Fluid Analysis of Early Message Discarding Policy under Heavy Traffic

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Abstract— **A message(or a frame) is a group of consecutive packets (or cells ATM terminology). Often a loss of one packet from the message can result in the loss of the whole message. Selective Message Discarding Policies have been proposed to achieve the twin goal of increased goodput and reduced network congestion by discarding the packets which do not belong to (or have potentials of not belonging to) *good* messages. In this paper we provide a fluid analysis of Early Message Discard. We obtain approximations for performance metrics which are valid for heavy traffic conditions. In particular, we obtain exact expressions for the distribution of the stationary workload process and the fluid *Goodput ratio*. Our analytical results shall be quite helpful in selecting an optimal discarding threshold, in studying the sensitivity of goodput ratio to different parameters and also in problems related to buffer and/or capacity dimensioning to achieve a desired Quality of Service.**

I. INTRODUCTION

Often a set of consecutive packets are grouped into a frame, and loss of one packet results in the loss of the whole frame. This is the case in ATM where a transport layer protocol (known as AAL) is responsible for this grouping, see for e.g., Chapter 5 in [11]. Also in packet over SONET technique data units are grouped into smaller size frames (of variable lengths see [8]) and these are then transmitted over optical links. Selective message discarding policies, as a means for congestion avoidance, were proposed in [6] and [7]. These policies are implemented at network elements and act independently of the users or the behavior of other network elements. Two main policies are the Partial Message Discard (PMD) and the Early Message Discard (EMD). In PMD if the buffer is full when a packet arrives to the network element, this packet and all successive packets that belong to the same message are discarded. A much efficient policy is EMD in which in addition to forced discard when the buffer overflows (like PMD) there is also threshold discarding. In other words if a message begins to arrive when the buffer occupancy is above this threshold the message is discarded. EMD has been employed for e.g., in Cisco Bp 8600 (wide area switches) series [1] and also in Fujitsu switches.

In [7] and [14], the basic performance measure for the study of discarding policies is the *effective throughput*, which is the ratio of good packets on the outgoing link to the total outgoing flow. However as argued in [16], a more suitable performance measure is the *Goodput Ratio*, defined as the ratio of good packets out of the total number of packets that arrive at the network element's input.

Though there have been works on studying the performance of EMD policy numerically e.g., [16], [10], [9] exact expressions for performance metrics are not available. Recently in [5] an algebraic approach for obtaining closed form expressions for the queue length distribution and the goodput ratio was provided (for the packet model of [16]). However the resulting expressions were quite algebraically involved and so we turn to fluid models with the purpose of obtaining simpler expressions that can be used for dimensioning purposes. In [3] we analysed a queue fed with bursty fluid and having a threshold message discarding policy [1] and obtained closed form expressions for the workload process (which were quite complicated) but no expressions for goodput were obtained. The goal of this paper is to present *explicit and simple expressions* for the goodput of the EMD policy using fluid approximations (valid for heavy traffic conditions) which can provide valuable guidelines for network dimensioning. The input process (which may be quite general) is approximated by a fluid with a constant rate. We obtain explicit expressions for the stationary workload process and of the goodput [2] for the fluid approximation. Our analytical results shall be quite helpful in selecting an optimal discarding threshold, in studying the sensitivity of the goodput to different parameters and also in problems related to buffer and/or capacity dimensioning to achieve a desired Quality of Service.

The structure of the paper is as follows. We present the model as well as the description of the system evolution in the next section (Section II). We then derive the Laplace-Stieltjes transform and the probability distribution of the stationary workload process in Section III. The Goodput is then defined, derived and studied in Section IV. We include a stability analysis in Section V and numerical examples in Section VI.

