On Fluid Analysis of Queues with Selective Message Discarding Policies

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Abstract

In this paper we analyse a fluid queue with an on-off Markov modulated fluid process as the arrival process. We call each on-period of the arrival process a message. The queue incorporates a selective message discarding policy to prevent network congestion. We work within the framework of Poisson Counter Driven Stochastic Differential Equations and obtain the moment generating function and hence the probability distribution function of the stationary workload process. We then comment on the stability of this fluid queue.

1 Introduction

In this paper we analyse a fluid queue having some selective message discarding policy. A message (or a frame) is a group of packets. Often, a loss of a single packet in a message can result in the loss of the whole message. Thus packets of a message which arrive after a packet is lost from the same message or has the potential of getting lost, due to buffer overflow, have no use any more and it is advantageous to discard them to avoid the occurrence of future congestion due to useless packets (packets of a corrupted message) traversing the network. Such ideas to prevent network congestion were presented in [3] and [4].

Queues with such policies arise in switches incorporating selective message discarding policies. We work in the framework of fluid queues. The input source is modeled as an on-off Markov modulated source with constant fluid arrival rate in on-state; the server has a constant capacity. We call a message the total fluid arriving in an on-period. The fluid queue adopts the policy of accepting fluid corresponding to only those messages, at whose commencement epoch, the queue length (alternatively the work-load process) is less than some threshold level. We consider the case of an infinite buffer. By commencement epoch of a message we mean the time instant when the modulating process goes from off to on state.

The paper is organised as follows. In Section 2 we describe our model and state the main results. In Section 3 we analyse our model and prove our main results. We do the stability analysis of our queue in Section 4 and finally conclude in Section 5.

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2 Model and Main Results

Our fluid source is a Markov modulated on-off source. Let θ denote the state of the Markov chain. The source sends fluid at rate h when in on($\theta = 1$) state and at rate 0 when in off($\theta = 0$ state). The off and on periods are exponentially distributed with rate λ_1 and λ_2 respectively. The sojourn time in the on-state corresponds to the length of a message. Thus our message lengths are exponentially distributed with parameter λ_2 and silence time (when there is no fluid coming) between messages is exponentially distributed with parameter λ_1.

The discarding policy is such that if at the commencement epoch of a message the workload process $v(t)$ is less than some threshold, the message is admitted, otherwise not.

The fluid source is input to an infinite buffer served with a rate c. To remove trivialities we assume that $c < h$. Here we admit a message fully if at its starting epoch, the workload v is less than B. We write the dynamics of the system in terms of *Poisson Counter Driven Stochastic Differential Equations* (PCDSDE)[1]. Let N_1 and N_2 be Poisson counters with parameters λ_1 and λ_2. We define a new variable x as the *virtual arrival-process* to the buffer. The process $x(t)$ captures the behavior of the discarding policy. The dynamics of $\theta(t)$, $x(t)$ and $v(t)$ are,

$$d\theta(t) \;=\; (-\theta(t) + 1)dN_1 - \theta(t)dN_2$$

$$dx(t) \;=\; (-x(t) + 1)dN_1\mathbf{I}(v(t) < B) - x(t)dN_2 \qquad (1)$$

$$dv(t) \;=\; -c\mathbf{I}(v(t) > 0)dt + hx(t)dt \qquad (2)$$

Remark 1 *Observe that $\theta(t)$ is a two-state MMFP with rates λ_1 and λ_2. Since the policy does not accept those messages at whose commencement epoch the workload process is greater than or equal to B, hence even if at time t, θ shifts from off to on state but $v(t) \geq B$, the workload process will continue evolving as if $\theta = 0$. Thus, we define a* virtual arrival-process *$x(t)$ which captures this effect. In fact x is the arrival process being admitted to the queue and is not Markovian. A pictorial view of the dynamics of $\theta(t)$, $x(t)$ and $v(t)$ is provided in Figure 1.*

We will now work with the above equations for finding a recursive formula for the average workload moments $E[v^n]$ for all positive integers n. The Laplace-Stieltjes transform, $V(s)(= E[e^{-sv}])$ of v will then be obtained from which we shall obtain the stationary probability density function, $\rho(v)$ of v. We first present our main results.

