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ANALYSIS OF A TWO-NODE ALOHA-NETWORK WITH INFINITE CAPACITY BUFFERS

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Considered is a two-node ALOHA-network with infinite capacity buffers. Time is slotted and at the beginning of each time-slot a station sends a packet, if any, to a central station with a constant probability. When two transmissions occur in the same time-slot both messages have to be retransmitted in a later time-slot. For geometric arrivals we obtain the generating function in closed form of the stationary joint queue length distribution with the aid of the theory of boundary value problems for regular functions. Ergodicity conditions are derived and exact numerical results are provided for the mean response time of each station.

INTRODUCTION

We consider a radio packet-switching model of the ALOHA-type [1,2] consisting of two stations each with an infinite capacity buffer and a single server. Packets have equal length and the time is divided into slots corresponding to the transmission time of a packet.

At the beginning of each slot, station j, (j=1,2) if it is non-empty, transmits a packet with probability r_j to a central station. If both stations send a packet during the same time-slot, there is a collision, and the two packets have to be retransmitted in a later time-slot, following the above procedure.

Our model differs from the numerous related works because the packets may enter the system whatever the station state may be. Indeed, the common assumption in this area is that a station cannot store more than one packet at a time ([8,14,15] among others).

Although the classical model allows detailed analyses for an arbitrary number of stations and also for more sophisticated transmission protocols [22,23], it cannot provide the real response time of the system, namely the time which elapses between the arrival of a packet to the station until its successful transmission (including the waiting time before the first transmission); however this quantity is of main interest for the user.

The model we consider in this paper has also been investigated by **SIDI** and **SEGALL** [20] (see also [21] for a related model) in the symmetrical case, namely $r_1 = r_2$ and identical distributions for the arrival processes. By taking advantage of the symmetry of the model they were able to derive the mean response times without explicitely computing the generating function for the stationary joint queue-length distribution.

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This paper is devoted to the resolution of the non-symmetrical case, (at least for particular arrival processes). The key point is the resolution of a functional equation of the following type : (cf.[20,eq.4]) $K(x,y)F(x,y) = a(x,y)F(0,y) + b(x,y)F(x,0) + c(x,y)F(0,0)$ for $|x| \leq 1$, $|y| \leq 1$, where the functions a,b,c,K are known and where F(x,y) is the generating function for the stationary joint queue-length distribution. Indeed, it turns out that in the non-symmetrical case the knowledge of $F_x^{(1)}(1,1)$ and $F_y^{(1)}(1,1)$ -which allows for instance the computation of the mean response time of each station using Little's formula- is equivalent to the knowledge of F(x,y) for $|x| \leq 1$, $|y| \leq 1$ (here $F_x^{(1)}(1,1) = \frac{d}{dx} F(x,y)|_{(1,1)}$ and similarly for $F_y^{(1)}$).

It is now well known that, for particular a,b,c,K (or equivalently for particular arrival process distributions), this type of functional equation can be solved by formulating a boundary value problem. This has been first shown by **FAYOLLE** and **IASNOGORODSKI** [9] and their method has been extended (to more general random walks) by **COHEN** and **BOXMA** [6]. Up to now several queueing systems have been successfully investigated using this machinery [3,4,7,10,16,18], all these studies leading to a fair methodology in this field.
The paper is organized as follows : the model is precisely defined in Section 1 and the related functional equation for the generating function of the stationary joint queue-length distribution is established. Then some basic properties of the kernel of this equation are derived in Section 2, leading to the formulation (Section 3) and to the resolution (Section 4) of two boundary value problems (Dirichlet and Riemann-Hilbert problems). The sought generating function can then be obtained in closed form as well as the ergodicity conditions. Exact numerical results are given for the mean response time of each station (Section 5).

1 - The model and the related functional equation.

We now defined more precisely the queueing model under consideration.
Let $\{A_j(t)\}_{t \in \mathbb{N}^*}$ be a sequence of i.i.d. random variables where $A_j(t)$ represents the number of packets which arrive to station j (j=1,2) in the time interval (t,t+1]. We first assume that $A_1(t)$ and $A_2(s)$ are independent random variables whenever $t \neq s$ and possibly correlated when t = s. We define H(x,y) the generating function for the joint distribution of the number of arrivals in any slot, that is

(1.1) $$H(x,y) = E\{x^{A_1(t)} y^{A_2(t)}\}, \ |x|, |y| \leq 1, \ t \in \mathbb{N}^*.$$

Let $N_j(t)$ be the number of packets in station j at the beginning of the t-th slot.
From the description of the model it is readily seen that $\{(N_1(t),N_2(t)),t=1,2,...\}$ is a homogeneous Markov chain with irreducible state space $\mathbb{N} \times \mathbb{N}$ and that the following relation is satisfied for $t \in \mathbb{N}^*$, $|x|, |y| \leq 1$:

(1.2) $$E\{x^{N_1(t+1)} y^{N_2(t+1)}\} = H(x,y)[P(N_1(t) = N_2(t) = 0)$$
$$+ (\frac{r_1}{x} + 1 - r_1) E\{x^{N_1(t)} 1(N_1(t) > 0, N_2(t)=0)\}$$
$$+ (\frac{r_2}{y} + 1 - r_2) E\{y^{N_2(t)} 1(N_1(t)=0, N_2(t) > 0)\}$$
$$+ (\frac{r_1(1-r_2)}{x} + \frac{(1-r_1)r_2}{y} + r_1r_2 + (1-r_1)(1-r_2)) E\{x^{N_1(t)} y^{N_2(t)} 1(N_1(t) > 0, N_2(t) > 0)\}]$$ where

