsically non-ergodic, with expected maximum throughput

$$\mu(K)\,\mathbb{P}(\mathcal{W}(0) = 0) = \lambda_{\min}.$$

To prove Theorem 4 below, we need the following technical result.

Lemma 11 *For any $b > 0$, let $\mu_b(K)$ be the inverse of the Lyapunov exponent associated with the system without cross flows, where the service times of the controlled customers are changed into the θ_A^n–compatible sequences $\{\alpha_n^1 \mathbb{I}_{\{\alpha_n^1 \le b\}}\}_n, \dots, \{\alpha_n^N \mathbb{I}_{\{\alpha_n^N \le b\}}\}_n$. We have*

$$\lim_{b \to +\infty} \mu_b(K) = \mu(K).$$

Proof. By monotonicity, we have

$$\forall b > 0, \quad \mu_b(K) \ge \mu(K).$$

Denote respectively by U_n and U_n^b the departure time of the n–th controlled customer from the input queue, when the systems (without cross flows) with original service times and with service times changed into $\{\alpha_n^1 \mathbb{I}_{\{\alpha_n^1 \le b\}}\}, \dots, \{\alpha_n^N \mathbb{I}_{\{\alpha_n^N \le b\}}\}$ are saturated. Since

$$\forall n \ge 1, \quad U_n^b + \sum_{k=1}^{n} \sum_{i=1}^{N} \alpha_k^i \mathbb{I}_{\{\alpha_k^i > b\}} \ge U_n,$$

we have in view of Remark 5,

$$\frac{1}{\mu_b(K)} + \sum_{i=1}^{N} \mathbb{E}_A^0(\alpha_0^i \mathbb{I}_{\{\alpha_0^i > b\}}) \ge \frac{1}{\mu(K)}.$$

The result follows then from the fact that, by dominated convergence,

$$\forall i = 1, \dots, N, \quad \lim_{b \to +\infty} \mathbb{E}_A^0(\alpha_0^i \mathbb{I}_{\{\alpha_0^i > b\}}) = 0.$$

$\square$

Theorem 4 *The maximum throughput of the controlled connection tends to its minimum value $\lambda_{\min}$ when the cross flows are scaled in time and space by a factor which tends to infinity, that is*

$$\lim_{c \to +\infty} \bar{\lambda}(c) = \lambda_{\min}.$$

Proof. Denote by S_n^i and V_n^i respectively the beginning and the end of service of the n–th cross customer at station i in the original system without controlled customers. When the cross flows are scaled in time and space by the factor c, the beginning and the end of service of the n–th cross customer at station i occur at times cS_n^i and cV_n^i, respectively.

For a fixed $b > 0$, consider the system where the input queue is saturated (i.e. the arrival point process of the controlled customers is $\bar{A}$), the service times of the controlled customers are given by $\{\alpha_n^1 \mathbb{I}_{\{\alpha_n^1 \leq b\}}\}_n, \ldots, \{\alpha_n^N \mathbb{I}_{\{\alpha_n^N \leq b\}}\}_n$, the arrival times of the cross customers are given by $\{cT_n^1\}_n, \ldots, \{cT_n^N\}_n$, and their service times are recursively defined by

$$\forall i = 1, \ldots, N, \quad \tilde{\beta}_n^i = \begin{cases} \beta_n^i & \text{if} \quad n \leq 0, \\ \max(cV_n^i - \tilde{S}_n^i, 0) & \text{if} \quad n \geq 1, \end{cases}$$

where $\tilde{S}_n^i$ denotes the beginning of service of the n–th cross customer at station i in this system, namely if $\tilde{D}_0 = \{\tilde{U}_n^0\}, \ldots, \tilde{D}_N = \{\tilde{U}_n^N\}$, denote the corresponding departure point processes of the controlled customers,

$$\tilde{S}_n^i = \sup_{u \leq cT_n^i} \left\{ u + \sum_{m \geq 1:\ u \leq \tilde{U}_m^{i-1} \leq cT_n^i} \alpha_m^i \mathbb{I}_{\{\alpha_m^i \leq b\}} + \sum_{m:\ u \leq cT_m^i < cT_n^i} \tilde{\beta}_m^i \right\}.$$

Denote by $\tilde{V}_n^i$ the departure time of the n–th cross customer from station i in this system. For all $n \leq 0$, we have $\tilde{V}_n^i = cV_n^i$, and for all $n \geq 1$,

$$\tilde{V}_n^i = \tilde{S}_n^i + \tilde{\beta}_n^i = \max(cV_n^i, \tilde{S}_n^i) \geq cV_n^i,$$

so that for all $n \in \mathbb{Z}$,

$$\tilde{S}_n^i \geq \max(\tilde{V}_{n-1}^i, cT_n^i) \geq c \max(V_{n-1}^i, T_n^i) = cS_n^i.$$

Therefore, $\tilde{\beta}_n^i \leq c\beta_n^i$ for all $n \in \mathbb{Z}$, and from Corollary 3, using the monotonicity properties of Lemmas 2 and 3, we get

$$\bar{\lambda}(c) \leq \tilde{\lambda}(c) = \limsup_{n \to +\infty} \frac{n}{\tilde{U}_n^0}.$$

In addition, since the number of controlled customers in this system is always bounded by K, we get for all $i = 1, \ldots, N$,

$$\forall n \geq 1, \quad \tilde{S}_n^i \leq \sup_{u \leq cT_n^i} \left\{ u + \sum_{m:\ u \leq cT_m^i < cT_n^i} \tilde{\beta}_m^i \right\} + Kb \leq cS_n^i + Kb.$$

In particular, no controlled customer is served at station i in any interval (possibly empty) of the form

$$[cS_n^i + Kb, cV_n^i) \subset [\tilde{S}_n^i, \tilde{V}_n^i), \quad n \in \mathbb{Z}.$$