II. MODEL AND ITS MOTIVATION

Our fluid source always has messages to send and the capacity of the fluid buffer is finite, say B. The fluid buffer is served with a capacity c. The fluid arrives at a constant rate of h. The length of a message is assumed to be exponentially distributed with parameter η. If during the arrival of a message, the workload process $V(t)$ (alternatively the queue length, i.e., amount of fluid in the fluid buffer) reaches B, then the whole message becomes corrupted and therefore all the remaining fluid corresponding to this message is dropped, as it has no use anymore. In addition, the fluid already queued or that which had already left the system and that belongs to that message is also considered to be useless. It therefore seems plausible not to start to accept fluid of a new message if it seems likely that this message will eventually become corrupted, which is the case if when the header (the beginning) of the message arrives, the buffer is already close to B. This is the idea behind EMD policy. There-

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[1] This was not EMD as in EMD there is threshold discarding of a full message and also partial message discarding like PMD.

[2] In the paper the words goodput and goodput ratio are used interchangeably.

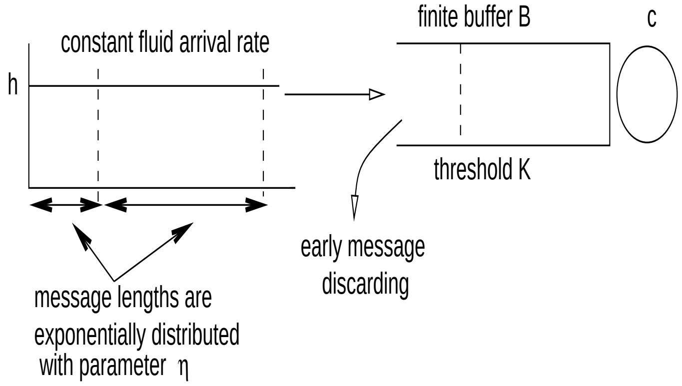

Fig. 1. The system model for the analysis

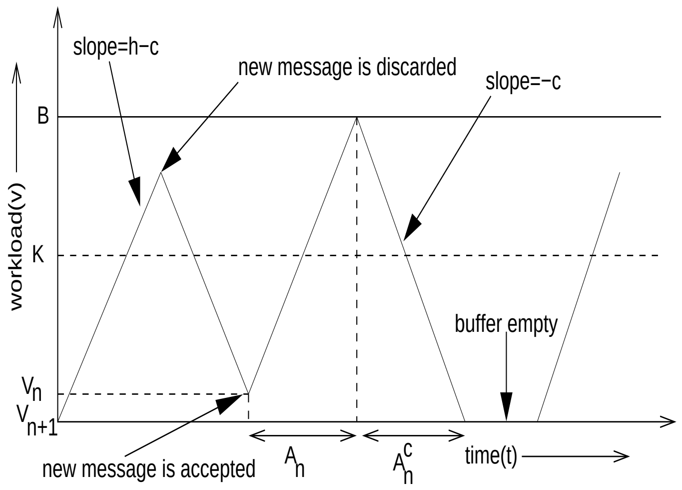

Fig. 2. A typical evolution of the workload process, $V(t)$ in our fluid model

fore an incoming message is only accepted if at the epoch of the arrival of the header of the message, $V(t)$ is less than some threshold K ($K < B$). Fig. 1 shows the system model for our analysis.

Remark 1: The fluid limit can be seen as a weak limit of the packet model in [16] through a standard scaling. More precisely, consider n models, and add n as a superscript to the parameters of the nth model. Then the scaling is obtained as follows.

- Arrival rate: Poisson with $\lambda^{(n)} := n\lambda$;
- Service rate: exponential with $\mu^{(n)} := n\mu$;
- Size of messages: geometrically distributed with parameter $q^{(n)} := q/n$;
- Buffer size: $N^{(n)} = nN$.
- Threshold: $K^{(n)} = nK$.

Let $X^{(n)}(t)$ be the queue length of the nth model. Then, as $n \to \infty$, the process $X^{(n)}(t)/n$ weakly converges to our fluid process $V(t)$, with $h = \lambda$, $c = \mu$, and with $\eta = q\lambda$. The convergence analysis is outside the scope of this paper, for more details refer to [15], [13].

A typical evolution of $V(t)$ in our model is shown in Fig. 2. Also, let A be the event that the incoming fluid is accepted. To remove trivialities we assume that $c < h$ [3].

[3]When $c > h$, then the workload will be 0 w.p. 1.

