Relaying in Mobile Ad Hoc Networks

R.B. Groenevelt, E. Altman and P. Nain

INRIA, 2004 route des Lucioles,
06902 Sophia Antipolis, France
{robin.groenevelt, eitan.altman, philippe.nain}@sophia.inria.fr

Abstract. Mobile ad hoc networks are characterized by a lack of a fixed infrastructure and by node mobility. In these networks data transfer is accomplished by using mobile nodes as relay nodes. As a result, transmission power and movement pattern of nodes have a key impact on the performance. In this work we focus on the impact of node mobility through the analysis of a simple one-dimensional ad hoc network topology. Nodes move in adjacent segments with reflecting boundaries according to Brownian motions. Communications (or relays) between nodes may only occur when they are within transmission range of each other. We determine the expected time to relay a message and compute the probability density function of relaying locations. We also provide an approximation formula for the expected relay time between any pair of mobiles.

1 Introduction

Ad hoc networks can be deployed when a fixed network structure is not available. The lack of a fixed infrastructure may arise in emergency situations, remote regions, hostile areas or, as is often the case, due to the financial costs involved in the deployment of a fixed infrastructure.

As a consequence of the absence of a fixed infrastructure, components (or nodes) of an ad hoc network need to behave as routers by relaying messages in order to improve communications. Instances of nodes in ad hoc networks are laptops, planes [11], cars, electronic tags on animals [10], etc. If nodes are mobile then operating these networks become even more complex, as mobility will impact routing protocols, control of transmission power, quality of service (e.g. interference, path loss, shadowing effects), battery usage, etc.

As long as data does not have to be transferred directly between two mobiles and that nodes are willing to relay messages, their mobility may have a positive impact on the performance, as shown in [6]. This has led to the design of protocols that take advantage of node mobility to enhance the performance of some applications (e.g. messaging applications in [7]). Data relaying cuts down transmission power, interferences and increases battery usage. On the other hand, it may increase latency - since the existence at any time of a “path” between two mobiles is not guaranteed - even if (intermediary) nodes can be used as routers to convey a message from its source to its destination.

In this paper we study the impact of mobility on the latency in the case of nodes acting as relay nodes. This is done for one-dimensional ad hoc network topologies and under the assumption that nodes move according to (independent) Brownian motions.

A natural approach (but not the only one, see [9] for an another approach) to modeling a mobile ad hoc network with relaying nodes consists of looking down at the earth and representing it as a finite two-dimensional plane. If two mobiles are within a fixed transmission range of each other then a message can be relayed/transmitted (see Figure 1). Furthermore, mobiles move according to a certain movement pattern. Unfortunately, this simple model of an ad hoc network (no physical restrictions in the area covered by the nodes, nodes are homogeneous, etc.) is extremely difficult to analyze, even with simple movement patterns such as, for example, the Random Waypoint Mobility (RWM) model [12]. For instance, finding the stationary distribution of the location of the mobiles under the RWM is, to the best of our knowledge, an open problem.

Obtaining any results characterizing the first instance of time when two mobiles come within transmission range of each other is a problem of even greater complexity. For this reason, this paper focuses on a one-dimensional topology - a model that has already reveales interesting properties. Its extension to two dimensions is an open problem.

When analyzing a mobile ad hoc network, an important consideration is the movement pattern. Are mobiles restricted in their movement by roads, physical objects, waterways, or mountains? Do they

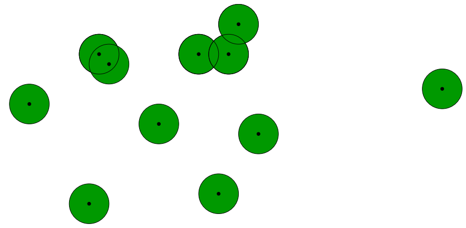

Figure 1. Graphical representation of an ad hoc network

roam around a central point? It has been shown that this is the case for RWM, where there is a higher concentration of mobiles around a central region [2].

The following scenarios are addressed in this paper. In Section 2 we consider the situation where two mobiles move along a segment with reflecting boundaries (see Figure 2). Both mobiles move along the segment according to independent Brownian motions. We are interested in computing the expected time until both mobiles come within communication range of each other. This quantity is computed for any given initial locations (Proposition 1) as well as for the case where each Brownian motion is initially in steady-state (Proposition 3). It is known (see Section 3) that the latter assumption implies that both mobiles are uniformly distributed over the segment. The uniform spatial distribution over the coverage area has attracted attention lately and several fundamental results [1, 6] have been obtained in this setting. However, our model is different from the models considered in those papers.

In Section 3, we consider I mobiles and I segments, one mobile per segment, as depicted in Figure 5. The mobiles move along their respective segment (with reflecting boundaries) according to independent Brownian motions. The goal is to determine the expected transfer time between the first and last mobile in the sequence (Proposition 5). As an additional result, we identify the probability density function (pdf) of the position of a mobile at a relay epoch (Proposition 4). Numerical results are reported in Section 4. These results suggest an accurate and scalable approximation for the expected transfer time (see (15)).

2 Two mobiles moving along a line segment

We consider two mobiles (say mobiles X and Y) moving along segment $[0, L]$, (see Figure 2). Communications between these two mobiles may only occur when the distance between them is less than or equal to $r < L$. The objective of this section is to determine the expected *transfer time*, defined as the first time when both mobiles come with a distance r of each other.

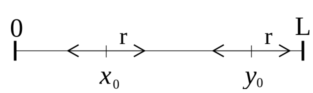

Figure 2. Two mobiles moving along $[0, L]$ with transmission range r.