Proposition 1 *$V(s)$ is given by*

$$V(s) \;=\; \frac{k_2(1-p)}{(1+k_1 s)}\left(\frac{\lambda_2 e^{-Bs}}{\lambda_2 + (h-c)s} - 1\right) + 1 - \frac{shk_1E[x]}{c(1+k_1 s)} \qquad (3)$$

where $p = Prob(v < B)$ and

$$k_1 = \left[\frac{(h-c)}{(\lambda_1+\lambda_2)}\left(1 - \frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] \quad k_2 = \left[1 - \tfrac{\lambda_2}{\lambda_1+\lambda_2}\left(1 - \tfrac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] \qquad (4)$$

and

$$E[x] \;=\; \frac{\lambda_1}{\lambda_1+\lambda_2}\left(p(1-\frac{c}{h}) + \frac{c}{h}\right)$$

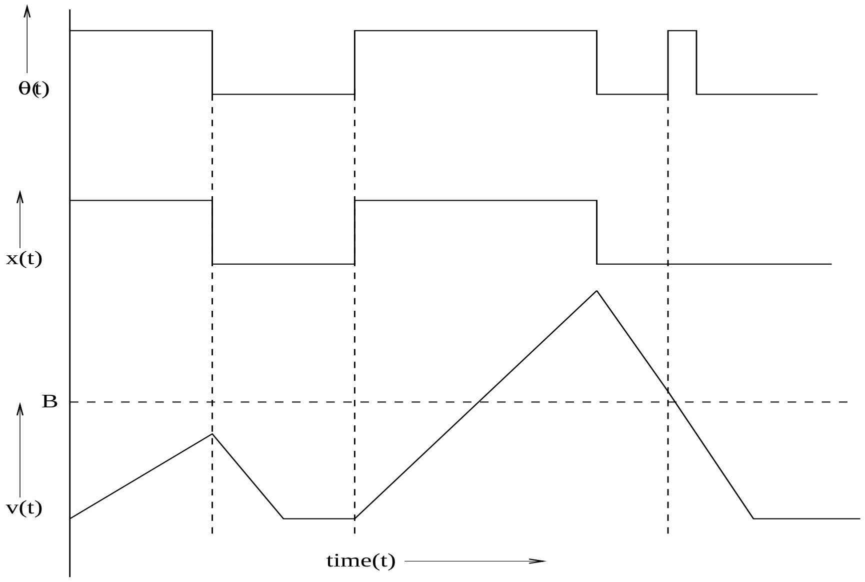

Figure 1: A realization of $\theta(t)$, $x(t)$ and $v(t)$ process

Proposition 2 *The stationary probability density function $\rho(.)$ of $v(t)$is given by,*

$$\rho(m) \;=\; \begin{cases} \rho_1(m) & \textit{for } 0 \le m < B \\ \frac{k_2(1-p)\frac{a\lambda_2}{(h-c)}}{\left(a-\frac{\lambda_2}{(h-c)}\right)}\left(e^{-\frac{\lambda_2}{(h-c)}(m-B)} - e^{-a(m-B)}\right) + \rho_1(m) & \textit{for } m \ge B \end{cases}$$

where,

$$\rho_1(m) \;=\; \delta(m)\left(1-\frac{h}{c}E[x]\right) + \left(\frac{h}{ck_1}E[x] - \frac{k_2}{k_1}(1-p)\right)e^{-am}$$

k_1, k_2 and $E[x]$ as defined in Proposition 1 with

$$p = \frac{1}{1+\frac{\alpha}{1-\frac{\alpha c}{h}}}$$

where,

$$\alpha \;=\; \frac{\lambda_1 e^{-aB}h}{(\lambda_1+\lambda_2)c}\left(1-k_2+k_2e^{-aB}\right)^{-1}$$

and $a = k_1^{-1}$.