(A) denotes the indicator function of the event A.
By assuming that the system is stable and by introducing

(1.3) $$F(x,y) = \lim_{t \to \infty} E\{x^{N_1(t)} y^{N_2(t)}\} \quad \text{for } |x|, |y| \leq 1,$$

we obtain from (1.2) the following functional equation :

(1.4) $$K(x,y)F(x,y) = a(x,y)F(0,y) + b(x,y)F(x,0) + c(x,y)F(0,0) \quad \text{for } |x|, |y| \leq 1 \text{ where}$$

(1.5) $$K(x,y) = H^{-1}(x,y) - 1 + r_1(1-r_2)(1-\frac{1}{x}) + (1-r_1)r_2(1-\frac{1}{y}),$$

(1.6) $$a(x,y) = r_1[(1-r_2)(1-\frac{1}{x}) - r_2(1-\frac{1}{y})],$$

(1.7) $$b(x,y) = r_2[(1-r_1)(1-\frac{1}{y}) - r_1(1-\frac{1}{x})],$$

(1.8) $$c(x,y) = r_1r_2[2 - \frac{1}{x} - \frac{1}{y}].$$

Some interesting relations can be immediately derived from (1.4). Taking y = 1, dividing by (x-1) and then taking x = 1 in (1.4) and vice versa yields the following "conservation of flow" relations

(1.9) $$\lambda_1 = r_1(1-r_2)[1-F(0,1)] + r_1r_2[F(1,0) - F(0,0)],$$

(1.10) $$\lambda_2 = (1-r_1)r_2[1-F(1,0)] + r_1r_2[F(0,1) - F(0,0)],$$

where $\lambda_j \triangleq E\{A_j(t)\}$ for j=1,2. We assume $\lambda_j > 0$, j=1,2.

For $\underline{r_1 + r_2 = 1}$ the above relations give

(1.11) $$F(0,0) = 1 - \frac{\lambda_1}{r_1} - \frac{\lambda_2}{r_2}.$$

From this we immediately deduce that

(1.12) $$\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} < 1$$

is a necessary condition for the stability of the system if $r_1+r_2 = 1$. From now on we will assume that this condition is satisfied if $r_1+r_2 = 1$.

In the case $\underline{r_1 + r_2 \neq 1}$ we get from (1.9) and (1.10) :

(1.13a) $$F(0,1) = \frac{1-r_1 - \frac{\lambda_1(1-r_1)}{r_1} - \lambda_2 - r_2 F(0,0)}{1-r_1-r_2},$$

(1.13b) $$F(1,0) = \frac{1-r_2-\lambda_1 - \frac{\lambda_2(1-r_2)}{r_2} - r_1F(0,0)}{1-r_1-r_2}.$$

When both queues are non-empty the successful transmission rate for station 1 is clearly $r_1(1-r_2)$, while this transmission rate for station 2 is $(1-r_1)r_2$.
Consequently it is seen from classical results on the single server queue [5] that under the following condition

(1.14) $$\lambda_1 \geq r_1(1-r_2) \ \underline{\text{and}} \ \lambda_2 \geq (1-r_1)r_2$$

both queues remain unbounded with probability 1.

Therefore it turns out that the following conditions

(1.15) $\lambda_1 < r_1(1-r_2)$ <u>or</u> $\lambda_2 < r_2(1-r_1)$

are necessary conditions for the system to be stable. (In particular the system is always unstable if $r_1=r_2=1$ which is obvious since in that case a collision will be infinitely repeated).

From now on, we will assume without loss of generality that

(1.16) $\lambda_1 < r_1(1-r_2)$.

We now turn to our objective of resolving the functional equation (1.4) under the condition $\lambda_1 < r_1(1-r_2)$. It is easily seen that the <u>"kernel"</u> $K(x,y)$ of eq.(1.4) can be rewritten as

(1.17) $K(x,y) = \dfrac{xy - \Psi(x,y)}{xyH(x,y)}$ for $|x|, |y| \le 1$,

where $\Psi(x,y)$ is a generating function of a proper probability distribution of two $\mathbb{N}$-valued random variables $\underline{x}$ and $\underline{y}$, with

$$E\{\underline{x}\} = 1+\lambda_1 - r_1(1-r_2) \quad , \quad E\{\underline{y}\} = 1+\lambda_2 - (1-r_1)r_2, \text{ cf.[19]}.$$

Multiplying both sides of eq.(1.4) by $xy\,H(x,y)$, it is then seen that the resulting functional equation is of the same type as the one considered in [6,p.81]. However the method developed by **COHEN** and **BOXMA** for solving this type of functional equation requires that $E\{\underline{x}\} < 1$ <u>and</u> $E\{\underline{y}\} < 1$ or equivalently that

(1.18) $\lambda_1 < r_1(1-r_2)$ <u>and</u> $\lambda_2 < (1-r_1)r_2$. This is a rather strong restriction for our model, since clearly condition (1.18) is not a necessary condition for the ergodicity of the considered Markov chain, as we will show in a particular case.

We solve the functional equation (1.4) only for a particular distribution of the arrival processes at both stations, namely the <u>geometric distribution</u>. We also assume that both arrival processes are independent.