Let $x(t)$ and $y(t)$ be the position of mobiles X and Y, respectively, at time t. We assume that $\mathbf{X} = \{x(t), t \geq 0\}$ and $\mathbf{Y} = \{y(t), t \geq 0\}$ are identical and independent Brownian motions with drift 0 and diffusion coefficient[1] D, both moving along the segment $[0, L]$ with *reflecting* boundaries at the edges. Let $T_{L,r}$ be the transfer time, namely,

$$T_{L,r} = \inf\{t \geq 0 \, : \, |y(t) - x(t)| \leq r\}. \tag{1}$$

Set $x(0) = x_0$ and $y(0) = y_0$. By convention we assume that $T_{L,r} = 0$ if $|y_0 - x_0| \leq r$. From now on we assume that $|y_0 - x_0| > r$.

We are interested in

$$T_{L,r}(x_0, y_0) := \mathbb{E}[T_{L,r} \,|\, x(0) = x_0, y(0) = y_0], \quad 0 < x_0, y_0 < L,$$

[1] i.e $x(t + h) - x(t)$ (resp. $y(t + h) - y(t)$) is normally distributed with mean 0 and variance $2Dh$ for all $h > 0$, and non-overlapping time intervals are indepent of each other.

the expected transfer time given that mobiles X and Y are located at position x_0 and y_0, respectively, at time $t = 0$. The following result holds:

Proposition 1 (Expected transfer time with given initial positions).
For $0 \leq x_0 < y_0 \leq L$ with $x_0 + r < y_0$ and $0 \leq r \leq L$

$$T_{L,r}(x_0, y_0) = \frac{32(L-r)^2}{D\pi^4} \sum_{\substack{m \geq 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n \geq 1 \\ n \text{ odd}}}^{\infty} \frac{\sin\left(\frac{m\pi(y_0+x_0-r)}{2(L-r)}\right) \sin\left(\frac{n\pi(y_0-x_0-r)}{2(L-r)}\right)}{mn(m^2+n^2)}. \tag{2}$$

$\diamond$

The proof of Proposition 1 is based on the following intermediary result that gives the expected time for a two-dimensional Brownian motion $\mathbf{Z}$ evolving in a R by R square to hit any boundary of the square.

Proposition 2 (Two Brownian motions in a square).
Consider two independent and identical one-dimensional Brownian motions $\{u(t),\, t \geq 0\}$ and $\{v(t),\, t \geq 0\}$, with zero drift and diffusion coefficient D. Define the two-dimensional Brownian motion $\mathbf{Z} = \{z(t) = (u(t), v(t)), t \geq 0\}$. Set $u_0 = u(0)$ and $v_0 := v(0)$ and assume that $0 < u_0 < R$ and $0 < v_0 < R$.
Let

$$\tau_R := \inf\{t > 0 \,:\, u(t) \in \{0, R\} \text{ or } v(t) \in \{0, R\}\}$$

be the first time when the process $\mathbf{Z}$ hits the boundary of a square of size R by R.
Define $\tau_R(u_0, u_0) = \mathbb{E}[\tau_R \,|\, z(0) = (u_0, v_0)]$. Then,

$$\tau_R(u_0, v_0) = \frac{16R^2}{D\pi^4} \sum_{\substack{m \geq 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n \geq 1 \\ n \text{ odd}}}^{\infty} \frac{\sin\left(\frac{m\pi u_0}{R}\right) \sin\left(\frac{n\pi v_0}{R}\right)}{mn(m^2+n^2)}. \tag{3}$$

$\diamond$

The proof of Proposition 2 is given in Appendix A. We are now in a position to prove Proposition 1.

Proof of Proposition 1.
Let $x_0 + r < y_0 \leq L$. An equivalent way to view the Brownian motions $\mathbf{X}$ and $\mathbf{Y}$ at time $t = 0$ is to consider that the point (x_0, y_0) is located in the upper triangle in Figure 3 delimited by the lines $x = 0$, $y = L$ and $y = x + r$. If we assume that the boundaries $x = 0$ and $y = L$ are reflecting boundaries in Figure 3, then we see that $T_{L,r}(x_0, y_0)$ is nothing but the expected time needed for the two-dimensional Brownian motion $\{(x(t), y(t)), t \geq 0\}$ to hit the diagonal of the triangle (i.e. to hit the line $y = x + r$) given that $(x(0), y(0)) = (x_0, y_0)$. (The process $\{(x(t), y(t)), t \geq 0\}$ is a two-dimensional Brownian motion since $\{x(t),\, t \geq 0\}$ and $\{y(t), t \geq 0\}$ are both independent Brownian motions.)

By using the classical method of images (see e.g. [8, p. 81]), it can be seen that this time is itself identical to the expected time needed to hit the boundary of the square of size $\sqrt{2}(L-r)$ by $\sqrt{2}(L-r)$ shown in Figure 4 given that $(x(0), y(0)) = (x_0, y_0)$.

In order to apply the result in Proposition 2, we need to compute the coordinates (x_0', y_0') of (x_0, y_0) in the new system of coordinates (x', y') depicted in Figure 4. We find $(x_0', y_0') = ((y_0 + x_0 - r)/\sqrt{2}$ and $(y_0 - x_0 - r)/\sqrt{2})$ and we may conclude, from Proposition 2, that

$$T_{L,r}(x_0, y_0) = \tau_{\sqrt{2}(L-r)}\left((y_0 + x_0 - r)/\sqrt{2}, (y_0 - x_0 - r)/\sqrt{2}\right). \tag{4}$$

By using (3) in the r.h.s. of (4) we see that (2) holds. ∎

The expected transfer time $T_{L,r}(x_0, y_0)$ is displayed in Figure 6 (see Section 4 for comments).