Now consider the same system but without cross flows, and where the services of all controlled customers are preempted at any time t such that

$$t \in \bigcup_{n\geq 1} \bigcup_{i=1}^{N} [cS_n^i + NKb, cV_n^i).$$

Denoting by $\tilde{D}$ the departure point process of the controlled customers from the input queue in this system, we have $\tilde{D} \prec \tilde{D}_0$. The number of points of $\tilde{D}$ between 0 and t is given by

$$\forall t \geq 0, \quad \tilde{D}(t) = D(\psi(t)),$$

where $D(t)$ is the number of controlled customers which leave the input queue for the same system but without preemption, and $\psi(t)$ is the cumulative time during which the controlled customers are not preempted between time 0 and t. Noting that the process $\{\mathcal{V}(t)\}$ defined on the probability space $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$ by

$$\forall t \in \mathbb{R}, \quad \mathcal{V}(t) = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{N} \mathbb{I}_{\{cS_n^i + NKb \leq t < cV_n^i\}},$$

is compatible with the flow $\{\theta_B(t)\}$, we get

$$\frac{\psi(t)}{t} \xrightarrow{a.s.} \mathbb{P}(\mathcal{V}(0) = 0) \quad \text{when} \quad t \to +\infty.$$

Defining $\mu_b(K)$ as in Lemma 11, we obtain

$$\frac{\tilde{D}(t)}{t} \xrightarrow{a.s.} \mu_b(K)\, \mathbb{P}(\mathcal{V}(0) = 0) \quad \text{when} \quad t \to +\infty.$$

Therefore,

$$\forall c > 0, \quad \bar{\lambda}(c) \leq \tilde{\lambda}(c) \leq \mu_b(K)\, \mathbb{P}(\mathcal{V}(0) = 0),$$

and since

$$\mathcal{V}(0) = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{N} \mathbb{I}_{\{S_n^i + \frac{NKb}{c} \leq 0 < V_n^i\}},$$

we obtain by dominated convergence,

$$\lambda_{\min} \leq \limsup_{c\to+\infty} \bar{\lambda}(c) \leq \mu_b(K)\, \mathbb{P}(\mathcal{W}(0) = 0).$$

The result follows from Lemma 11 by letting b tend to infinity. □

5 Examples

In all examples studied below, the number of stations is $N = 2$ and the window is $K = 1$. The service times are deterministic, unless specified all equal to 1, i.e. $\alpha_n^1 = \alpha_n^2 = 1$ and $\beta_n^1 = \beta_n^2 = 1$, for all $n \in \mathbb{Z}$. The arrival process of the cross customers at station 1 is periodic, with period $\tau \geq 1$. More precisely, we denote by $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ the probability space defined as follows: Ω_1 is the interval $(0, \tau]$, $\mathcal{F}_1$ is the trace of the Borel σ–field on Ω_1 and $\mathbb{P}_1$ is the uniform measure on Ω_1. The point process B_1, of intensity $\lambda_1 = \tau^{-1}$, is defined by

$$\forall \omega \in \Omega_1, \, \forall n \in \mathbb{Z}, \quad T_n^1 = \omega + (n-1)\tau.$$

Note that B_1 is $\theta_1(t)$–compatible, where $\{\theta_1(t)\}$ is the measure–preserving and ergodic flow defined on this space by

$$\forall t \in \mathbb{R}, \, \forall \omega \in \Omega_1, \quad \theta_1(t)\omega = \omega + t \mod \tau.$$

In the following, we consider different point processes B_2, and we use the saturation rule of Theorem 2 to derive the maximum throughput $\bar{\lambda}$ of the controlled flow. Note that this maximum throughput does not depend on the arrival process of the controlled customers (see Theorem 1). In particular, we do not specify the point process A, which can be any stationary and ergodic point process in view of Remark 3.

5.1 Non–monotonic, non–convex and fractal stability region

In this example, B_2 is the null point process. We consider the saturated system, that is the system where the arrival point process of the controlled customers is $\bar{A}$, and evaluate the departure rate of the controlled customers, which coincide with the maximum throughput $\bar{\lambda}$. We distinguish between three cases, depending on the value of the integer value $[\tau]$ of τ.

Case $[\tau] = 1$

We will show that station 1 is never empty in this case, so that the departure rate of the controlled customers is simply given by

$$\bar{\lambda} = 1 - \lambda_1.$$

Assume that station 1 is empty at some time $t > 0$. The controlled customer which is in station 2, began its service at station 1 after time $t-2$ and no later than time $t-1$. Hence, the cross customer arrived in station 1 between time $t - \tau$ and time t, cannot have been served before this controlled customer: This cross customer is in station 1 at time t, a contradiction.

Case $[\tau]$ even

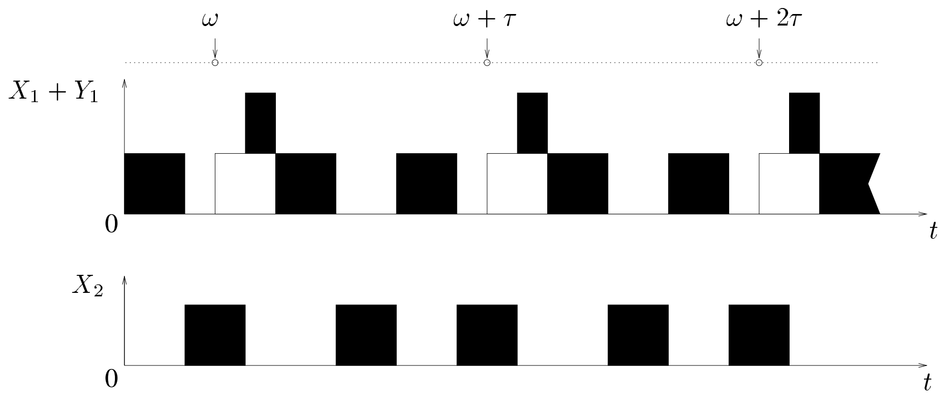

Figure 4: Case $[\tau]$ even ($\tau = 4.5$)

As illustrated in Figure 4, the saturated system reaches a periodic steady state of period τ. Since $[\tau]/2$ controlled customers leave the system each period of time τ, we get

$$\bar{\lambda} = \frac{[\tau]}{2\tau} = \frac{1}{2}\left[\frac{1}{\lambda_1}\right]\lambda_1.$$