3 Model Analysis and Proofs of the Main Results

Observe that, by using the Itô [1] calculus we get from Equations (1) and (2),

$$\begin{aligned} dv^2(t) &= -2cv(t)\mathbf{I}v(t)>0)dt + 2v(t)hx(t)dt \\ d(v(t)x(t)) &= v(t)dx(t) + x(t)dv(t) \\ &= v(t)\left((-x(t)+1)dN_1\mathbf{I}(v(t)<B) - x(t)dN_2\right) + x(t)\left(-c\mathbf{I}(v(t)>0)dt + hxdt\right) \end{aligned}$$

[1]For related background refer to the Appendix, see also [1]

We denote by $p = Prob(v < B)$. From the last two equations we get [2]

$$
\begin{aligned}
dE[v^2] &= -2cE[v] + 2h\left(E[vx|v < B]p + E[vx|v \geq B](1-p)\right) \\
dE[vx] &= \lambda_1 dt E[v(-x+1)\mathbf{I}(v < B)] - \lambda_2 dt E[vx] - cE[x\mathbf{I}(v > 0)]dt + hE[x^2]dt \\
&= (h-c)E[x]dt + E[v|v < B]p\lambda_1 dt - E[vx|v < B]\lambda_1 p dt \\
&- E[vx|v < B]\lambda_2 p dt - E[vx|v \geq B]\lambda_2(1-p)dt
\end{aligned}
$$

Observe that, $E[x\mathbf{I}(v > 0)] = E[x]$, because v is positive whenever x is. Thus if a steady state exists then

$$
\begin{pmatrix} p\lambda_1 & 0 & -(\lambda_1+\lambda_2)p & -\lambda_2(1-p) \\ -cp & -c(1-p) & hp & h(1-p) \end{pmatrix} E \begin{pmatrix} (v|v < B) \\ (v|v \geq B) \\ (vx|v < B) \\ (vx|v \geq B) \end{pmatrix} + \begin{pmatrix} (h-c)E[x] \\ 0 \end{pmatrix} = 0 \qquad (5)
$$

Thus we shall have two equations in six unknowns (the four conditional expectations, $E[x]$ and p). We shall obtain expressions for $E[v|v > B]$, $E[vx|v > B]$ and $E[x]$ in the following section. Observe also the fact that the fluid level v never remains steady at B (visiting it only at isolated points in time) and hence we take the density function of v to be continuous at B. Note that the density function need not be *differentiable* at B. Thus we guess that the density function will be of the form

$$
\rho(m) = \begin{cases} C_0\delta(m) & \text{for } m = 0 \\ \rho_1(m) & \text{for } m \in (0, B) \\ \rho_2(m) & \text{for } m \in [B, \infty) \end{cases}
$$

3.1 Calculation of $E[v|v \geq B]$ and $E[vx|v \geq B]$

Observe that when $v(t) \geq B$ and the state of $\theta(t)$ changes from 0 to 1 for the first time (and succeeding times) then the incoming fluid is not accepted. This is done till time t when $v(t) < B$. A realization of the evolution of $v(t)$ is shown in Figure 2. Observe that $v(t)$ can only cross the threshold of B at any time t, if $x(t) = x(t^-) = 1$. Since the sojourn time in a state is exponentially distributed, the excess time the Markov chain spends in state 1 after time t is again exponentially distributed given that the chain was in state 1 at time t. The fluid level will rise steadily with a rate $(h-c)$ from t until the Poisson process N_2 causes x to change from 1 to 0. Then v will start decreasing at a steady rate of c until the buffer level is B. During this period (when $v \geq B$), even if the process θ changes to 1 (and fluid starts arriving at rate h), the fluid is not accepted (the fact captured by the presence of an indicator function $\mathbf{I}(v < B)$ in the dynamics of $x(t)$ by Equation 1).