We conclude this section by considering the situation where both mobiles are uniformly distributed over the segment $[0, L]$ at time $t = 0$. (We will see in the next section that this case corresponds to the situation where both Brownian motions $\mathbf{X}$ and $\mathbf{Y}$ are in steady-state at time $t = 0$.)

The next result gives the expected transfer time.

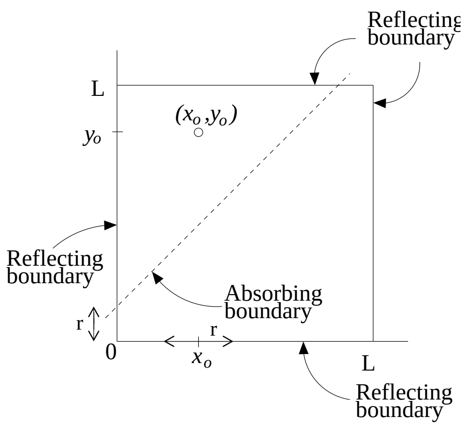

Figure 3. When mobiles X and Y are at a distance r of each other they are located on the line $y = x+r$ $(y_0 > x_0 + r)$.

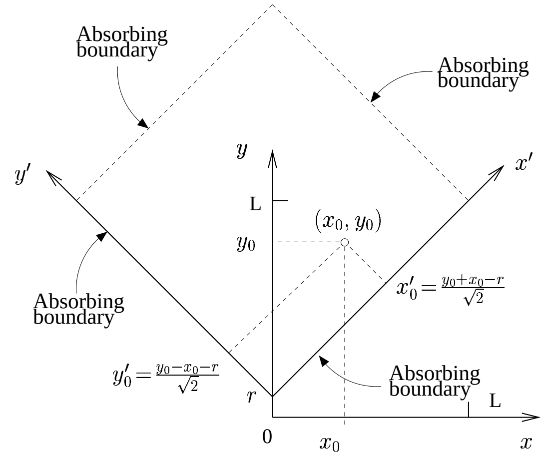

Figure 4. Since reflecting barriers at $x = 0$ and $y = L$ act as mirrors, the method of images turns the problem into a 2D Brownian motion inside four absorbing barriers.

Proposition 3 (Expected transfer time for uniformly distributed initial positions). *Assume that both mobiles X and Y are uniformly distributed over $[0, L]$ at time $t = 0$ and $0 \le r \le L$. The expected transfer time $\mathbb{E}[T_{L,r}]$ is*

$$\mathbb{E}[T_{L,r}] = \frac{128(L-r)^4}{D\pi^6 L^2} C_0, \tag{5}$$

where C_0 is a constant given by $C_0 = \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{m^2 n^2 (m^2+n^2)} \approx 0.52792664$. ⋄

Proof. Since **X** and **Y** are uniformly distributed at $t = 0$, we have

$$\begin{aligned}
\mathbb{E}[T_{L,r}] &= \frac{1}{L^2} \int_0^L \int_0^L \mathbb{E}[T_{L,r} \,|\, x(0) = x_0, y(0) = y_0] dx_0 dy_0 \\
&= \frac{1}{L^2} \int_{x_0+r<y_0 \le L} T_{L,r}(x_0, y_0)\, dx_0 dy_0 + \frac{1}{L^2} \int_{y_0+r<x_0 \le L} T_{L,r}(y_0, x_0)\, dx_0 dy_0 \\
&= \frac{2}{L^2} \int_{x_0+r<y_0 \le L} T_{L,r}(y_0, x_0)\, dx_0 dy_0 \\
&= \frac{64(L-r)^2}{D\pi^4 L^2} \int_{x_0+r<y_0 \le L} h(y_0+x_0-r, y_0-x_0-r) dx_0\, dy_0.
\end{aligned}$$

where

$$h(u, v) := \sum_{\substack{m \ge 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n \ge 1 \\ n \text{ odd}}}^{\infty} \frac{\sin(mu\beta)\sin(nv\beta)}{mn(m^2+n^2)}, \qquad \beta := \frac{\pi}{\sqrt{2}(L-r)}.$$

Define the new variables $u = (y_0 + x_0 - r)/\sqrt{2}$ and $v = (y_0 - x_0 - r)/\sqrt{2}$. We find

$$\mathbb{E}[T_{L,r}] = \frac{64(L-r)^2}{D\pi^4 L^2} \left[\int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=0}^{u} h(u,v)|J(u,v)|\, dv\, du + \int_{u=\frac{L-r}{\sqrt{2}}}^{\sqrt{2}(L-r)} \int_{v=0}^{\sqrt{2}(L-r)-u} h(u,v)|J(u,v)|\, dv\, du \right] \tag{6}$$

where $|J(u, v)|$ (=1) is the determinant of the Jacobian matrix

$$J(u,v) = \begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

It remains to evaluate the two integrals in (6). By using the identity $h(u,v) = h(\sqrt{2}(L-r)-u, v)$ we see that both integrals in the r.h.s. of (6) are equal, since

$$\int_{u=\frac{L-r}{\sqrt{2}}}^{\sqrt{2}(L-r)} \int_{v=0}^{\sqrt{2}(L-r)-u} h(u,v)dvdu = \int_{u=\frac{L-r}{\sqrt{2}}}^{\sqrt{2}(L-r)} \int_{v=0}^{\sqrt{2}(L-r)-u} h(\sqrt{2}(L-r)-u, v)dvdu = \int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=0}^{u} h(u,v)dvdu.$$