Case $[\tau]$ odd, $[\tau] \neq 1$

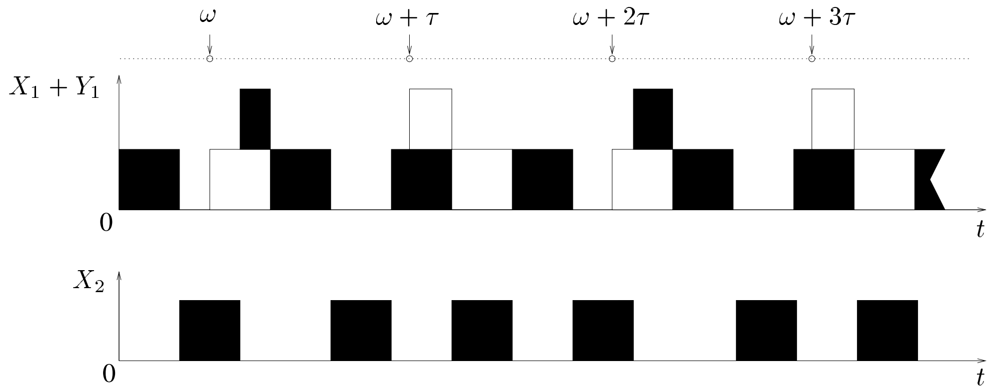

Figure 5: Case $[\tau]$ odd, $[\tau] \neq 1$ ($\tau = 3.3$, $p = 2$)

Let p be the smallest integer q such that $[q\tau]$ is even. We have

$$p = \left[\frac{1}{[\tau] + 1 - \tau}\right] + 1.$$

As illustrated in Figure 5, the saturated system reaches a periodic steady state of period $p\tau$. Since $[p\tau]/2$ controlled customers leave the system each period of time $p\tau$, we get

$$\bar{\lambda} = \frac{[p\tau]}{2p\tau} = \frac{1}{2p}\left[\frac{p}{\lambda_1}\right]\lambda_1.$$

Stability region

The stability region obtained, as well as the bounds on the maximum throughput, given in view of (9) by

$$\lambda_{\min} = \frac{1}{2}(1 - \lambda_1) \quad \text{and} \quad \lambda_{\max} = \min\left(\frac{1}{2}, 1 - \lambda_1\right),$$

are shown in Figure 6. Note that the maximum throughput $\bar{\lambda}$ is not monotone in the intensity λ_1 of the cross flow. In particular, an *increase* of the intensity λ_1 of the cross flow, may result in an *increase* of the maximum throughput $\bar{\lambda}$ of the controlled connection. Consider for instance the case where the arrival point process of the controlled customers is of intensity $\lambda = 0.45$. The system which is *unstable* when $\lambda_1 = 0.4$ becomes *stable* when $\lambda_1 = 0.5$. This unexpected behavior is a consequence of the non-monotonicity of the system with respect to the cross traffic (see Remark 1).

In addition, the maximum throughput $\bar{\lambda}$ as a function of λ_1 is neither convex nor continuous. The upper bound is reached when

$$\frac{1}{2} \leq \lambda_1 \leq 1 \quad \text{or} \quad \lambda_1 = \frac{1}{2k}, \quad k \in \mathbb{N},\ k \neq 0,$$

whereas the lower bound is "almost" reached, when

$$\lambda_1 \longrightarrow \left(\frac{1}{2k+1}\right)^+, \quad k \in \mathbb{N}.$$

Finally, the restriction of stability region to the region $\mathcal{R}$ such that $0 < \lambda_1 < 1/2$, which is invariant by the similarity of center $(0, 1/2)$ and ratio $(1-2\lambda_1)^{-1}$ restricted to $\mathcal{R}$, is *fractal*.

Remark 9 *Due to the convention that when a controlled customer and a cross customer arrive at a given station at the same time, the controlled customer is served first (see §2.2), the maximum throughput* $\bar{\lambda}$ *as a function of* λ_1 *is left-continuous with right-hand limits. With the converse convention, this function would have been right-continuous with left-hand limits.*