More precisely, it will be assumed from now on that

(1.19) $H(x,y)=[(1+\lambda_1(1-x))\,(1+\lambda_2(1-y))]^{-1}$ for $|x|, |y| \le 1$.

We have chosen the geometric distribution for sake of convenience. A similar analysis to the one developed in the forthcoming sections could also be carried out for other arrival distributions (e.g. Bernoulli distribution,...).

More generally, the key property for solving eq.(1.4) is the following : the right-hand side of (1.4) must vanish whenever the "kernel" $K(x,y)$ vanishes for $|x| \le 1$, $|y| \le 1$, since $F(x,y)$ is sought analytic in $|x| < 1$, $|y| < 1$ and continuous in $|x| \le 1$, $|y| \le 1$.

Consequently, the equation $K(x,y) = 0$ has to be carefully studied.

In connection with this, it therefore turns out that the geometric (or Bernoulli) distribution is of main interest, since they are first, a natural one in radio packet-switching area, and also because in this case $T(x,y) \triangleq xyK(x,y)$ is a polynomial of second degree w.r.t. each variable x and y, which therefore allows an explicit analysis of the kernel.

(For a study dealing with a polynomial $T(x,y)$ of third degree w.r.t. each variable x and y, the interested reader is referred to [16]).

The next sections are devoted to the resolution of eq.(1.4) when the arrival processes in both stations are <u>independent</u> with <u>geometric distributions</u>.

2 - Analysis of the kernel

In this section we obtain some preparatory results in view of the resolution of the functional equation (1.4). We first focus our attention on the kernel $K(x,y)$ of equation (1.4).

This kernel is given by cf.(1.5), (1.19),

(2.1) $K(x,y) = \lambda_1(1-x)+\lambda_2(1-y)+\lambda_1\lambda_2(1-x)(1-y) + r_1(1-r_2)(1-\frac{1}{x})+(1-r_1)r_2(1-\frac{1}{y})$.

Solving for x the equation $K(x,y) = 0$, we get

(2.2) $X(y) = \dfrac{\lambda_1((1+\lambda_2(1-y))+r_1(1-r_2)+A(y) \pm |\Delta(y)|^{1/2}e^{i\,[\arg\Delta(y)]/2}}{2\lambda_1(1+\lambda_2(1-y))}$

where

(2.3) $A(y) = \lambda_2(1-y)+(1-r_1)r_2(1-\frac{1}{y})$, $(-\pi < \arg\Delta(y) \le \pi)$

(2.4) $\Delta(y) = t(y,0)t(y,\pi)$ with

(2.5) $t(y,\phi) = \lambda_1(1+\lambda_2(1-y))+r_1(1-r_2)+A(y)-2\cos\phi\sqrt{\lambda_1 r_1(1-r_2)(1+\lambda_2(1-y))}$.

From the implicit function theorem [11,p.10] and (2.2) we get that the equation $K(x,y) = 0$ has one root $x = x(y)$ which is an analytic function of y in the complex plane cut along $[y_1,y_2] \cup [y_3,y_4]$, where y_1,y_2,y_3,y_4 are the four zeros of $\Delta(y)$ (the branch points of $X(y)$).

In order to locate the zeros of $\Delta(y)$ we state the following general lemma :

<u>Lemma 2.1</u>

For $\phi \in [0,2\pi]$, the equation $t(y,\phi) = 0$ has exactly two (real) roots $y=h_1(\phi)$ and $y=h_2(\phi)$ with $0 < h_1(\phi) < 1 < h_2(\phi) < (1+\lambda_2)/\lambda_2$.

<u>Proof</u>

We have $t(0^+,\phi)=-\infty$, $t(1,\phi)=\lambda_1+r_1(1-r_2)-2\cos\phi\sqrt{\lambda_1 r_1(1-r_2)} > 0$ and

$$t(\frac{1+\lambda_2}{\lambda_2},\phi) = -1+r_1(1-r_2) + \frac{(1-r_1)r_2}{1+\lambda_2} < 0.$$

Consequently, it is seen that for $\phi \in [0,2\pi]$, $t(y,\phi)$ has (at least) two real roots $h_1(\phi)$ and $h_2(\phi)$ which satisfy the following inequalities

(2.6) $0 < h_1(\phi) < 1 < h_2(\phi) < (1+\lambda_2)/\lambda_2$.

Noting now that $y^2 t(y,\phi)t(y,\phi+\pi)$ is a polynomial of degree four in the variable y, we deduce from the previous results that $t(y,\phi)$ has exactly two real roots $h_1(\phi)$ and $h_2(\phi)$ satisfying (2.6). □

From this lemma and (2.5), (2.6), it is readily seen that $y_1=h_1(\pi)$, $y_2=h_1(0)$, $y_3=h_2(0)$, $y_4=h_2(\pi)$ with

(2.7) $0 < y_1 < y_2 < 1 < y_3 < y_4 < \dfrac{1+\lambda_2}{\lambda_2}$.

As a second result it is shown in Appendix A that the equation $K(x,y) = 0$ has for $|y| = 1$ exactly one zero $x = x(y)$ such that $|x(y)| \le 1$. Let us denote $x(y)$ the algebraic branch defined by $K(x,y) = 0$ which satisfies the condition $|x(y)| \le 1$ for $|y| = 1$. The other zero of the equation $K(x,y) = 0$ is denoted by $x^\sigma(y)$. By writting

$$\arg\Delta(y) = \sum_{i=1}^{4} \arg(y-y_i) - 2\arg(y),$$ it turns out that for $y \in \mathbb{C}/([y_1,y_2] \cup [y_3,y_4])$

the plus and minus signs in (2.2) correspond to $x(y)$ and $x^\sigma(y)$ respectively (compute $x(1)$ and $x^\sigma(1)$).