The first integral can be evaluated by using the symmetry $h(u,v) = h(v,u)$. This gives

$$\int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=0}^{u} h(u,v)dvdu = \int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=0}^{u} h(v,u)dvdu = \int_{v=0}^{\frac{L-r}{\sqrt{2}}} \int_{u=v}^{\frac{L-r}{\sqrt{2}}} h(v,u)dudv = \int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=u}^{\frac{L-r}{\sqrt{2}}} h(u,v)dvdu.$$

Hence,

$$\int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=0}^{u} h(u,v)dvdu = \frac{1}{2} \int_{u=0}^{\frac{L-r}{\sqrt{2}}} \int_{v=0}^{\frac{L-r}{\sqrt{2}}} h(u,v)dvdu$$

so that

$$\mathbb{E}[T_{L,r}] = \frac{64(L-r)^2}{D\pi^4 L^2} \int_0^{\frac{L-r}{\sqrt{2}}} \int_0^{\frac{L-r}{\sqrt{2}}} h(u,v)dvdu. \tag{7}$$

Since the double series in $h(u,v)$ are uniformly bounded in the variables $u, v \in [0, \sqrt{2}(L-r)]$ (its absolute value is bounded from above by $(\sum_{k\geq 1} 1/k^2)^2 = \pi^4/36$), we may invoke the bounded convergence theorem to interchange the integral and summation signs in (7). This gives

$$\begin{aligned}
\mathbb{E}[T_{L,r}] =& \frac{64(L-r)^2}{D\pi^4 L^2} \sum_{\substack{m\geq 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n\geq 1 \\ n \text{ odd}}}^{\infty} \frac{1}{mn(m^2+n^2)} \int_{u=0}^{\frac{L-r}{\sqrt{2}}} \sin(mu\beta)du \int_{v=0}^{\frac{L-r}{\sqrt{2}}} \sin(nv\beta)dv \\
=& \frac{128(L-r)^4}{D\pi^6 L^2} \sum_{\substack{m\geq 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n\geq 1 \\ n \text{ odd}}}^{\infty} \frac{1}{m^2 n^2 (m^2+n^2)}.
\end{aligned}$$

The last line follows because $cos(\frac{j\pi}{2}) = 0$ for j odd. ∎

3 A chain of relaying mobiles

We consider the situation depicted in Figure 5. There are I adjacent segments, each of length L, and there is a single mobile per segment. We denote by X_i the mobile in segment i. Let $0 \leq x_i(t) \leq L$ ($i = 1, \ldots, I$) be the *relative* position of the i-th mobile in its segment. We assume that the process $\mathbf{X}_i = \{x_i(t),\, t \geq 0\}$ is a Brownian motion with zero drift and diffusion coefficient D and that $\mathbf{X}_1, \ldots, \mathbf{X}_I$ are mutually independent processes. Last, we assume that each segment has reflecting boundaries at the ends. Let $T_1 = \inf\{t \geq 0 \,:\, x_1(t) + r \geq L + x_2(t)\}$ be the transfer time between mobiles X_1 and X_2,

Figure 5. A chain of relaying mobiles.

that is T_1 is the first time when X_1 and X_2 are located at a distance less than or equal to r from each other. The relay times $T_2 \leq \cdots \leq T_{I-1}$ between mobiles X_2 and X_3, ..., X_{I-1} and X_I, respectively, are recursively defined by

$$T_i = \inf\{t \geq T_{i-1} \,:\, x_i(t) + r \geq L + x_{i+1}(t)\}, \quad i = 2, \ldots, I-1.$$

Our objective in this section is to compute $\mathbb{E}[T_i]$ for $i = 1, \ldots, I-1$.

Throughout this section we assume that $L < r < 2L$. This assumption is made for the sake of mathematical tractability. Indeed, a few seconds of reflexion will convince the reader that when $L <$

$r < 2L$ and $(x(0), y(0)) = (x_0, y_0)$ the transfer time needed to transfer a message between two adjacent segments is the same as $T_{2L,r}(x_0, y_0 + L)$, the expected transfer time obtained in Section 2 for a segment of length $2L$ (with the given initial conditions). This observation allows us to find at once the expected transfer time between mobiles X_1 and X_2 for any initial conditions $x_1(0)$ and $x_2(0)$. We find

$$\mathbb{E}[T_1 \,|\, x_1(0) = x, x_2(0) = y] = \mathbb{E}[T_{2L,r}(x, y + L)]. \tag{8}$$

The difficulty arises when trying to find the expected transfer time between mobiles X_i and X_{i+1} for $i = 2, \ldots, I-1$, since the position of X_i when the transfer between X_{i-1} and X_i takes place is not uniform in $[iL, (i+1)L]$.

To overcome this difficulty, we assume that the Brownian motions $\mathbf{X}_1, \ldots, \mathbf{X}_I$ are all in steady-state at time $t = 0$. This assumption implies,[2] in particular, that the position of each mobile at time $t = 0$ is uniformly distributed over its segment (i.e. the pdf of $x_i(0)$ is uniform over $[0, L]$). The same holds of course at any arbitrary time (i.e. the pdf of $x_i(t)$ is uniform over $[0, L]$ if t is arbitrary).

Another consequence of this assumption is that the position of mobile X_{i+1} at time T_{i-1} (i.e. when X_i receives a message from X_{i-1}) is still uniformly distributed over $[0, L]$. This property will be used later on.