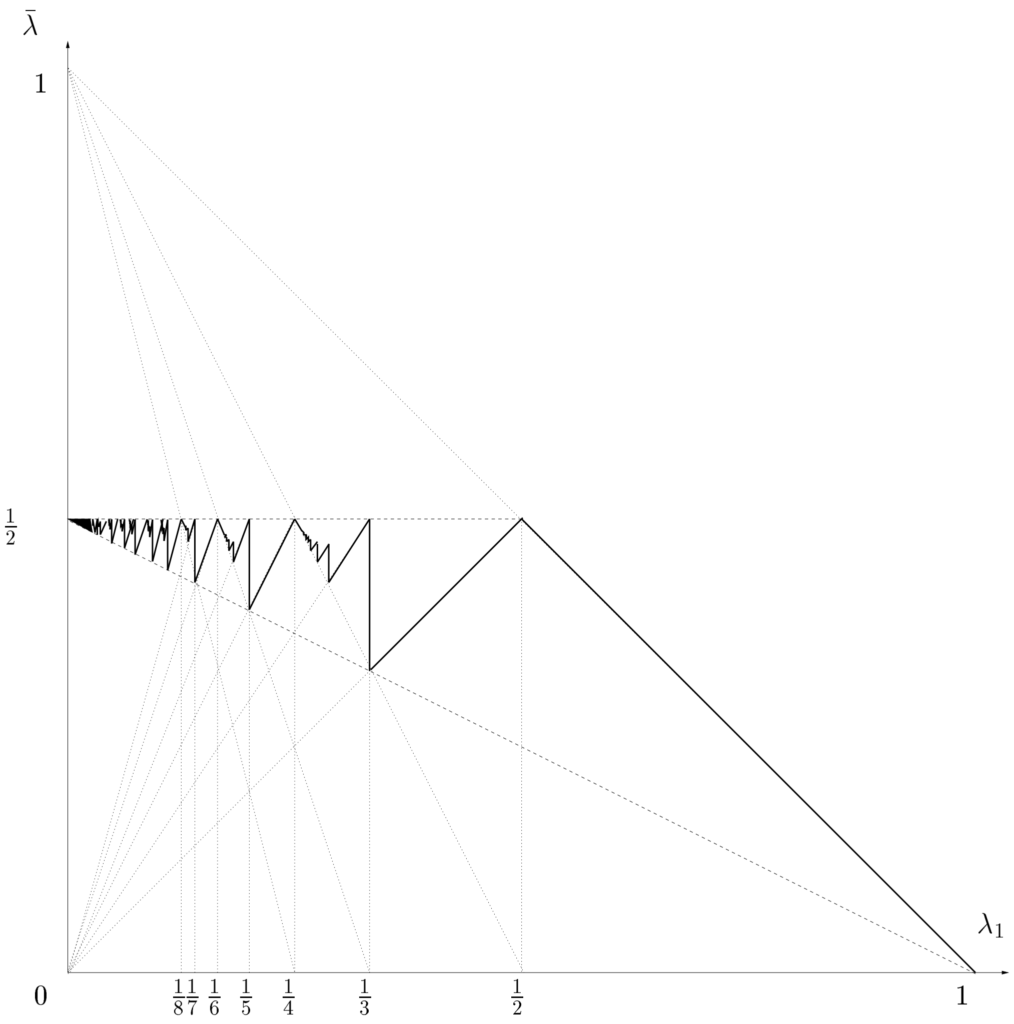

Figure 6: Stability region

In the two following examples, the service times of the cross customers are taken equal to $\tau/2$, i.e. $\beta_n^1 = \beta_n^2 = \tau/2$, for all $n \in \mathbb{Z}$, and τ is supposed to be larger than 2.

5.2 Non–mutually independent cross flows

Let B_2 be the periodic point process defined by

$$\forall \omega \in \Omega_1,\ C \in \mathcal{B}(\mathbb{R}), \quad B_2(\omega, C) = B_1\left(\omega, \frac{\tau}{2} + C\right).$$

The point processes B_1 and B_2 are $\theta_1(t)$–compatible on the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, but not mutually independent.

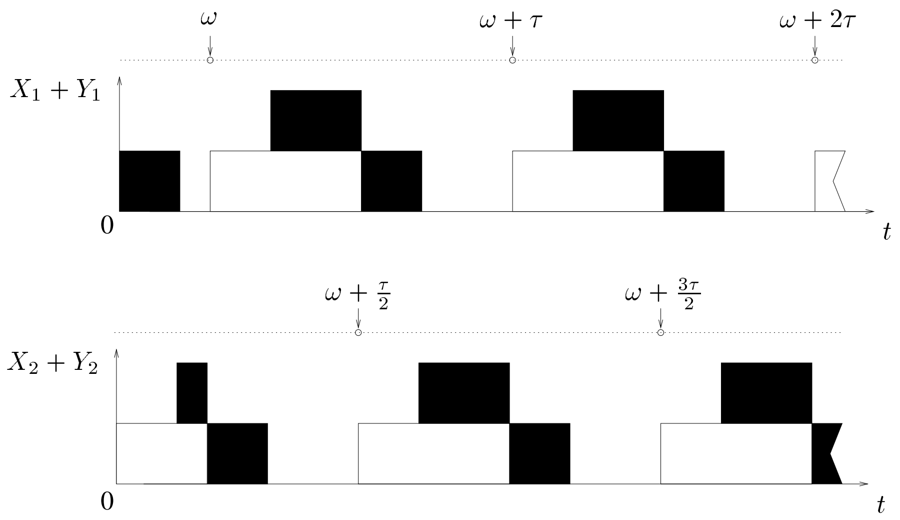

Figure 7: Non-mutually independent cross flows ($\tau = 5$)

As illustrated in Figure 7, the saturated system reaches a periodic steady state, where one controlled customer leaves the system each period of time τ, so that

$$\bar{\lambda} = \frac{1}{\tau}.$$

In particular, this maximum throughput tends to zero when τ tends to infinity, although the traffic intensities of the cross flows remain constant, equal to $\rho_1 = \rho_2 = 1/2$.

Remark 10 *This result is not surprising in view of Theorem 4, since in the absence of controlled customers, the system is never empty, so that lower bound is equal to* $\lambda_{\min} = 0$.

5.3 Non–jointly ergodic cross flows

Let $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be a replica of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, on which a flow $\{\theta_2(t)\}$ similar to $\{\theta_1(t)\}$ is defined. Let B_2 be the point process defined on the probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ in the same way as B_1 is defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$. Note that the point processes B_1 and B_2 are mutually independent on the product space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{F} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2) \quad \text{and} \quad \mathbb{P} = \mathbb{P}_1 \mathbb{P}_2,$$

but not jointly ergodic on this space since the joint flow $\{\theta(t)\} = \{\theta_1(t) \times \theta_2(t)\}$ is not ergodic. In this case, the maximum throughput is no more a constant but a random variable (see Remark 4).