Next, we describe some recursive equations for the evolution of the workload. To that end, we first present some definitions:

- Let V_n be the random variable denoting the queue length (in terms of amount of fluid) at the end of nth non-acceptance period.
- Let A_n be the random variable denoting the length (in time units) of the $n+1$th acceptance event.
- Let A_n^c be the minimum of the length (in time units) of the $n+1$th non-acceptance event and B/c.
- Let X_n be the remaining length (in time units) of the incoming message at the instant when $V(t)$ reaches K in the $n+1$th acceptance event.

With these definitions, we have the following relations:

$$V_{n+1} = V_n + (h-c)A_n - cA_n^c, \tag{1}$$

$$A_n = \frac{K - V_n}{h-c} + \min\left(\frac{B-K}{h-c}, X_n\right). \tag{2}$$

To express A_n^c, we define first $M_n^{(1)}$ (and $M_n^{(2)}$, respectively) as the remaining lengths of the incoming message (which is being discarded) when the $n+1$th non-acceptance period crosses the level K and the condition $X_n < \frac{B-K}{h-c}$ holds, (respectively, when the condition $\frac{B-K}{h-c} < X_n < \frac{B-K}{h-c} + \frac{B-K}{c}$ holds). A_n^c is then given by

$$\begin{cases} \frac{X_n(h-c)}{c} + Y_n^{(1)}, & X_n < \frac{B-K}{h-c} \\ \frac{B-K}{c} + Y_n^{(2)}, & \frac{B-K}{h-c} < X_n < \frac{B-K}{h-c} + \frac{B-K}{c} \\ X_n - \frac{B-K}{h-c}, & \frac{B-K}{c} < X_n - \frac{B-K}{h-c} < \frac{B}{c} \\ \frac{B}{c}, & \frac{B-K}{h-c} + \frac{B}{c} < X_n \end{cases}$$

where $Y_n^{(i)} := \min\left(M_n^{(i)}, K/c\right), i = 1, 2$.

Next we provide the expectations of A_n and A_n^c which will be useful later on.

$$\begin{aligned} E[A_n] &= \frac{K - E[V_n]}{h-c} + \frac{B-K}{h-c} e^{-\eta\frac{B-K}{h-c}} \\ &\quad + \frac{1}{\eta}(1 - e^{\frac{-\eta(B-K)}{h-c}}) - \frac{B-K}{h-c} e^{\frac{-\eta(B-K)}{h-c}} \\ &= \frac{K - E[V_n]}{h-c} + \frac{1}{\eta}(1 - e^{\frac{-\eta(B-K)}{h-c}}) \end{aligned} \tag{3}$$

We note that $M_n^{(1)}$ and $M_n^{(2)}$ are both exponentially distributed with parameter η and that $M_n^{(i)}, i = 1, 2,\ n \in N$ are i.i.d. Hence, $E[A_n^c]$ is given by

$$\begin{aligned} &\frac{(h-c)}{c}\left(\frac{1}{\eta}(1 - e^{\frac{-\eta(B-K)}{h-c}}) - \frac{B-K}{h-c} e^{\frac{-\eta(B-K)}{h-c}}\right) \\ &- \left(\frac{B-K}{c} + E[Y_n^{(2)}]\right)(e^{-\eta\left(\frac{B-K}{h-c} + \frac{B-K}{c}\right)} \\ &- e^{-\eta\left(\frac{B-K}{h-c}\right)}) + \left[\left(\frac{(B-K)h}{(h-c)c}\right) e^{-\eta\left(\frac{B-K}{h-c} + \frac{B-K}{c}\right)} \right. \\ &\left. - \left(\frac{B-K}{h-c} + \frac{B}{c}\right) e^{-\eta\left(\frac{B-K}{h-c} + \frac{B}{c}\right)}\right] \end{aligned}$$