Proposition 4 below addresses the location of a mobile at the time when a relay occurs. For later reference, we state the result in a general form. Consider two adjacents segment, each of length L, with a single mobile in each segment (mobile X in the first segment and Y in the second segment). Both mobiles move in their segment (with reflecting boundaries) according to independent and identical Brownian motions with drift 0 and coefficient diffusion D. We assume that the Brownian motion representing the movement of Y is in steady state at time $t = 0$. As usual, a relay will occur the first time when both mobiles comes at a distance r of each other, with $L < r < 2L$.

Proposition 4 (Pdf of location at relay epoch).

Fix $L < r < 2L$. Let $q(y; x)$, $y \in [0, L]$, be the pdf of the (relative) position of mobile Y at the relay epoch, given that at time $t = 0$ the mobile X is at position x and the position of mobile Y is uniform.

We have

$$q(y; x) = \frac{\mathbf{1}_{\{y \le x + r - L\}} + f(x, y)\,\mathbf{1}_{\{y \ge r - L, x < 2L - r\}}}{L}, \tag{9}$$

where

$$\begin{aligned} f(x, y) = {} & \frac{4}{\pi^2} \sum_{m \ge 1}^{\infty} \sum_{\substack{n \ge 1 \\ n \ne m}}^{\infty} \frac{n\,(a_{m,n} + b_{m,n} + c_{m,n})}{m^2 + n^2} \sin\left(\frac{m\pi(y - r + L)}{2L - r}\right) \\ & + \frac{2}{\pi(2L - r)} \sum_{m \ge 1}^{\infty} \frac{d_m + e_m}{m} \sin\left(\frac{m\pi(y - r + L)}{2L - r}\right), \end{aligned}$$

and

$$\begin{aligned} a_{m,n} = {} & \frac{2m \sin(n\theta) - 2n \sin(m\theta)}{m^2 - n^2}, \quad b_{m,n} = \frac{\sin\Big((m-n)\pi + n\theta\Big) + \sin\Big((m-n)\pi - m\theta\Big)}{m - n} \\ c_{m,n} = {} & -\frac{\sin\Big((m+n)\pi - n\theta\Big) + \sin\Big((m+n)\pi - m\theta\Big)}{m + n}, \quad d_m = 2(2L - r - x)\cos(m\theta) \\ e_m = {} & \frac{2L - r}{m\pi}\left(\sin(m\theta) - \sin\Big(2m\pi - m\theta\Big)\right), \qquad \theta = \frac{\pi x}{2L - r}. \end{aligned}$$

◇

The proof of Proposition 9 is sketched in Appendix B. We are now in a position to compute the expected transfer times $\mathbb{E}[T_i]$ for $i = 1, \ldots, I-1$.

Define $f_i(x)$ $(0 \le x \le L)$ as the pdf of $x_i(T_{i-1})$ for $i = 1, \ldots, I-1$ (i.e. $P(x_i(T_{i-1}) < y) = \int_0^y f_i(x)\,dx$). Note that $f_1(x) = 1/L$ for $x \in [0, L]$ thanks to the assumption that mobile X_1 is in steady-state at time $t = 0$ (recall that $T_0 = 0$ by convention). Let us first compute $\mathbb{E}[T_1]$. We find

$$\mathbb{E}[T_1] = \frac{1}{L^2}\int_0^L \int_0^L \mathbb{E}\left[T_1 \,|\, x_1(0) = x, x_2(0) = y\right]\,dx\,dy = \frac{1}{L^2}\int_{\{x + r < y + L\}} T_{2L,r}(x, y + L)\,dx\,dy, \tag{10}$$

[2] Hint: let $p(x)$ be the stationary density probability that the mobile is in position $x \in [0, L]$. Solving the diffusion equation $D\partial^2 p(x)/dx^2 = 0$ with the reflecting conditions $dp(x)/dx = 0$ for $x \in \{0, L\}$ and the normalizing condition $\int_0^L p(x)dx = 1$ yields $p(x) = 1/L$ for $x \in [0, L]$ – see e.g. [3, p. 223].

by using (8) and $T_{2L,r}(x, y+L) = 0$ if $x + r \le y + L$. Similar to the derivation of (5) we get

$$\mathbb{E}[T_1] = \frac{64\,(2L-r)^4}{D\pi^6 L^2}\, C_0. \tag{11}$$

We now compute $\mathbb{E}[T_i]$ for $i = 2, \ldots, I-1$. We have

$$\mathbb{E}[T_i] = \mathbb{E}[T_{i-1}] + \frac{1}{L}\int_0^L \int_0^L \mathbb{E}\left[T_i - T_{i-1}\,|x_i(T_{i-1}) = x, x_{i+1}(T_{i-1}) = y\right] f_i(x)\,dx\,dy \tag{12}$$

$$= \mathbb{E}[T_{i-1}] + \frac{1}{L}\int_{\{x+r<y+L\}} T_{2L,r}(x, y+L)\, f_i(x)\,dx\,dy, \tag{13}$$

where we have used (8) to derive (13). To derive (12) we have used the fact that the position of mobile X_{i+1} is uniformly distributed over its segment at time T_{i-1} (i.e. when the relay between mobiles X_{i-1} and X_i occurs), and that it is independent of the position of mobile X_{i-1} at time T_{i-1}. It remains to evaluate the functions $f_i(x)$ for $i = 2, \ldots, I-1$. Differentiating in y on both sides of the identity

$$P(x_i(T_{i-1}) < y) = \int_0^L P(x_i(T_{i-1}) < y \,|\, x_{i-1}(T_{i-2}) = x)\, f_{i-1}(x)\,dx,$$

and then using Proposition 9, gives

$$f_i(y) = \int_0^L q(y; x)\, f_{i-1}(x)\,dx, \quad 0 \le y \le L, \tag{14}$$

for $i = 2, \ldots, I-1$. These results are summarized in the next proposition.