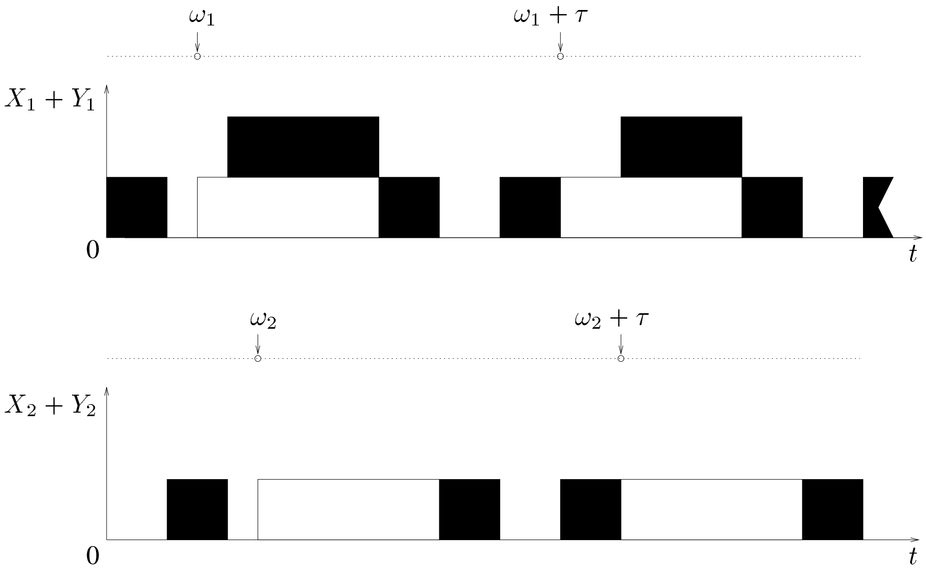

Figure 8: Cross flows "in phase" ($\Delta\omega = 1$)

We consider cases where

$$\tau = 2(2L - 1), \quad L \in \mathbb{N},\ L \neq 0.$$

As illustrated by Figures 8 and 9 in the case $\tau = 6$, the saturated system reaches a periodic regime of period τ. The number l of controlled customers which leave the system each period of time τ depends on the sample path $\omega = (\omega_1, \omega_2) \in \Omega$, and can take any value in

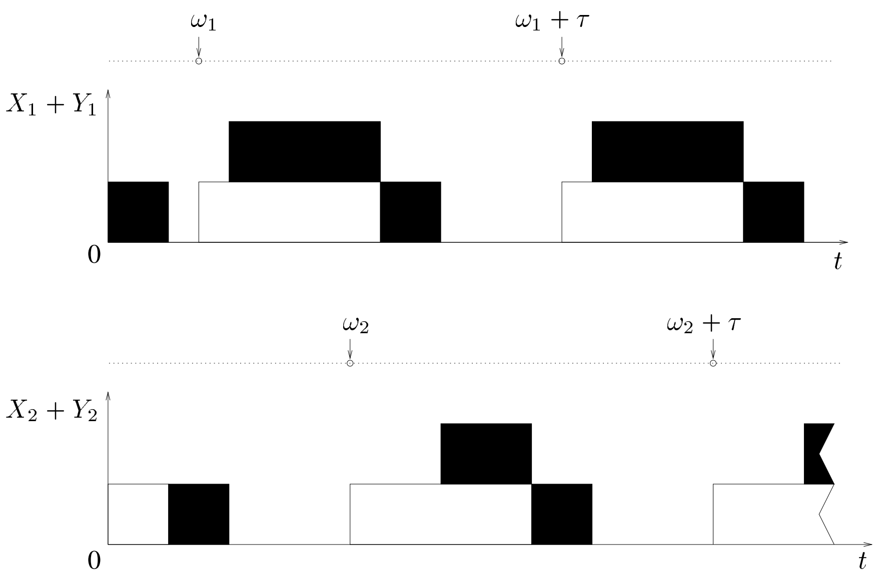

Figure 9: Cross flows "out of phase" ($\Delta\omega = 2.5$)

the finite set $\{1, \dots, L\}$. More precisely, defining $\Delta\omega$ as the unique element of the interval $(-\tau/2, \tau/2]$ such that

$$\omega_2 - \omega_1 = \Delta\omega \mod \tau,$$

we get

$$l = \begin{cases} L & \text{if} \quad 0 \le |\Delta\omega| \le 2, \\ L-1 & \text{if} \quad 2 < |\Delta\omega| \le 4, \\ \vdots & \qquad \vdots \\ 2 & \text{if} \quad 2(L-2) < |\Delta\omega| \le 2(L-1), \\ 1 & \text{if} \quad 2(L-1) < |\Delta\omega| \le 2L-1. \end{cases}$$

Therefore, there is a *spectrum* of L possible values for the maximum throughput of the controlled flow, depending on the relative phases of the cross flows. Noting that the probability of the event $\{|\Delta\omega| = 0\}$ is zero, we have a.s.

$$\bar{\lambda}(\omega) = \left[L + 1 - \frac{|\Delta\omega|}{2} \right] \lambda_1,$$

and

$$\mathbb{E}(\bar{\lambda}(\omega)) = \left(\sum_{l=1}^{L-1} \frac{2l}{2L-1} + \frac{L}{2L-1} \right) \lambda_1 = \frac{1}{2} \left(\frac{L}{2L-1} \right)^2.$$

Remark 11 *In this non–ergodic case,* $\lambda_{\min}$ *which is equal to*

$$\mu(K)(1-\rho_1)(1-\rho_2) = \frac{1}{8},$$

provides a lower bound on the expected *maximum throughput. In particular, Theorem 4 takes here the form*

$$\lim_{L\to+\infty} \mathbb{E}(\bar{\lambda}(\omega)) = \lambda_{\min}.$$

6 Conclusion

The performance of the window flow control in terms of *maximum throughput* of the controlled connection depends in a crucial way on the characteristics of the cross flows. A first surprising result is that this maximum throughput is neither monotone nor convex in the traffic of the cross flows. In particular, the performance of the flow control can improve when the intensity of the cross flows increases, or when these cross flows become more "random".