Let T_i $i = 1, 2, \ldots$ be the random variable denoting the time spent by v after crossing B at the ith cross. Thus

$$
T_i = X_i + \frac{X_i(h-c)}{c} = X_i\frac{h}{c} \qquad (6)
$$

[2]for notational convenience we shall not explicitly show t in parentheses

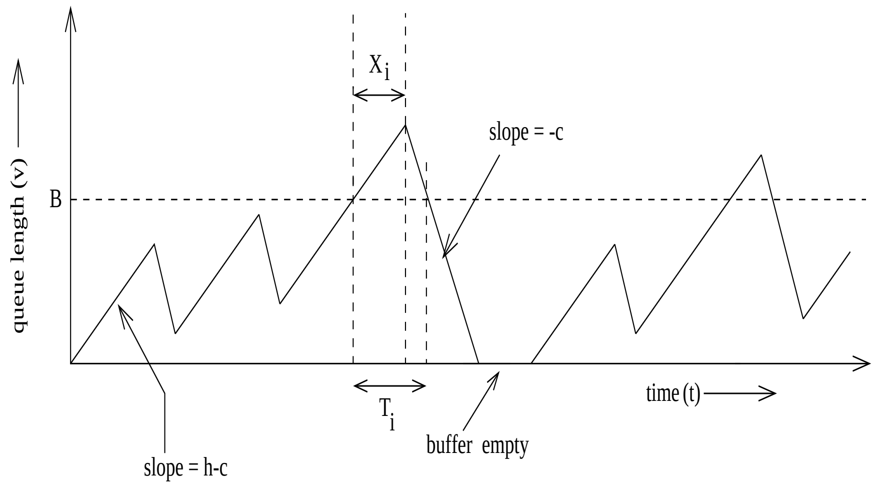

Figure 2: An evolution of the workload process $v(t)$

where X_i is the excess sojourn time in state 1. Notice that the condition $v(t) \geq B$ implies that at the time of crossing the level B, say at time $t_1 \leq t$, $x(t_1) = 1$. Thus, the sequence $\{T_i\}$ is i.i.d. ($X_i \sim \exp \lambda_1$) and the process $T = \sum_{i=1}^{n} T_i$ is a renewal process. We can write

$$\begin{aligned} E[v|v \geq B] &= E[v|v \geq B, \frac{dv}{dt} = (h-c)]P(\frac{dv}{dt} = (h-c)|v \geq B) \\ &+ E[v|v \geq B, \frac{dv}{dt} = -c]P(\frac{dv}{dt} = -c|v \geq B) \end{aligned} \quad (7)$$

Also from Equation (6) we get $X_i = \frac{c}{h} T_i$. Thus we have $P(\frac{dv}{dt} = (h-c)|v \geq B) = \frac{c}{h}$ and $P(\frac{dv}{dt} = -c|v \geq B) = 1 - P(\frac{dv}{dt} = (h-c)|v \geq B) = 1 - \frac{c}{h}$. Also, the process $\{X_i\}$ is i.i.d. and so in the space where $(\frac{dv}{dt} = h - c, v \geq B)$ the process $X = \sum_{i=1}^{n} X_i$ is renewal. By the *Reward Renewal Theorem* [6] we can easily show that,

$$E[v|v \geq B, \frac{dv}{dt} = (h-c)] = B + (h-c)\lambda_2^{-1}$$

And,

$$E[v|v \geq B, \frac{dv}{dt} = -c] \;=\; B + (h-c)\lambda_2^{-1}$$

Thus we have from Equation 7

$$E[v|v \geq B] \;=\; B + (h-c)\lambda_2^{-1}$$

We will now proceed to calculate $E[vx|v \geq B]$. By similar arguments, we can write

$$E[vx|v \geq B] \;=\; E[vx|v \geq B, \frac{dv}{dt} = (h-c)]\frac{c}{h} + E[vx|v \geq B, \frac{dv}{dt} = -c](1 - \frac{c}{h})$$