$$+\frac{1}{\eta}\left(e^{-\eta\left(\frac{B-K}{h-c}+\frac{B-K}{c}\right)}-e^{-\eta\left(\frac{B-K}{h-c}+\frac{B}{c}\right)}\right)$$
$$-\frac{B-K}{h-c}\left(e^{-\eta\left(\frac{B-K}{h-c}+\frac{B-K}{c}\right)}-e^{-\eta\left(\frac{B-K}{h-c}+\frac{B}{c}\right)}\right)$$
$$+\frac{B}{c}e^{-\eta\left(\frac{B-K}{h-c}+\frac{B}{c}\right)}+E[Y_n^{(1)}](1-e^{\frac{-\eta(B-K)}{h-c}})$$
$$= \frac{1}{\eta}\left(\frac{h}{c}(1-e^{\frac{-\eta(B-K)}{h-c}})+(e^{\frac{-\eta(B-K)}{h-c}}-e^{\frac{-\eta K}{c}})\right) \quad (4)$$

Remark 2: $E[V_n]$ in steady state can also be directly calculated from Equation (1). We have

$$E[V_{n+1}] = E[V_n] + (h-c)E[A_n] - cE[A_n^c]$$

which in steady state simplifies to

$$E[A] = \frac{c}{(h-c)}E[A^c] \quad (5)$$

Substituting expressions from Equations (3) and (4) in the last equation we can solve for $E[V]$.

Observe that we have defined the system dynamics via a difference equation (Equation (1)) associated with an embedded point process. We will next proceed to obtain the stationary probability distribution of the point process $\{V_n\}$ after first obtaining its Laplace-Stieltjes Transform (LST) in the next section.

III. The workload: Laplace-Stieltjes Transform and distribution

Denote the LST of V_n by $V_p(s)$[4] in steady state. Let ρ_p be the probability density of V_n in steady state.

Lemma 1: The LST and probability density of V_n in steady state are given by:

$$V_p(s) = \frac{\eta}{\eta - sc}\left(e^{-sK} - \frac{sc}{\eta}e^{\frac{-\eta K}{c}}\right) \quad (6)$$

$$\rho_p(v) = \begin{cases} e^{-\frac{\eta K}{c}}\delta(v) + \frac{\eta}{c}e^{\frac{\eta(v-K)}{c}} & \text{for } v \in [0, K) \\ 0 & \text{for } v \geq K \end{cases} \quad (7)$$

Proof: Observe that, from Equation (1) we can write,

$$\begin{aligned} V_{n+1} = {} & (K - cY_n^{(1)})I\left(X_n < \frac{B-K}{h-c}\right) \\ & +(K - cY_n^{(2)})I\left(\frac{B-K}{h-c} < X_n < \frac{B-K}{h-c}\right. \\ & \left.+\frac{B-K}{c}\right) + \left[B - cX_n + c\left(\frac{B-K}{h-c}\right)\right] \times \\ & I\left(\frac{B-K}{c} < X_n - \frac{B-K}{h-c} < \frac{B}{c}\right) \\ & +0 \cdot I\left(\frac{B-K}{h-c} + \frac{B}{c} < X_n\right) \end{aligned}$$

And thus the generating function,

$$E[e^{-sV_{n+1}}] =$$
$$E[e^{-s(K-cY_n^{(1)})}](1-e^{\frac{\eta(B-K)}{h-c}})$$
$$+E[e^{-s(K-cY_n^{(2)})}](e^{\frac{-\eta(B-K)}{h-c}}-e^{-\eta(\frac{B-K}{h-c}+\frac{B-K}{c})})$$
$$+e^{-s(B+c(\frac{B-K}{h-c}))}\eta\int_{\frac{B-K}{h-c}+\frac{B-K}{c}}^{\frac{B-K}{h-c}+\frac{B}{c}} e^{-x(\eta-sc)}dx$$
$$+\eta\int_{\frac{B-K}{h-c}+\frac{B}{c}}^{\infty} e^{-\eta x}dx$$
$$= \eta\left[\frac{e^{-sK}-e^{-\frac{K\eta}{c}}}{\eta-sc}+\frac{e^{\frac{-\eta K}{c}}}{\eta}\right](1-e^{-\eta(\frac{(B-K)h}{c(h-c)})})$$
$$+e^{-\eta\left(\frac{B-K}{h-c}+\frac{B}{c}\right)}\left[\eta\left(\frac{e^{\frac{K}{c}(\eta-sc)}-1}{\eta-sc}\right)+1\right]$$

which gives (6). The inversion of the LST [5] implies (7).