Similar results also hold when x is fixed. In that case $y(x)$ will denote the root of $K(x,y)$ which satisfies the condition $|y(x)| \le 1$ for $|x| = 1$, while $y^\sigma(x)$ will denote the other root. $y(x)$ is analytic in $\mathbb{C}$ cut along $[x_1,x_2] \cup [x_3,x_4]$, where x_1,x_2,x_3,x_4 denote the four (real) branch points of $y(x)$. These points satisfy the following inequalities

$$0 < x_1 < x_2 \le 1 < x_3 < x_4 < \frac{1+\lambda_1}{\lambda_1} \quad (x_2 = 1 \text{ iff. } \lambda_2 = (1-r_1)r_2).$$

For concluding this section, we now investigate the image of the cut $[y_1,y_2]$ by the branch $x(y)$.

Define

$$\rho(\phi) = \left(\frac{r_1(1-r_2)}{\lambda_1(1+\lambda_2(1-h_1(\phi)))}\right)^{1/2}.$$ We have the following

Lemma 2.2

For $\phi \in [0,2\pi]$, $x(h_1(\phi)) = \rho(\phi)e^{i\phi}$.

Proof

One easily shows that the point $y=h_1(\phi)$ sweeps twice the cut $[y_1,y_2]$ when ϕ traverses the real interval $[0,2\pi]$. Therefore for $y=h_1(\phi)$, $\Delta(y)^2 < 0$ and consequently $x(y)$ and $x^\sigma(y)$ are conjugate complex numbers whose the product satisfies the relation, cf (2.1) :

$$x(h_1(\phi))\, x^\sigma(h_1(\phi)) = |x(h_1(\phi)|^2 = \rho^2(\phi).$$

Using now the definition of the algebraic branch $x(y)$, we readily get that $x(h_1(\phi)) = \rho(\phi)e^{i\phi}$, $\phi \in [0,2\pi]$. ◻

Define $L_x \stackrel{\Delta}{=} \{x \in \mathbb{C} : x = \rho(\phi)e^{i\phi},\ \phi \in [0,2\pi]\}$.

It is easily verified that L_x is a smooth closed contour symmetric w.r.t. the real axis, with $0 \in L_x^+$. (L_x^+ denotes the interior of L_x).

Finally the following important relation will be used in the next sections :

(2.8) $y(x(y)) = y$ for $y \in [y_1,y_2]$ (cf. Appendix B).

3 - Formulation of the boundary value problem

We follow the procedure given in [9].

For pairs (x,y) with $K(x,y) = 0$, $|x| \le 1$, $|y| \le 1$, the following relation between $F(x,0)$ and $F(0,y)$ must hold (cf.(1.4)-(1.8)) :

(3.1) $$r_2\left[(1-r_1)(1-\frac{1}{y})-r_1(1-\frac{1}{x})\right]F(x,0)+r_1\left[(1-r_2)(1-\frac{1}{x})-r_2(1-\frac{1}{y})\right]F(0,y) + r_1r_2\left[2-\frac{1}{x}-\frac{1}{y}\right]F(0,0) = 0.$$

For $r_1 + r_2 = 1$ this equation reduces to

(3.2) $$(1-r_1)F(x,0)-r_1F(0,y) + \frac{(1-r_1)c(x,y)}{b(x,y)}F(0,0) = 0,$$

where $F(0,0)$, $a(x,y)$ and $c(x,y)$ are given by (1.11), (1.6), (1.8) respectively.

For $r_1 + r_2 \ne 1$ equation (3.1) can be rewritten as

(3.3) $G(x,0)\, b(x,y) + G(0,y)\, a(x,y) = 0,$

where

(3.4) $$G(x,0) = F(x,0) + \frac{r_1F(0,0)}{1-r_1-r_2},$$

(3.5) $$G(0,y) = F(0,y) + \frac{r_2F(0,0)}{1-r_1-r_2}.$$

Next define $D = \{y \in \mathbb{C} / |y| \le 1, |x(y)| \le 1\}$, $\bar{D} = \{y \in \mathbb{C} / |y| \le 1, |x(y)| > 1\}$.

Note that D is non-empty from Appendix A.

We have :

for $r_1 + r_2 = 1$

(3.6) $$(1-r_1)F(x(y),0)-r_1F(0,y) - \frac{(1-r_1)c^1(y)}{b^1(y)}F(0,0) \quad \text{for } y \in D,$$

for $r_1 + r_2 \ne 1$

(3.7) $b^1(y)G(x(y),0) + a^1(y)G(0,y) = 0$ for $y \in D$,

where

(3.8) $a^1(y)=a(x(y),y)$, $b^1(y)=b(x(y),y)$, $c^1(y)=c(x(y),y)$.

In the light of eqs. (3,6), (3.7), we may write some results concerning the ergodicity conditions of the system.

Indeed for $y \in D^* \stackrel{\Delta}{=} D/[y_1,y_2]$ the functions $F(0,y)$ and $F(x(y),0)$ are both analytic. This entails from (3.6), (3.7) that $a^1(y)$ and $b^1(y)$ must not vanish in D^* otherwise $F(0,y)$ and/or $F(x,0)$ would have poles in $|x| \le 1$, $|y| \le 1$.