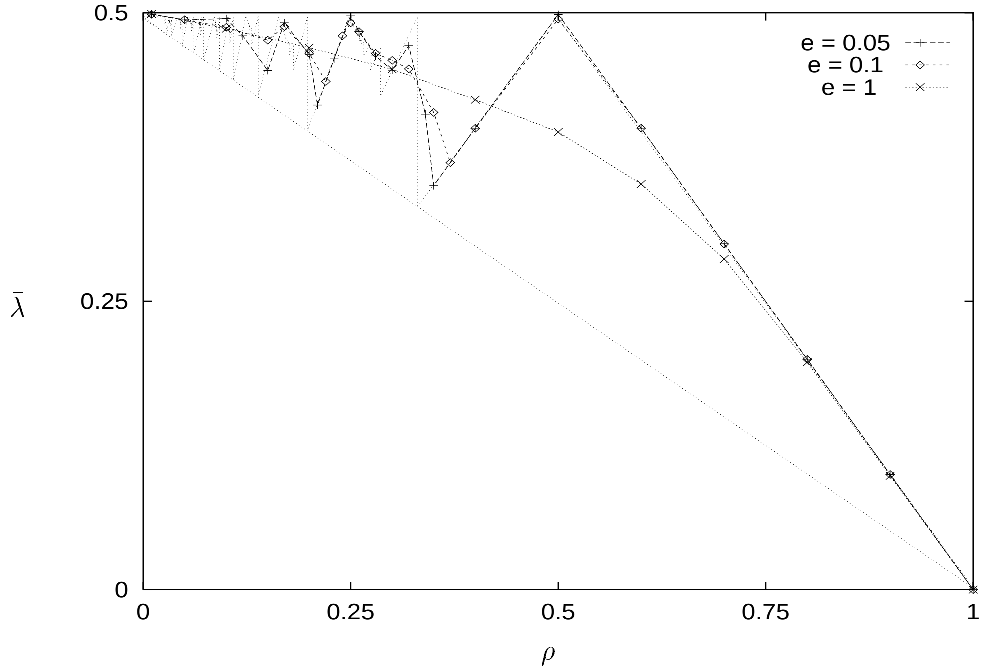

Figure 10: Effect of randomization

To illustrate this latter fact, consider the model of §5.1, where the periodic cross flow is replaced by a renewal point process with interarrival times uniformly distributed on the interval $[\tau(1-e), \tau(1+e)]$, with $0 \leq e \leq 1$ and $\tau \geq 1$. Figure 10 shows how the maximum throughput as a function of the traffic intensity of the cross flow $\rho = \tau^{-1}$, is "smoothed" when, starting from the periodic case $e = 0$, the randomization factor e takes the values 0.05, 0.1 and 1. These results were obtained by simulation of the associated saturated system, by a QNAP program available in [17].

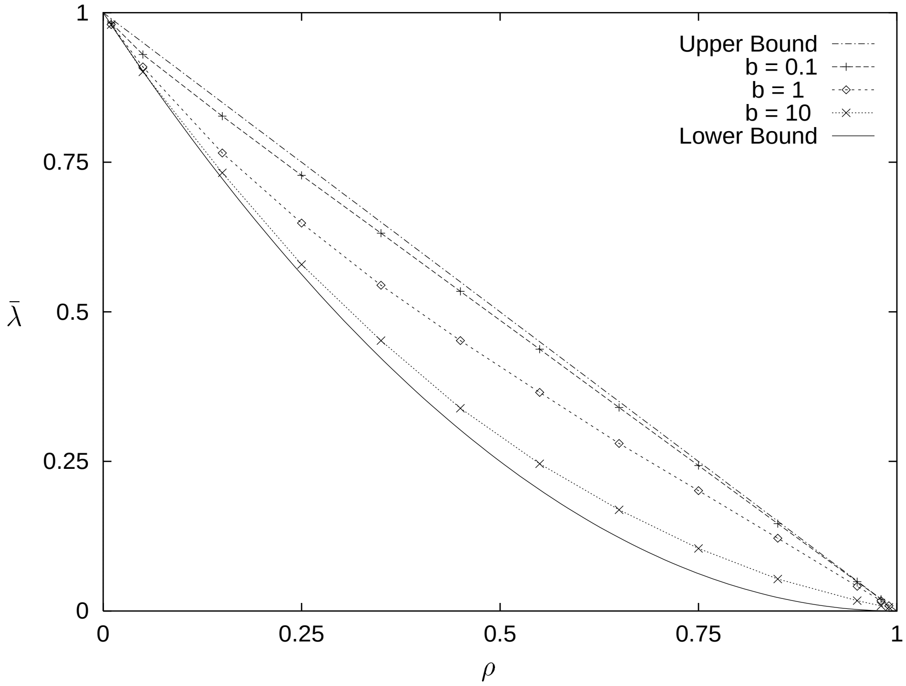

Figure 11: Impact of the burstiness of the cross flows ($N = 2$)

Another interesting result is that the maximum throughput of the controlled connection is very sensitive to the *burstiness* of the cross flows. Consider for instance the case where the cross flows are independent Poisson processes with deterministic service times equal to $b > 0$, and traffic intensities $\rho_1 = \ldots = \rho_N = \rho$. If the service times of the controlled customers are deterministic and equal to 1, and the window is $K = N$, the upper bound and the lower bound on the maximum throughput of the controlled flow are respectively given by

$$\lambda_{\max} = 1 - \rho \quad \text{and} \quad \lambda_{\min} = (1 - \rho)^N.$$

We have shown (see Theorems 3 and 4) that $\bar{\lambda}$ tends to these bounds when b tends to zero and to infinity, respectively. This is illustrated by the simulation results of Figures 11 and 12, where the maximum throughput $\bar{\lambda}$ and the utilization $U = \bar{\lambda}/\mu$ are represented as functions of the traffic intensity of the cross flows ρ, for different values of b (see [17] for the corresponding QNAP program). Thus when the cross flows consist of a *fluid* stream of small packets (generated by Telnet connections for instance), the utilization of the network resources by the controlled connection, given in view of Remark 6, by

$$U = \frac{\bar{\lambda}}{1-\rho},$$

is very close to its maximum value $U_{\max} = 1$, whereas when the cross flows consist of a *bursty* stream of large packets (generated by batch arrivals of packets due to scene changes in a video sequence for instance), the utilization of the network resources is close to its minimum value