In the above Lemma we obtained the workload LST and the probability density at the end of non-acceptance periods. Next we shall obtain these quantities at an arbitrary time, i.e., the LST and the probability density of the stationary workload. To that end, we first define B_n^c as the length of the nth discarding period. It can thus be represented as

$$B_n^c = \begin{cases} \frac{X_n(h-c)}{c} + M_n^{(1)}, & X_n < \frac{B-K}{h-c} \\ \frac{B-K}{c} + M_n^{(2)}, & \frac{B-K}{h-c} < X_n < \frac{(B-K)h}{(h-c)c} \\ X_n - \frac{B-K}{h-c} & \text{otherwise} \end{cases}$$

and $E[B_n^c]$ is given by

$$\frac{(h-c)}{c}\left[\frac{1}{\eta}(1-e^{\frac{-\eta(B-K)}{h-c}})-\right.$$
$$\left.\left(\frac{B-K}{h-c}\right)e^{\frac{-\eta(B-K)}{h-c}}\right]+\frac{1}{\eta}(1-e^{\frac{-\eta(B-K)}{h-c}})$$
$$+\left(\frac{B-K}{c}+\frac{1}{\eta}\right)(e^{\frac{-\eta(B-K)}{h-c}}-e^{-\eta(\frac{B-K}{h-c}+\frac{B-K}{c})})$$
$$+e^{-\eta\left(\frac{B-K}{h-c}+\frac{B-K}{c}\right)}\left(\frac{B-K}{c}+\frac{1}{\eta}\right)$$
$$= \frac{h-(h-c)e^{\frac{-\eta(B-K)}{c}}}{\eta c} \quad (8)$$

Proposition 1: The LST $\mathcal{V}(s)$ and the probability density $\rho(v)$ of $V(t)$ in stationary regime are given by:

$$\begin{aligned} \mathcal{V}(s) = \Big(& \frac{h}{s(h-c)c} \\ & \times(E^0[e^{-sV_0}] - E^0[e^{-s[K+(h-c)\min(\frac{B-K}{h-c},X_n)]}]) \\ & +E[B_0^c] - E[A_0^c]\Big)R \end{aligned} \quad (9)$$

$$\rho(v) = \quad (10)$$

[4]The subscript p indicates the point process

[5]By inverse of LST of a random variable we mean the density function of the random variable, i.e., if A is a random variable with density $f(a)$ and $\mathcal{A}(s)$ is the LST of A, then denoting by LST^{-1} the inverse LST

$$LST^{-1}\mathcal{A}(s) = f(a)$$

$$\begin{cases} R(E[B_0^c]-E[A_0^c])\delta(v), & \text{if } v=0 \\ R\alpha e^{\frac{\eta(v-K)}{c}} & \text{if } v\in(0,K] \\ R\alpha e^{-\frac{\eta(v-K)}{h-c}}, & \text{if } v\in(K,B) \\ 0, & \text{if } v\ge B. \end{cases}$$

where, $\alpha = \frac{h}{(h-c)c}$, $R = (E[A_0]+E[B_0^c])^{-1}$ and $E[A_0]$, $E[A_0^c]$ and $E[B_0^c]$ were computed in (3), (4) and (8), respectively. Also $E^0[.]$ is the expectation associated with the Palm distribution ([2]).