Observe that, $x = 1$ when $\left(\frac{dv}{dt} = (h-c), v \geq B\right)$ and $x = 0$ when $\left(\frac{dv}{dt} = -c, v \geq B\right)$. Thus,

$$\begin{aligned} E[vx|v \geq B] &= E[v|v \geq B, \frac{dv}{dt} = (h-c)]\frac{c}{h} \\ &= \left(B + (h-c)\lambda_2^{-1}\right)\frac{c}{h} \end{aligned}$$

Having obtained $E[v|v \geq B]$ and $E[vx|v \geq B]$ we now return to the steady-state condition (Equation 5). From steady state conditions we have

$$-c\left(E[v|v < B]p + E[v|v \geq B](1-p)\right) + h\left(E(vx|v < B)p + E(vx|v \geq B)(1-p)\right) = 0$$
$$(h-c)Ex + E[v|v < B]p\lambda_1 - E[vx|v < B]\lambda_1 p - E[vx|v < B]\lambda_2 p - E[vx|v \geq B]\lambda_2(1-p) = 0$$

Observe that we also need an expression for $E[x]$. In the next Lemma we obtain an expression for $E[x]$ in terms of p.

Lemma 1 *E[x] is given by,*

$$E[x] \quad = \quad \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(p(1 - \frac{c}{h}) + \frac{c}{h}\right) \tag{8}$$

Proof:
In steady state from Equation 1, we get

$$\lambda_2 E[x] \quad = \quad p\lambda_1 - E[x|v < B]p\lambda_1 \tag{9}$$

But,

$$\begin{aligned} E[x] &= E[x|v < B]p + E[x|v \geq B](1-p) \\ &= E[x|v < B]p + \left(E[x|v \geq B, \frac{dv}{dt} = h - c]P(\frac{dv}{dt} = h - c|v \geq B)\right. \\ &\quad \left. + E[x|v \geq B, \frac{dv}{dt} = -c]P(\frac{dv}{dt} = -c|v \geq B)\right)(1-p) \\ &= E[x|v < B]p + \frac{c}{h}(1-p) \end{aligned} \tag{10}$$

The last equality follows as $x = 0$ for the case $\left(v \geq B, \frac{dv}{dt} = -c\right)$ and $P(\frac{dv}{dt} = h - c|v \geq B) = \frac{c}{h}$ from preceding discussion. From Equations (9) and (10) we get (8).

3.2 Proof of Proposition 1:

From the steady-state equations, after some calculations we get,

$$\begin{aligned} E[v] &= \left[\frac{h(h-c)}{c(\lambda_1+\lambda_2)}\left(1 - \frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] E[x] \\ &+ E[v|v \geq B](1-p)\left[1 - \frac{\lambda_2}{\lambda_1+\lambda_2}\left(1 - \frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] \end{aligned} \tag{11}$$

Similarly we can write the steady state equations for any n, $n \geq 1$ by Itô rule. Thus, in general for the nth moment we get [3]

$$E[v^n|v < B] \quad = \quad \left[n\frac{h(h-c)}{c(\lambda_1+\lambda_2)}E[v^{n-1}x] - E[v^n|v \geq B](1-p)\frac{\lambda_2}{\lambda_1+\lambda_2}\right]\left(p - \frac{hp\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}$$

[3] For details see [2]

Observe that we can obtain expressions for $E[v^n|v \geq B]$ and $E[v^n x|v \geq B]$ for any $n > 1$ by similar arguments as for $n = 1$. Also for any $n \geq 1$, $E[v^n x] = \frac{c}{h}E[v^n]$. Thus we get for $n \geq 2$

$$\begin{aligned}
E[v^n|v < B] &= \left[n\frac{(h-c)}{(\lambda_1+\lambda_2)}E[v^{n-1}] - E[v^n|v \geq B](1-p)\frac{\lambda_2}{\lambda_1+\lambda_2}\right]\left(p - \frac{hp\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1} \\
\Rightarrow E[v^n] &= \left[n\frac{(h-c)}{(\lambda_1+\lambda_2)}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right]E[v^{n-1}] \\
&+ E[v^n|v \geq B](1-p)\left[1-\frac{\lambda_2}{\lambda_1+\lambda_2}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] \qquad (12)
\end{aligned}$$

and for $n = 1$

$$\begin{aligned}
E[v^n] &= \left[n\frac{h(h-c)}{c(\lambda_1+\lambda_2)}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right]E[v^{n-1}x] \\
&+ E[v^n|v \geq B](1-p)\left[1-\frac{\lambda_2}{\lambda_1+\lambda_2}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] \qquad (13)
\end{aligned}$$