To this end it is shown in [19, Appendix D] that for $\lambda_1 < r_1(1-r_2)$ the conditions $a^1(y) \ne 0$ and $b^1(y) \ne 0$ are satisfied for $y \in D^*$ iff.

(3.9) $\lambda_2 < r_2(1-r_1)$ or $[\lambda_2 \ge r_2(1-r_1)$ and $\lambda_1r_2+\lambda_2(1-r_2) < r_2(1-r_2)]$.

Consequently, conditions (3.9) are necessary conditions for the stability of the system, and in the following we shall assume that they are satisfied. (actually conditions (3.9) are also sufficient, see Section 4).

Remark : Note that (1.16), (3.9), reduce to condition (1.12) when $r_1+r_2=1$.

We now proceed with the analytic continuation of the function $F(x,0)$ outside the unit disk, which turns out to be a crucial point of the method used hereafter (cf. the pioneering paper [9]).

When y is in the region $|y| \leq 1$, where $F(0,y)$ is analytic, $x(y)$ is in a region containing the curve L_x. Consequently (3.6) [resp.(3.7)] can be used to continue $F(x,0)$ [resp. $G(x,0)$] as a meromorphic function up to $\bar{L}_x$. The eventual poles of $F(x,0)$ [resp. $G(x,0)$] are the zeros of $b^1(y)$ for $y \in \bar{D}$.

It is shown in Appendix C that $b^1(y)$ has no zeros for $y \in \bar{D}$. Consequently, $F(x,0)$ [resp. $G(x,0)$] can be continued analytically up to L_x using (3.7) [resp.(3.8)]. Taking $y \in [y_1,y_2]$ in (3.6) [resp.(3.7)], then multiplying the relation by the complex number i^2 and using the fact that $F(0,y)$ has in $|y| \leq 1$ a power series expansion with positive coefficients, it comes up using Lemma 2.2 and (2.8) that:

<u>for $r_1 + r_2 \neq 1$</u>

(3.10) $$\mathrm{Re}\{i \frac{b^2(x)}{a^2(x)} G(x,0)\} = 0 \qquad \text{for } x \in L_x,$$

<u>for $r_1 + r_2 = 1$</u>

(3.11) $$\mathrm{Re}\{i F(x,0)\} = \mathrm{Re}\{-i \frac{c^2(x)}{b^2(x)}\}F(0,0) \qquad \text{for } x \in L_x,$$

where

(3.12) $$a^2(x) = a(x,y(x)),\ b^2(x)=b(x,y(x)),\ c^2(x)=c(x,y(x)).$$

The equation (3.10) defines an <u>homogeneous Riemann-Hilbert</u> boundary value problem, cf.[13,p.220], [17,p.99], <u>that is find a function $G(x,0)$</u> analytic in L_x', continuous in $L_x^+ \cup L_x$, satisfying (3.10) where $\frac{b^2(x)}{a^2(x)}$ is a non-vanishing function on L_x. (This last property is a consequence of the results obtain in [19, Appendix D].

Similarly the equation (3.11) defines a <u>Dirichlet</u> boundary value problem, cf.[13,p.221], [17,p.107], that is find a <u>function</u> $F(x,0)$ analytic in L_x' continuous in $L_x \cup L_x$, satisfying (3.11) where $\frac{c^2(x)}{b^2(x)}$ is a non-vanishing function on L_x. (This function only vanishes for $x = 1$ which does not belong to L_x under the condition (1.16)).

4. The solution of the boundary value problems

The solutions of the two boundary value problems formulated in the previous section are known whenever L_x is the unit circle. Therefore we must transform the boundary conditions (3.10) and (3.11) conformally to the unit circle.

To this end we have the following

<u>Lemma 4.1</u>

The conformal mapping $\gamma_0(z)$ from the unit circle onto the curve L_x satisfying $\gamma_0(0) = 0$ and $\gamma_0(z) = \overline{\gamma_0(\bar{z})}$ is uniquely determined by :

(4.1) $$\gamma_0(z) = z \exp[\frac{1}{2\pi}\int_0^{2\pi} \log\rho(\theta(\xi)) \frac{e^{i\xi}+z}{e^{i\xi}-z} d\xi] \text{ for } |z| < 1,$$

where $\theta(\omega)$ is the unique continuous and strictly increasing solution in $[0,2\pi]$ of the following <u>Theodorsen integral equation</u> : for $\xi \in [0,2\pi]$

(4.2) $$\theta(\xi) = \xi - \frac{1}{2\pi}\int_0^{2\pi} \log(\rho(\theta(\omega))) \cot(\frac{1}{2}(\omega-\xi))d\omega,$$

(4.3) $$\theta(\xi) = -\theta(-\xi).$$

Moreover γ_0 maps conformally $\{|z| = 1\}$ onto L_x and

(4.4) $$\gamma_0(e^{i\xi}) = \rho(\theta(\xi))e^{i\theta(\xi)}.$$

<u>Proof</u>

These results can be found in [3,p.70-73]. Note that since L_x is symmetric w.r.t. the real axis, $\gamma_0(z)$ can be chosen such that $\gamma_0(z) = \overline{\gamma_0(\bar{z})}$ for $|z| \leq 1$. □

We will denote by $\gamma(z)$ the inverse of $\gamma_0(z)$. Using the above lemma the unique solution (up to a constant) of the Dirichlet boundary value problem formulated by (3.11) reads, cf.[13,p.221], [17,p.108],