$$U_{\min} = \frac{\lambda_{\min}}{1-\rho} = (1-\rho)^{N-1}.$$

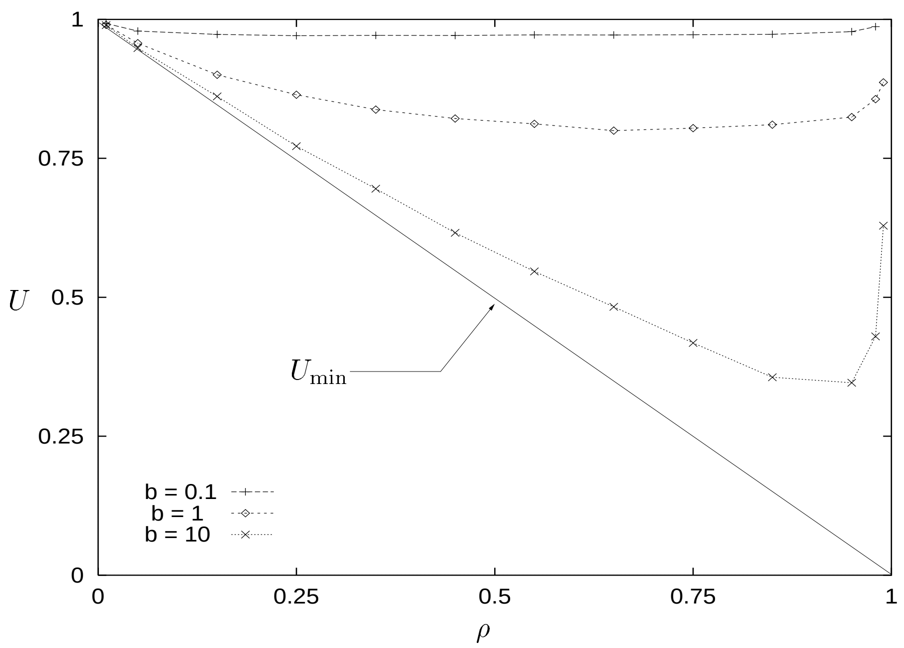

Figure 12: Utilization of the network resources ($N = 2$)

In particular, the effect of bursty cross flows on the performance of the controlled connection grows with the number of nodes N on the round–trip path followed by the packets of the controlled connection.

It is worth noting that these essential features of the window flow control are not captured by models with a *single bottleneck*, for which the utilization of the network resources is always equal to 1, and the maximum throughput of the controlled connection is simply given by the available bandwidth on the (single) shared link.

Acknowledgements. The authors would like to thank S. Foss for fruitful discussions on the present paper, and in particular for having suggested Remark 4.

Appendix A gives the main definitions on stationary and ergodic point processes used in this paper. For a more complete presentation on the subject, we refer the reader to [3]. Appendix B extends the results obtained in this paper to more general models, including the presence of propagation delays or the case of a multicast connection for instance.

A Stationary and ergodic point processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, on which a flow $\{\theta(t)\}_{t\in\mathbb{R}}$ is defined:

(i) $(t, \omega) \to \theta(t)\omega$ is measurable with respect to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ and $\mathcal{F}$,

(ii) $\theta(t)$ is bijective for all $t \in \mathbb{R}$,

(iii) $\theta(t) \circ \theta(s) = \theta(t+s)$ for all $t, s \in \mathbb{R}$.

Let $A = \{T_n\}_{n\in\mathbb{Z}}$ be a point process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the usual convention $T_0 \le 0 < T_1$, and $T_n \le T_{n+1}$, for all $n \in \mathbb{Z}$. We assume that A is *simple*, that is $\mathbb{P}(T_n < T_{n+1}) = 1$, for all $n \in \mathbb{Z}$. For any sample path $\omega \in \Omega$, we denote by $A(\omega, C)$ the number of points of A belonging to the Borel set C. We say that A is *compatible* with the flow $\{\theta(t)\}$, if

$$\forall t \in \mathbb{R}, \quad A(\theta(t)\omega, C) = A(\omega, t + C).$$

When the flow $\{\theta(t)\}$ is measure-preserving, that is $\mathbb{P} \circ \theta(t) = \mathbb{P}$ for all $t \in \mathbb{R}$, the point process A is *stationary*. The intensity of A is then defined by

$$\lambda = \mathbb{E}(A(0, 1]).$$

If λ is finite and non-null, we can define the Palm probability associated with A by

$$\forall F \in \mathcal{F}, \quad \mathbb{P}^0(F) = \lambda^{-1}\mathbb{E}\left[\int_{(0,1]} \mathbb{I}_F \circ \theta(t)\ A(dt)\right].$$

Note that $\mathbb{P}^0(T_0 = 0) = 1$, and on $\Omega_0 = \{T_0 = 0\}$, $\theta(T_n) = \theta(T_1)^n$, for all $n \in \mathbb{Z}$. Defining $\theta = \theta(T_1)$, we obtain then a *discrete* measure–preserving flow $\{\theta^n\}_{n\in\mathbb{Z}}$ on the Palm probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$.

We say that a sequence of random variables $\{\alpha_n\}_{n\in\mathbb{Z}}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence of *marks* for A, if

$$\forall n \in \mathbb{Z}, \quad \alpha_n(\omega) = \alpha_0(\theta(T_n)\omega).$$

Any sequence of marks of A (e.g. the interarrival times $\{T_{n+1} - T_n\}_n$) is *stationary* on the Palm probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$, since it is compatible with the discrete flow $\{\theta^n\}$.

The flow $\{\theta(t)\}$ is *ergodic* if for all $F \in \mathcal{F}$,

$$(\forall t \in \mathbb{R}, \quad \theta(t)F = F) \quad \Longrightarrow \quad \mathbb{P}(F) = 0 \text{ or } \mathbb{P}(F) = 1.$$

In this case, we know from Birkhoff Ergodic Theorem that for any random variable X defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\frac{1}{T}\int_0^T X \circ \theta(t)\ dt \quad \xrightarrow{a.s.} \quad \mathbb{E}(X) \quad \text{when} \quad T \to \pm\infty.$$

The ergodicity of $\{\theta(t)\}$ with respect to $\mathbb{P}$ implies the ergodicity of the discrete flow $\{\theta^n\}$ with respect to $\mathbb{P}^0$ (the converse is also true), so that

$$\frac{1}{N}\sum_{n=0}^{N-1} X \circ \theta(T_n) \quad \xrightarrow{a.s.} \quad \mathbb{E}^0(X) \quad \text{when} \quad N \to \pm\infty.$$

Hence, for any $\{\theta_t\}$–compatible process $\{X(t)\}$, that is

$$\forall t \in \mathbb{R}, \quad X(t) = X(0) \circ \theta(t),$$

the expectation of $X(0)$ under the Palm probability $\mathbb{P}^0$ is the empirical average of $\{X(t)\}$ *at events*, or *event average*, whereas the expectation of $X(0)$ under $\mathbb{P}$ is the usual *time average* of this process.