Thus we have obtained recursive relation between $E[v^n]$ and $E[v^{n-1}]$ for $n \geq 1$ in Equations (12) and (13). Observe that, for $n \geq 0$, $E[v^n|v \geq B] = E[v^n|v \geq B, \frac{dv}{dt} = (h-c)] = E[v^n|v \geq B, \frac{dv}{dt} = -c]$, hence we calculate

$$E[v^n|v \geq B] = E\left[\frac{(B+X_1(h-c))^{n+1} - B^{n+1}}{(n+1)(h-c)}\right]\lambda_2 \qquad (14)$$

Substituting the expression for $E[v^n|v \geq B]$ from Equation (14) in Equation (12) and then multiplying both the sides of the resulting equation by $\frac{(-s)^n}{n!}$ and summing from $n = 2$ to ∞, we can write,

$$\begin{aligned}
\sum_{n\geq 2}\frac{(-s)^n}{n!}E[v^n] &= \left[\frac{(h-c)}{(\lambda_1+\lambda_2)}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right](-s)\sum_{n\geq 2}\frac{(-s)^{n-1}}{n-1!}E[v^{n-1}] \\
&+ (1-p)\left[1-\frac{\lambda_2}{\lambda_1+\lambda_2}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right]E\left[\frac{1}{(h-c)(-s)}\sum_{n\geq 2}\right. \\
&\left.[(B+X_1(h-c))^{n+1} - B^{n+1}]\frac{(-s)^{n+1}}{n+1!}\right]\lambda_2 \qquad (15)
\end{aligned}$$

Adding to Equation (15) $(-s)\times$ *Equation* (13) we get,

$$\begin{aligned}
\sum_{n\geq 1}\frac{(-s)^n}{n!}E[v^n] &= k_1(-s)\sum_{n\geq 2}\frac{(-s)^{n-1}}{n-1!}E[v^{n-1}] + (1-p)k_2 E\left[\frac{1}{(h-c)(-s)}\sum_{n\geq 1}\right. \\
&\left.[(B+X_1(h-c))^{n+1} - B^{n+1}]\frac{(-s)^{n+1}}{(n+1)!}\right]\lambda_2 + \frac{hk_1}{c}E[x](-s) \quad (16)
\end{aligned}$$

where k_i are given in Equation (4). First we will look at the second expectation term on the right side of Equation (16).

$$E\left[\frac{1}{(h-c)(-s)}\sum_{n\geq 1}[(B+X_1(h-c))^{n+1} - B^{n+1}]\frac{(-s)^{n+1}}{n+1!}\right] = E\left[\frac{e^{-Bs}}{(h-c)s}\left(1-e^{-X_1(h-c)s}\right) - X_1\right]$$

From which we get,

$$E\left[\frac{e^{-Bs}}{(h-c)s}\left(1-e^{-X_1(h-c)s}\right)-X_1\right] = \frac{e^{-Bs}}{(h-c)s}\lambda_2\int_0^\infty\left(1-e^{-x(h-c)s}\right)e^{-\lambda_2 x}dx-\lambda_2^{-1}$$
$$= \frac{e^{-Bs}}{\lambda_2+(h-c)s}-\frac{1}{\lambda_2}$$

Using $V(s)=\sum_{n\geq 0}\frac{(-s)^n Ev^n}{n!}$ we get Equation (3) from Equation (16).