<u>for $r_1 + r_2 = 1$</u>

(4.5) $$F(x,0) = \frac{-F(0,0)}{2\pi}\int_{|t|=1} f(t)(\frac{t+\gamma(x)}{t-\gamma(x)})\frac{dt}{t} + C \qquad \text{for } x \in L_x^+,$$

where

C is a constant,

(4.6) $$f(t) = \mathrm{Re}\{-i \frac{c^2(\gamma_0(t))}{b^2(\gamma_0(t))}\}, \quad |t| = 1.$$

Using (1.11) and $\gamma(0) = 0$, cf. Lemma 4.1, we get

(4.7) $$C = \left(1 - \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2}\frac{1}{2\pi}\int_{|t|=1} f(t)\frac{dt}{t}\right).$$

The constant C is determined as follows. First, note that $\gamma_0(e^{i\phi})=\rho(\theta(\phi))e^{i\theta(\phi)}$, cf.(4.4) and that $y(\gamma_0(e^{i\phi})) = h_1(\theta(\phi))$ with (2.8), (4.4) and Lemma 2.2. So from (3.12), (4.6), an easy calculation yields

$$f(e^{i\phi}) = \frac{r_1 \sin(\theta(\phi))\,(1-h_1(\theta(\phi))^{-1})}{\rho(\theta(\phi))[\{(1-r_1)(1-\frac{1}{h_1(\theta(\phi))})-r_1(1-\frac{\cos(\theta(\phi))}{\rho(\theta(\phi))})\}^2+\{r_1\frac{\sin(\theta(\phi))}{\rho(\theta(\phi))}\}^2]}.$$

From (4.3) we deduce that $f(e^{i\phi})$ is an odd function of ϕ.

This implies with (4.7) that $C = 1 - \frac{\lambda_1}{r_1} - \frac{\lambda_2}{r_2}$ and finally that

(4.8) $$\boxed{F(x,0)=(1-\frac{\lambda_1}{r_1}-\frac{\lambda_2}{r_2})(\frac{-2\gamma(x)}{\pi}\int_0^{\pi}\frac{f(e^{i\phi})\sin\phi\, d\phi}{1-2\gamma(x)\cos\phi+\gamma(x)^2} + 1)} \qquad \text{for } x \in L_x^+.$$

Let us now consider the case $r_1 + r_2 \neq 1$.

Define

$\chi = \frac{-1}{\pi}\left[\arg \frac{b^2(x)}{a^2(x)}\right]_{x \in L_x}$ the index of the homogeneous Riemann-Hilbert boundary value problem defined by eq.(3.10). ($[\arg\alpha(t)]_{t\in C}$ denotes the variation of the argument of the function $\alpha(t)$ when t moves along any closed contour C in the positive direction, provided that $\alpha(t) \neq 0$ for $t \in C$).

As a result, cf.[13,p.221], [17,p.104], we have that (3.10) has a unique solution $G(x,0)$ (up to a constant) which is analytic in L_x^+, continuous in $L_x^+ \cup L_x$ iff. $\chi = 0$.

Let us show that $\chi = 0$ under (1.16), (3.9).

It is shown in [19, Appendix D] that if (1.16), (3.9) hold then $x(y_2)>1$. With this result and the fact that $x(y_1)<0$, we immediately deduce that, cf. Lemma 2.2

$$\mathrm{sgn}\left(\frac{b^1(y_2)}{a^1(y_2)}\right).\mathrm{sgn}\left(\frac{b^1(y_1)}{a^1(y_1)}\right) > 0,$$ where sgn(.) denotes "the sign of".

Using now the one to one mapping of $[y_1,y_2]$ onto L_x, cf.(2.8), as well as (3.8) and (3.12), we readily deduce that $\chi = 0$.

The solution of (3.10) reads using (3.4) :

$$(4.9)\quad F(x,0)=D\exp\left[\frac{1}{2i\pi}\int_{|t|=1}\frac{\log g(t)}{t-\gamma(x)}dt\right] - \frac{r_1}{1-r_1-r_2}F(0,0) \text{ for } x\in L_x^+,$$

where

D is a constant,

$$g(t) = \frac{\overline{g_1(t)}}{g_1(t)}, \quad g_1(t) = \frac{b^2(\gamma_0(t))}{a^2(\gamma_0(t))} \quad \text{for } |t| = 1.$$

Making x=0 in (4.9) we obtain the constant D in term of F(0,0). Then combining (1.13b) and (4.9) for x=1, we get the constants D and F(0,0). Putting these results into (4.9) finally gives :

$$(4.10)\quad \boxed{\begin{aligned} F(x,0)=&\left(\frac{1-\frac{\lambda_1}{1-r_2}-\frac{\lambda_2}{r_2}}{1-r_1-r_2}\right)\Bigg[(1-r_2)\exp\left\{\frac{\gamma(x)-\gamma(1)}{2i\pi}\int_{|t|=1}\frac{\log g(t)dt}{(t-\gamma(x))(t-\gamma(1))}\right.\\ &- r_1\exp\left\{\frac{-\gamma(1)}{2i\pi}\int_{|t|=1}\frac{\log g(t)}{t-\gamma(1)}\frac{dt}{t}\right\}\Bigg]\end{aligned}}$$

for $x \in L_x^+$.