Proposition 5 (Expected transfer times).
The expected transfer times $\mathbb{E}[T_i]$ *for* $i = 1, \ldots, I-1$, *are given by (11) and (13), where the functions* $f_i(x)$, $i = 2, \ldots, I-1$, *satisfy the recursion (14) with* $f_1(x) = 1/L$. *In particular,*

$$\mathbb{E}[T_1] = \frac{64\,(2L-r)^4}{D\pi^6 L^2}\, C_0.$$

◇

4 Numerical results and discussion

The expected transfer time $T_{L,r}(x_0, y_0)$ is displayed in Figure 6 as a function of the initial position x_0 and y_0 of the mobiles, for $L = 30$, $r = 5$ and $D = 1/4$ (recall that D is the diffusion coefficient of the Brownian motions $\mathbf{X}$ and $\mathbf{Y}$). The figure shows that the expected transfer time grows (roughly) linearly as the initial distance between both mobiles increases and neither of the mobiles is near the boundaries of the interval $[0, L]$. We used (14) to determine the mapping $x \to f_2(x)$ for $0 \le x \le L$, the pdf of the location of mobile X_2 when the relay with X_1 occurs. This mapping is plotted in Figure 7 for different values of the starting position of mobile X_1 ($x_1(0) = 5, 10, 15, 20$) and for $L = 30$, $r = 35$, $D = 1/4$. It is interesting to observe that $f_2(x)$ is uniform in $[0, x_1(0)]$. This is easily explained by the fact that if X_2 is located in $[0, r-L]$ at time T_1 then it was necessarily located in this interval prior to time T_1, since otherwise the relay would have occured before T_1. Each peak corresponds to the most likely value y in $[0, L]$ where mobile $X2$ will be located at time T_1. This value is given by $y = x_1(0) + r$.

Figure 8 displays mappings $x \to f_i(x)$ for $i \in \{2, 3, 100\}$ (evaluated from (14) – case where initial locations are uniformly distributed). It is worth observing that these functions converge very rapidly (already $f_2(x)$ and $f_3(x)$ are very close to each other).

Figure 9 displays mappings $r \to \mathbb{E}[T_{100}]$, $r \to 100 \times \mathbb{E}[T_2 - T_1]$ and $r \to 100 \times \mathbb{E}[T_1]$. This figure carries two important messages. First, it shows for different values of the transmission range r, that the approximation $\mathbb{E}[T_{100}] \sim 100 \times \mathbb{E}[T_2 - T_1]$ is very close to the exact result $\mathbb{E}[T_{100}]$ (derived from Proposition 5)), thereby suggesting the approximation

$$\mathbb{E}[T_i] \sim i \times \mathbb{E}[T_2 - T_1] \tag{15}$$

for the expected time to relay a message from mobile X_1 to mobile X_{i+1}. We have indeed checked (the results are not reported in this paper; see [5] for more information) that (15) is accurate for small values of i as well as for large values (i.e. larger than 100). Second, it shows that the approximation $\mathbb{E}[T_{100}] \sim 100 \times \mathbb{E}[T_1]$ may not be accurate for small transmission ranges, thereby ruling out the approximation $\mathbb{E}[T_i] \sim i \times \mathbb{E}[T_1]$. This is so because the latter approximation does not account for the fact that mobile X_i does not start from a "uniform location" at time T_{i-1} (as opposed to mobile X_1 whose position is uniformly distributed over $[0, L]$ at time $t = 0$).

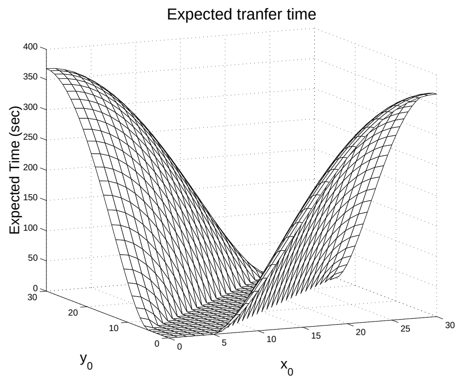

Figure 6. The mapping $(x_0, y_0) \rightarrow T_{L,r}(x_0, y_0)$ (expected transfer time between mobiles X and Y starting from x_0 and y_0, respectively, at $t = 0$. See (2)) for $L = 30$, $r = 5$, $D = 1/4$.

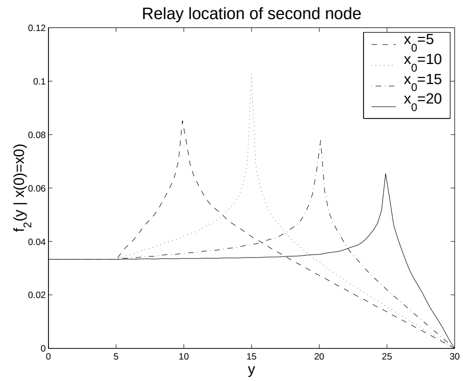

Figure 7. Mapping $x \rightarrow f_2(x)$ (pdf of location of mobile X_2 at the relay epoch) for when mobile X_1 is at position $x_0 \in \{5, 10, 15, 20\}$ at time $t = 0$, for $D = 1/4$, $L = 30$, $r = 35$.