3.3 Proof of Proposition 2:

From Proposition (1) we have

$$V(s)=\frac{k_2(1-p)\lambda_2}{k_1(h-c)}e^{-Bs}\frac{1}{(\frac{\lambda_2}{(h-c)}+s)(s+k_1^{-1})}-\frac{k_2(1-p)}{k_1(s+k_1^{-1})}+1-\frac{h}{c}E[x]+\frac{hE[x]}{ck_1(k_1^{-1}+s)} \tag{17}$$

Also viewing the first term on the right side of Equation (17) as the product of two LSTs $\frac{1}{s+\frac{\lambda_2}{(h-c)}}$ and $\frac{1}{s+k_1^{-1}}$ and denoting by L^{-1} the inverse LST [4] we get (with $*$ denoting *convolution* and $a=k_1^{-1}$)

$$L^{-1}\left(\frac{1}{\left(s+\frac{\lambda_2}{(h-c)}\right)(s+a)}\right) = e^{-\frac{\lambda_2}{(h-c)}m}*e^{-am}$$
$$= \frac{1}{\left(a-\frac{\lambda_2}{(h-c)}\right)}\left(e^{-\frac{\lambda_2}{(h-c)}m}-e^{-am}\right)$$

Thus for $m<B$ we have, (from the inverse of *Laplace-Stieltjes Transform* Equation (17)),

$$\rho(m)=\rho_1(m) = \delta(m)\left(1-\frac{hE[x]}{c}\right)+\left(\frac{h}{ck_1}E[x]-\frac{k_2}{k_1}(1-p)\right)e^{-am} \tag{18}$$

And for $m>B$

$$\rho(m)=\rho_2(m) = \frac{k_2(1-p)a\frac{\lambda_2}{(h-c)}}{\left(a-\frac{\lambda_2}{(h-c)}\right)}\left(e^{-\frac{\lambda_2}{(h-c)}(m-B)}-e^{-a(m-B)}\right)+\rho_1(m) \tag{19}$$

where, $\rho_2(m)$ is the continuous density function for $m\in[B,\infty)$. Observe that Equation (19) involves a term $1-p$ on the right side where $1-p=P(m\geq B)$. However $1-p=\int_B^\infty\rho_2(m)dm$. Thus we can write for $B>0$,

$$1-p = \int_B^\infty\left[\frac{k_2(1-p)a\frac{\lambda_2}{(h-c)}}{\left(a-\frac{\lambda_2}{(h-c)}\right)}\left(e^{-\frac{\lambda_2}{(h-c)}(m-B)}-e^{-a(m-B)}\right)+\left(\frac{h}{ck_1}E[x]-ak_2(1-p)\right)e^{-am}\right]dm$$
$$= \frac{k_2(1-p)a\frac{\lambda_2}{(h-c)}}{\left(a-\frac{\lambda_2}{(h-c)}\right)}\left(\frac{1}{\frac{\lambda_2}{(h-c)}}-\frac{1}{a}\right)+\left(\frac{h}{ck_1}E[x]-ak_2(1-p)\right)\frac{e^{-aB}}{a}$$
$$= k_2(1-p)+(\frac{h}{c}E[x]-k_2(1-p))e^{-aB}$$

[4]For a random variable X with distribution $F(x)$ and LST $L(s)(=\int_0^\infty e^{-sx}dF(x)dx)$ by L^{-1} we mean,

$$L^{-1}[L(s)]=\frac{dF(x)}{dx}$$

Thus we get solving for $(1-p)$

$$1-p \;=\; \frac{e^{-aB}hE[x]}{c}\left(1-k_2+k_2e^{-aB}\right)^{-1} \tag{20}$$

However we have the expression for $E[x]$ from Equation (8) in terms of p. Thus from Equation (8) and (20), solving for p gives,

$$p \;=\; \frac{1}{1+\frac{\alpha}{1-\frac{\alpha c}{h}}}$$

where, $\alpha = \frac{\lambda_1 e^{-aB}h}{(\lambda_1+\lambda_2)c}\left(1-k_2+k_2e^{-aB}\right)^{-1}$. Having obtained p we have the density function $\rho_1(.)$ and $\rho_2(.)$.