Remark : Similar arguments to those employed in the case $r_1+r_2=1$ allow to tranform the curvilinear integrals of (4.10) into integrals over the real interval $[0,\pi]$. This is not done here for sake of brevity (cf. [19]).

In the case where the unit disk is not entirely contained in L_x^+, we need to analytically continue relations (4.8), (4.10) up to the unit circle to obtain $F(x,0)$ for all $|x| \leq 1$.

If conditions (1.16), (3.9) are fulfilled, this analytic continuation can be carried out with the aid of Plemelj-Sokhotski formulae, cf. [19].

This in turn shows that conditions (1.16), (3.9) are also sufficient conditions for the stability of the system.

The ergodicity conditions can therefore be summarized as :

$$(4.11)\quad \left|\begin{array}{ll} \lambda_1 < r_1(1-r_2) \text{ and/or } \lambda_2 < r_2(1-r_1) & \\ \lambda_1 r_2 + \lambda_2(1-r_2) < r_2(1-r_2) & \text{if } \lambda_1 < r_1(1-r_2) \\ \lambda_1(1-r_1) + \lambda_2 r_1 < r_1(1-r_1) & \text{if } \lambda_2 < r_2(1-r_1) \end{array}\right.$$

where these conditions reduce to $\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} < 1$ if $r_1+r_2 = 1$.

We will not pay attention to the analytic continuation of eqs. (4.8), (4.10) in the following, both for sake of brevity and also because the point x=1 belongs to L_x if (1.16), (3.9) hold (it is shown in [19, Appendix D], that $x(y_2)<1$ in that case, which therefore ensures that $1 \in L_x^+$ by Lemma 2.2).

Consequently, the moments of the queue length processes can be computed from (1.4), (4.8), (4.10) by standard calculations. This is done for the expected queue lengths in the next section, both analytically and numerically.

5 - Mean response times

Define $a = \frac{\partial}{\partial x}F(x,0)\Big|_{x=1}$, $b = \frac{\partial}{\partial y}F(0,y)\Big|_{y=1}$.

Setting y = 1 in (1.4), then differentiating both sides of the equation twice in the variable x gives by making x = 1 :

$$(5.1)\quad N_1 = \frac{r_1r_2[F(1,0)-F(0,0)]+r_1(1-r_2)[1-F(0,1)]-r_1r_2a}{r_1(1-r_2)-\lambda_1},$$

where N_j denotes the expected queue-length in station j at steady state, j=1,2.

Introducing relation (1.9) into (5.1) yields

$$(5.2)\quad N_1 = \frac{\lambda_1-r_1r_2a}{r_1(1-r_2)-\lambda_1},$$ where a can be computed using (4.8) or (4.10) depending on the value of r_1+r_2.

Take now x = y in the functional equation. A straightforward calculation using (1.9), (1.10) leads to :

$$(5.3)\quad N_1+N_2 = \frac{\lambda_1+\lambda_2-\lambda_1\lambda_2+r_1(1-2r_2)b+r_2(1-2r_1)a}{r_1(1-r_2)+r_2(1-r_1)-\lambda_1-\lambda_2}.$$

From (5.1) and (5.3) we obtain N_2, the expected queue length in station 2. If $\lambda_1=\lambda_2=\lambda$ and $r_1=r_2=r$ (symmetrical case) then we immediately deduce from (5.2), (5.3) that $N_1=N_2 = \frac{\lambda(2(1-r)-\lambda r)}{2(r(1-r)-\lambda)}$.

Finally the mean response times T_j for both stations are computed with the aid of Little's formula, that is

$$(5.4)\quad T_j = \frac{N_j}{\lambda_j},\ j=1,2.$$

It is easily seen that the numerical computation of $T_j (j=1,2)$ only requires the determination of $\gamma(1)$ and $\gamma'(1)$. This, in turn, needs the computation of the angular deformation $\theta(.)$ defined in (4.2).

Following the numerical procedure proposed in [6, p.350], we have numerically computed N_1 and N_2 as well as T_1, T_2, $F(0,0)$, $F(0,1)$ and $F(1,0)$ for particular values of the parameters $\lambda_1,\lambda_2,r_1,r_2$. These results are given Table 1.

In a future paper we shall concentrate on the determination of "optimal couples" (r_1,r_2) which minimize -for given (λ_1,λ_2)- the total average line-length. Other problems will also be addressed, including the calculation of the mean number of collisions experienced by a packet before its successful transmission.

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APPENDIX A

Lemma : For $|y| = 1$, $y \neq 1$, the equation $K(x,y)=0$ has exactly one root $x=x(y)$ such that $|x(y)| < 1$. If $\lambda_1 < r_1(1-r_2)$ then $x(1) = 1$ is the only root of $K(x,1)$ in $|x| \leq 1$.

Proof :

For $|y| =1$, $y \neq 1$ and $|x| = 1$ then $|\Psi(x,y)| < 1 = |xy|$, where $\Psi(x,y) = E\{x^{X} y^{Y}\}$ has been introduced in Section 1. With Rouche's theorem this implies that for $|y| = 1$, $y \neq 1$ there exists exactly one x, $|x| < 1$, such that $xy - \Psi(x,y) = 0$. The first statement of the lemma is then proved by noting that $K(x,y) = \frac{xy-\Psi(x,y)}{xy\, H(x,y)}$, cf.(1.17) and that $H(x,y) \neq 0$ for $|x|, |y| \leq 1$, cf.(1.19).