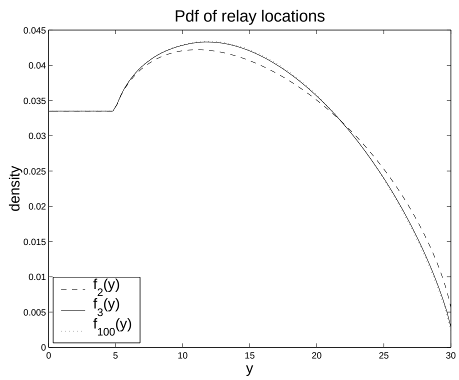

Figure 8. Mappings $x \rightarrow f_i(x)$ for $i \in \{2, 3, 100\}$ (pdf of starting location of mobiles after one or more hops) for $L = 30, r = 35$, and $D = 1/4$.

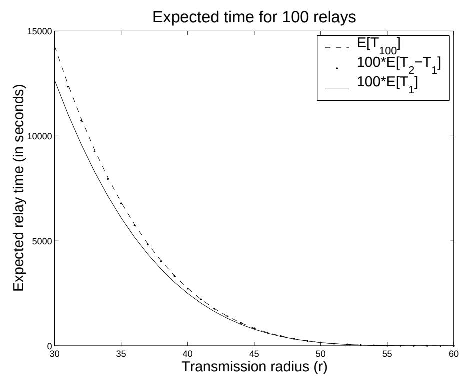

Figure 9. Comparison of mappings $r \rightarrow \mathbb{E}[T_{100}]$, $r \rightarrow 100 \times \mathbb{E}[T_2 - T_1]$, $r \rightarrow 100 \times \mathbb{E}[T_1]$ for $L = 30$ and $D = 1/4$.

5 Future Research

The question of power control is central in ad hoc networking. Ongoing research is concerned with determining the minimum transmission range that will ensure communication between mobiles (within a certain probability) before the battery power runs out, and with introducing utility functions in our model.

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Appendix A - A proof of Proposition 2 (2D-Brownian motion in a square)

The density probability $q(x,t;u_0)$ that the Brownian motion $\{u(t),\, t\geq 0\}$ is in position $x\in(0,R)$ at time t, given that $u(0)=u_0$, is [8, p. 255, formula (8.2.1)]

$$w(x,t;u_0)=\frac{2}{R}\sum_{n\geq 1} e^{-(n\pi/R)^2Dt}\,\sin\left(\frac{n\pi x}{R}\right)\sin\left(\frac{n\pi u_0}{R}\right).$$

Since $\{u(t),\, t\geq 0\}$ and $\{v(t),\, t\geq 0\}$ are independent and identical Brownian motions, we deduce from the above that the density probability $p(x,y,t;u_0,v_0)$ that the two-dimensional Brownian motion $\mathbf{Z}$ is in position (x,y) at time t is given by

$$p(x,y,t;u_0,v_0)=w(x,t;u_0)\,w(y,t;v_0). \quad 0<x,y<R. \tag{16}$$

Conditioned on $z(0)=(u_0,v_0)$, the probability $S(t;u_0,v_0)=P(\tau_R>t)$ that the process has not hit the boundaries at time t (often called the survival probability) is given by [8]

$$S(t;u_0,v_0)=\int_0^R\int_0^R p(x,y,t;u_0,v_0)dxdy.$$

Therefore,

$$S(t;u_0,v_0)=\int_0^R w(x,t;u_0)dx\int_0^R w(y,t;v_0)dy \tag{17}$$

$$=\frac{4}{R^2}\sum_{m\geq 1}e^{-(m\pi/R)^2Dt}\sin\left(\frac{m\pi u_0}{R}\right)\int_0^R\sin\left(\frac{m\pi x}{R}\right)dx\sum_{n\geq 1}e^{-(n\pi/R)^2Dt}\sin\left(\frac{n\pi v_0}{R}\right)\int_0^R\sin\left(\frac{n\pi y}{R}\right)dy$$

$$=\frac{16}{\pi^2}\sum_{\substack{m\geq 1\\ m\text{ odd}}}\sum_{\substack{m\geq 1\\ n\text{ odd}}}\frac{\sin\left(\frac{m\pi u_0}{R}\right)\sin\left(\frac{n\pi v_0}{R}\right)}{mn}\,e^{-\frac{\pi^2}{R^2}(m^2+n^2)Dt}, \tag{18}$$

where the uniform convergence of the series $w(x,t;\cdot)$ in $x\in[0,\infty)$ ($|w(x,t;\cdot)|\leq 1/(1-\exp(-(\pi/R)^2Dt))$) allows one to interchange integral and summation signs in (17). (Note that, as expected, $S(0;u_0,v_0)=1$ since $\sum_{i\geq 1}\sin((2i-1)x)/(2i-1)=\pi/4$ for all x [4, Formula 1.442.1].)