B Extensions to more general models

Our structural results do not depend on the topology of the network provided that in the absence of cross flows, the system is a single-input FIFO event graph [4]. In particular, these results apply in the presence of deterministic propagation delays $d_1, \ldots, d_N$ between the stations, or in the case of a multicast connection where the data packets are duplicated at some routers on the forward paths from the source to the destinations, and the acknowledgments aggregated at some other routers on the reverse paths from the destinations to the source. The corresponding models and the underlying event graphs are shown in Figures 13 and 14.

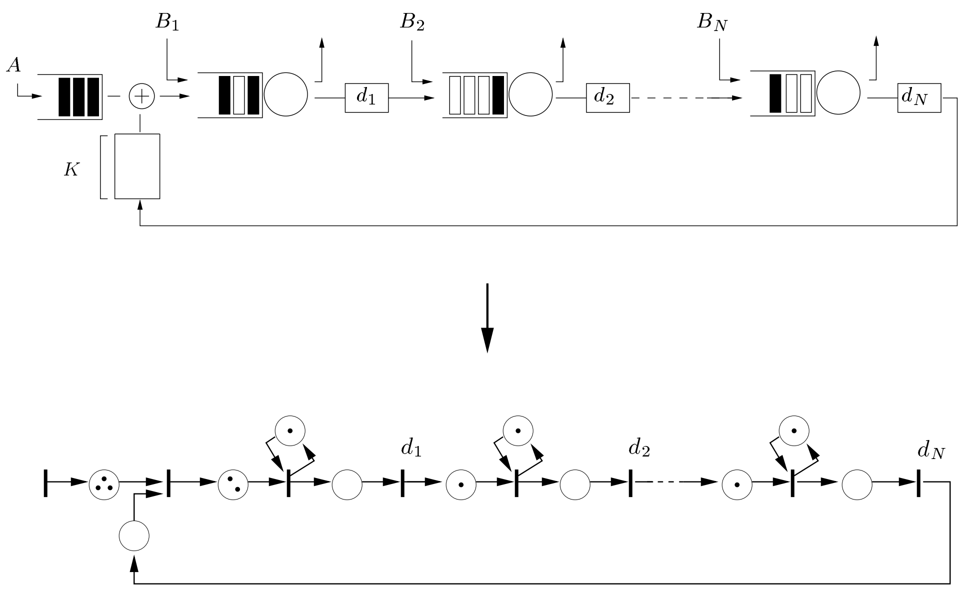

Figure 13: Model with propagation delays

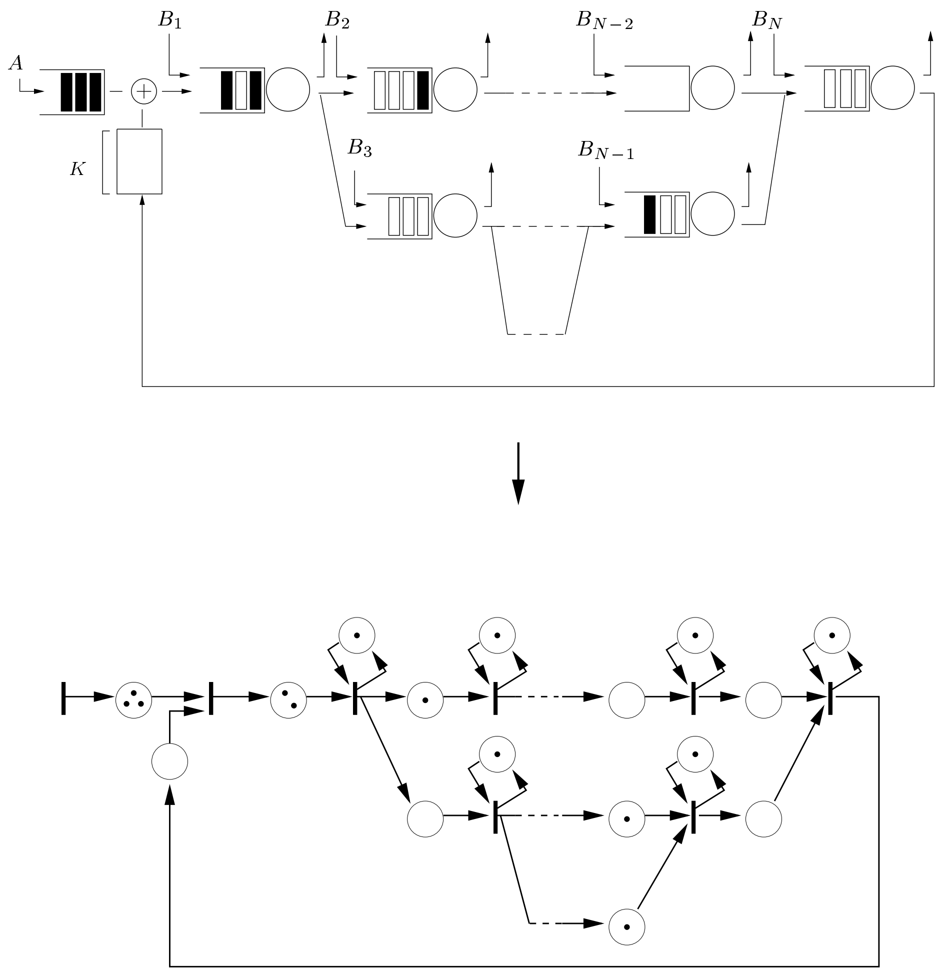

Figure 14: Multicast connection

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