For $y=1$, the equation $K(x,1) = 0$ reduces to, cf.(2.1)

$$(1-x)[\lambda_1 - \frac{r_1(1-r_2)}{x}] = 0$$ which concludes the proof. ◘

APPENDIX B

Lemma B.1 : The cut $[y_1,y_2]$ is contained inside the domain $L_y^+ \cup L_y$.

Proof :

Since $y_i>0$ for $i=1,2$ and that the curve L_x cuts the positive real axis at point $y=y(x_2)$ cf. Lemma 2.2, it suffices to show that $y(x_2) \notin]0,y_2[$.

First, we get from the relations $x(y(x_2))=x_2$ or $x^{\sigma}(y(x_2))=x_2$ that $y(x_2) \notin]y_1,y_2[$ (otherwise $x(y)$ and $x^{\sigma}(y)$ would be complex (conjugate) numbers, see Section 2, which would contradict the fact that x_2 is a real number).

On the other hand it is readily seen that $y(x_2)=y_2$ if $\lambda_1=r_1(1-r_2)$ and $\lambda_2=r_2(1-r_2)$. Then a continuity argument on the parameters $\lambda_1,\lambda_2,r_1,r_2$ shows that $y(x_2)\geq y_2$, which concludes the proof. ◘

Lemma B.2 : The algebraic branch $y^{\sigma}(x)$ lies entirely outside the domain L_y^+.

Proof

The function $\frac{1}{y^{\sigma}(x)}$ is analytic in $\mathbb{C} / [x_1,x_2] \cup [x_3,x_4]$ ($y^{\sigma}(x)=0$ entails $x=0$, cf.(2.1). But from the definition of y^{σ} it is readily seen that $y^{\sigma}(0) = \infty$ and $y(0) = 0$, which proves that $\frac{1}{y^{\sigma}(x)}$ is well-defined for $x=0$ and equal to 0 at this point).

Then by applying the maximum modulus principle [12,p.201] we have :

$$\left|\frac{1}{y^{\sigma}(x)}\right| \leq \max\left\{ \left|\frac{1}{y^{\sigma}(x)}\right|_{x\in[x_1,x_2]},\ \left|\frac{1}{y^{\sigma}(x)}\right|_{x\in[x_3,x_4]},\ \lim_{x\to\infty}\left|\frac{1}{y^{\sigma}(x)}\right| \right\}.$$

For $x \in [x_1,x_2] \cup [x_3,x_4]$, $|y^{\sigma}(x)| = \left(\frac{r_2(1-r_1)}{\lambda_2(1+\lambda_1(1-x))}\right)^{1/2}$ (cf.Section 2).

Consequently $|y^{\sigma}(x)|_{x\in[x_1,x_2]} < |y^{\sigma}(x)|_{x\in[x_3,x_4]}$.

On the other hand

$$\lim_{x\to\infty} |y^{\sigma}(x)| = \frac{1+\lambda_2}{\lambda_2}.$$

It remains to show that $|y^{\sigma}(x)|_{x\in[x_1,x_2]} < \frac{1+\lambda_2}{\lambda_2}$ or equivalently that $y^{\sigma}(x_2) < \frac{1+\lambda_2}{\lambda_2}$.

We have that $y(x_2)$ $(= y^{\sigma}(x_2)) \notin]y_3,y_4[$ (same proof as in Lemma B.1). Therefore necessarily $y^{\sigma}(x_2) \leq y_3$ by a continuity argument (since for $\lambda_2=r_2(1-r_1)$, $\lambda_1 \neq r_1(1-r_2)$ then $x_2=1$, and that $y^{\sigma}(1) = 1 < y_3$).

Finally $|y^{\sigma}(x)| \geq |y^{\sigma}(x)|_{x\in[x_1,x_2]}$ for all $x \in \mathbb{C}$, which concludes the proof. ◘

From Lemma B.2 we get

$y(x(y))=y$ and $y^{\sigma}(x(y)) \neq y$ for $y \in L_y^+$.

Then, in particular, $y(x(y)) = y$ for $y \in [y_1,y_2]$ using Lemma B.1 and the fact that $y^{\sigma}(x(y_2)) = y(x(y_2)) = y_2$ if $y_2 \in L_y$ (i.e. $y_2 = y(x_2)$).

APPENDIX C

Lemma : $b^1(y) = r_2[(1-r_1)(1-\frac{1}{y})-r_1(1-\frac{1}{x(y)})]$ has no root for $y \in \bar{D}$.

Proof

$b^1(y) = 0$ together with (2.1) entails

$$P(y) \stackrel{\Delta}{=} \lambda_2(-1+2r_1-\lambda_1(1-r_1))y^2+(1-r_1)((1+\lambda_1)(1+\lambda_2)-2r_1)y-(1-r_1)^2 = 0.$$

The discriminant Δ_P of the polynomial $P(y)$ reads

$$\Delta_P = (1+\lambda_1)^2\lambda_2^2 + 2\lambda_2[-1+2r_1-\lambda_1(1-r_1)+\lambda_1(1+\lambda_1-r_1)]+(1-2r_1+\lambda_1)^2.$$

An elementary study of this polynomial in λ_2 reveals that it is always positive for $\lambda_2 > 0$. Then $P(y)$ has always two real roots. For $-1 \leq y < 0$ then $|x(y)| \leq 1$ (apply Rouche's theorem to $K(x,y)=0$ with $y<0$). For $0<y\leq 1$, then obviously $b^1(y)$ cannot vanish in **D**, which concludes the proof. □

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