Finally,

$$\begin{aligned}
\tau_R(u_0,v_0)&=\int_0^\infty S(t;u_0,v_0)\,dt\\
&=\frac{16}{\pi^2}\int_0^\infty\sum_{\substack{m\geq 1\\ m\text{ odd}}}\sum_{\substack{m\geq 1\\ n\text{ odd}}}\frac{\sin\left(\frac{m\pi u_0}{R}\right)\sin\left(\frac{n\pi v_0}{R}\right)}{mn}\,e^{-\frac{\pi^2}{R^2}(m^2+n^2)Dt}\,dt \qquad (19)\\
&=\frac{16}{\pi^2}\sum_{\substack{m\geq 1\\ m\text{ odd}}}\sum_{\substack{m\geq 1\\ n\text{ odd}}}\frac{\sin\left(\frac{m\pi u_0}{R}\right)\sin\left(\frac{n\pi v_0}{R}\right)}{mn}\int_0^\infty e^{-\frac{\pi^2}{R^2}(m^2+n^2)Dt}dt \qquad (20)\\
&=\frac{16R^2}{D\pi^4}\sum_{\substack{m\geq 1\\ m\text{ odd}}}^{\infty}\sum_{\substack{n\geq 1\\ n\text{ odd}}}^{\infty}\frac{\sin\left(\frac{m\pi u_0}{R}\right)\sin\left(\frac{n\pi v_0}{R}\right)}{mn(m^2+n^2)},
\end{aligned}$$

where we have used the property that the series $S(t;\cdot,\cdot)$ is uniformly convergent in $[0,\infty)$ (since $S(t;\cdot,\cdot)\leq 1$ for all $t\geq 0$ by definition of $S(t;\cdot,\cdot)$) to interchange the summation and the integral signs in (19) which gives (20). This concludes the proof. ∎

Appendix B - Proof of Proposition 4 (Pdf of location at relay epoch)

Let $x(t)$ and $y(t)$ be the relative positions at time t of mobiles X and Y in $[0, L]$ and $[L, 2L]$, respectively. Let T the first time when $x(t) + r \geq y(t) + L$. Observe that $T = 0$ if $x(0) + r \geq y(0) + L$. We have

$$P(y(T) < y \,|\, x(0) = x_0) = \frac{1}{L}\int_0^L P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)\, dy_0$$
$$= \frac{1}{L}\int_0^L \mathbf{1}_{\{x_0+r\geq L+y_0\}}\mathbf{1}_{\{y>y_0\}}\, dy_0 + \frac{1}{L}\int_0^L \mathbf{1}_{\{x_0+r<L+y_0, y\geq r-L\}} P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)\, dy_0$$
$$= \frac{1}{L}\min(x_0 + r - L, y) + \frac{1}{L}\mathbf{1}_{\{y\geq r-L, x_0<2L-r\}}\int_{x_0+r-L}^L P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)\, dy_0,$$

where the indicator function $\mathbf{1}_{\{y\geq r-L\}}$ in the second integral in the second equality accounts for the fact that if the transfer does not take place at $t = 0$ (under the condition $x_0 + r < y + L$ then necessarily $T > 0$) then mobile Y can not be located in $[L, L - r)$ at time T as otherwise the relay would have occured before time T. Differentiating both sides of the above relation w.r.t. y gives

$$q(y; x_0) = \frac{1}{L}\mathbf{1}_{\{y\leq x_0+r-L\}} + \frac{1}{L}\mathbf{1}_{\{y\geq r-L, x_0<2L-r\}}\int_{x_0+r-L}^L g(y; x_0, y_0)\, dy_0, \tag{21}$$

with $g(y; x_0, y_0) := (\partial/\partial y)P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)$. It remains to evaluate $g(y; x_0, y_0)$. To this end, we will use again the method of images (see proof of Proposition 1).

Consider a square of size R by R, with $R = \sqrt{2}(2L - r)$, delimited by the (absorbing) boundaries $x' = 0$, $x' = R$, $y' = 0$ and $y' = R$. Starting from position (x'_0, y'_0) at time $t = 0$, the pdf $p(x', y', t; x'_0, y'_0)$ of the location of the mobile at time t is given by (see (16))

$$p(x', y', t; x'_0, y'_0) = \frac{4}{R^2}\sum_{n\geq 1}\sum_{n\geq 1} e^{-(m^2+n^2)(\pi/R)^2 Dt} \sin\left(\frac{m\pi x'}{R}\right)\sin\left(\frac{n\pi y'}{R}\right)\sin\left(\frac{m\pi x'_0}{R}\right)\sin\left(\frac{n\pi y'_0}{R}\right).$$

Let $\xi(x'; x'_0, y'_0)$, $0 \leq x\prime \leq R$, be the pdf of the absorption occuring at point $(x', 0)$. From [8, p. 25, p. 45] we obtain

$$\xi(x'; x'_0, y'_0) = D\int_0^\infty \frac{\partial p(x', y', t; x'_0, y'_0)}{\partial y'}|_{y'=0}\, dt \tag{22}$$
$$= \frac{4}{R\pi}\sum_{n\geq 1}\sum_{n\geq 1}\frac{n}{m^2+n^2}\sin\left(\frac{m\pi x'}{R}\right)\sin\left(\frac{m\pi x'_0}{R}\right)\sin\left(\frac{n\pi y'_0}{R}\right). \tag{23}$$

With the method of images (see [5] for details) we find

$$g(y; x_0, y_0) = \xi\left(\sqrt{2}(y + L - r); x'_0, y'_0\right) + \xi\left(\sqrt{2}(y + L - r); y'_0, x'_0\right)$$
$$+\xi\left(\sqrt{2}(y + L - r); \sqrt{2}(2L - rx'_0, 2L - r - y'_0\right) + \xi\left(\sqrt{2}(y + L - r); \sqrt{2}(2L - r - y'_0, 2L - rx'_0\right) \tag{24}$$

with $x'_0 = (x_0 + y_0 + L - r)/\sqrt{2}$ and $y'_0 = (y_0 - x_0 + L - r)/\sqrt{2}$. Plugging (24) into (21) yields (9) after tedious but elementary algebra.

■