4 Stability Analysis

Next, we show that the workload process is a renewal process. To that end we consider the Markov chain V_n which is the workload process at the nth transition of $\theta(t)$ from state 0 to state 1. In other words, it is the workload as seen by the commencement of the nth potential arriving message. We shall show that V_n is a Harris recurrent Markov chain, and that the empty state is recurrent whenever $c > 0$. This will then imply that the original workload process is a renewal process.

To that end, we recall the following sufficient condition for the Harris recurrence which follows from Theorem 14.0.1 in p. 330 of [5].

Lemma 2 *Assume that V_n is Ψ-irreducible (for some Ψ) and aperiodic. Let there be a function f, some $\epsilon > 0$, and some small set C such that*

$$E[f(V_{n+1}) - f(V_n)|V_n = v] < -\epsilon + \mathbf{I}\{v \notin C\}. \tag{21}$$

Then the Markov chain V_n is positive recurrent, it has a stationary probability π, and the n-step transition probabilities P^n converges to the π in total variation as $n \to \infty$.

In the above Lemma, a set C is *small* if there exists some integer n, a constant $g > 0$ and a probability measure ϕ over the state space R such that

$$[P^n]_{xA} > g\phi(A) \tag{22}$$

for all $x \in C$ and measurable set $A \subset R$.

Theorem 1 *Assume that $c > 0$. Then the process V_n is Harris recurrent.*

Proof: Define $C = [0, B]$ and $f(v) = v$. Let T be an exponentially distributed random variable with parameter λ_2. Then for all $v \notin C$,

$$E[V_{n+1} - V_n|V_n = v] < -cE[\min(T, B/c)] =: -\epsilon.$$

Thus (21) holds. Next we check that C is indeed a small set. Let Y_i be an exponentially random variable with parameter λ_i, $i = 1, 2$. Viewing Y_1 as the length of the off-period and Y_2 as the 0 of the on-period, we have for any $v \in C$ and measurable $A \subset R$

$$[P^2]_{vA} \geq P(\frac{B}{h-c} < Y_2 < \frac{2B}{h-c}, Y_1 > \frac{3B}{c})P_{oA}.$$

Thus, (22) holds with $\phi(A) = P_{0A}$ and $g = P(\frac{B}{h-c} < Y_2 < \frac{2B}{h-c}, Y_1 > \frac{3B}{c})$.

Observe that V_n is ϕ-irreducible (p. 70,87 of [5]) since the probability of eventually reaching any measurable set $A \subset R$ from any state x is greater that $g\phi(A)$. Finally, the aperiodicity follows since the probability to go from state 0 to state 0 is strictly positive.

5 Conclusion

In this paper we analysed an infinite buffer fluid queue which employs a threshold policy for message discarding. We obtain the exact Laplace-Stieltjes transform, $V(s)$ and hence the exact stationary probability density function, $\rho(.)$ of the stationary workload process. We then analysed the stability of this fluid queue and established that the Markov chain V_n(workload process at the nth transition of the source state from off to on) is Harris recurrent. We are currently working on finding exact expressions for the performance measures like goodput, throughput etc.

Appendix

Consider a stochastic differential equation driven by n independent Poisson Counters $N_1, \ldots, N_n$,

$$dx \quad = \quad f(x)dt + \sum_{i=1}^{n} g_i(x)dN_i, \qquad x \in R^n$$

Then we have the following Itô rule:
If $\phi : R^n \to R$ is differentiable function, then

$$d\phi(t) \quad = \quad < \frac{\partial \phi}{\partial x}, f(x) > dt + \sum_{i=1}^{n} [\phi(x(t) + g_i(x(t))) - \phi(x(t))] dN_i$$

Also, since $x(t)$ is continuous from the left and the Poisson counter is taken to be continuous from the right, we have

$$\frac{d}{dt}(E[x(t)]) \quad = \quad E[f(x(t))] + \sum_{i=1}^{n} (E[g_i(x(t)), t]) \lambda_i$$

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