Proof: We shall now use the inversion formula (see e.g., [2], Ch.1, Sec. 4) to obtain the LST for the stationary workload process $V(t)$ which we will invert to get the probability density function. Thus,

$$\begin{aligned}
&\mathcal{V}(s)-1= \\
&\quad\Big(E^0[\int_0^{A_0}(\exp\{-s(V_0+(h-c)t)\}-1)\,dt] \\
&\quad+E\Big[\int_{A_0}^{A_0+A_0^c}\Big(\exp\{-s(V_0+(h-c)A_0-c(t-A_0))\} \\
&\quad-1\Big)dt\Big]\Big)\,E^0[A_0+B_0^c]
\end{aligned}$$

From the last equation,

$$\begin{aligned}
\mathcal{V}(s) &= \Big(E^0\Big[e^{-sV_0}\int_0^{A_0}e^{-s(h-c)t}dt \\
&\quad+e^{-s(V_0+hA_0)}\int_{A_0}^{A_0+A_0^c}e^{sct}dt+B_0^c-A_0^c\Big]\Big)R \\
&= \Big(\frac{E^0[e^{-sV_0}]}{s(h-c)}+\frac{E^0[e^{-sV_1}]}{sc}-\frac{h}{s(h-c)c} \\
&\quad\times E^0\Big[\exp\{-s[K+(h-c)\min(\frac{B-K}{h-c},X_n)]\}\Big] \\
&\quad+E^0[B_0^c]-E^0[A_0^c]\Big)R
\end{aligned}$$

which gives (9). The inversion of LST implies (10).

Having obtained the density function we can easily calculate all the performance metrics. In particular all the moments of the process $V(t)$ (these can also easily be calculated from the *Generating Function*, $V(s)$) and the *Goodput Ratio*.

IV. The Goodput Ratio: $\mathcal{G}_f$

We proceed with the model of the previous section and in particular, we continue to assume that $c<h$ in order to avoid trivialities. We define, the fluid analog of the *Goodput Ratio* $\mathcal{G}_f$ (that was initially defined in [16], for a packet model), as the ratio of the total fluid comprising good messages (i.e., messages which do not suffer any fluid loss due threshold discarding or due to buffer overflow) exiting the node to the total arriving fluid at its input. Let $\mathcal{V}_f$ be the random variable representing the success of a message, $\mathcal{V}_f=1$ for a good message, and $\mathcal{V}_f=0$ for a message which has lost some fluid. Let W be the random variable denoting the length of a message. Let us define the sub-distribution function $F(w,1)$ as the probability that a message is of length $\le w$ and is good, i.e,

$$F(w,1) \;=\; P(W\le w,\mathcal{V}_f=1)$$

Then we can write the goodput ratio as

$$\mathcal{G}_f \;=\; \frac{\int_0^\infty w dF(w,1)}{\int_0^\infty w dF(w)} \tag{11}$$

where, $F(w)$ is the message length distribution ($\sim\ \exp\eta$). Again, writing $F(w,1)$ as,

$$\begin{aligned}
F(w,1) &= P(\mathcal{V}_f=1|W\le w)P(W\le w) \\
&= \int_0^B P(\mathcal{V}_f=1|W\le w,V_a=v)\rho_a(v)dv \\
&\quad\times\int_0^w f(u)du \\
&= \int_0^B P(\mathcal{V}_f=1|W\le w,V_a=v)\rho(v)dv \\
&\quad\times\int_0^w f(u)du
\end{aligned} \tag{12}$$

where, $f(u)$ is the message length density and V_a is the queue length at the arrival epoch of a message with $\rho_a(v)$ being the stationary density of V_a [6].

Lemma 2: $\mathcal{G}_f$ can be written as

$$\mathcal{G}_f=\eta\int_0^{\frac{B}{h-c}} w dF(w,1) \tag{13}$$

Proof: Refer to Appendix A.

Proposition 2: The expression for $\mathcal{G}_f$ is

$$\begin{aligned}
\mathcal{G}_f &= \eta^2R\,[I_0\,(E[B_0^c]-E[A_0^c] \\
&\quad-\frac{\alpha c}{\eta}e^{\frac{-\eta K}{c}}\Big)+\frac{\alpha c}{\eta}(I_1+e^{\frac{\eta(B-K)}{c}}I_2)\Big]
\end{aligned} \tag{14}$$

where,

$$I_0=\int_0^{\frac{B}{h-c}}we^{-\eta w}dw,\qquad I_1=\int_0^{\frac{B-K}{h-c}}we^{-\eta w}dw,$$

$$I_2=\int_{\frac{B-K}{h-c}}^{\frac{B}{h-c}}we^{\frac{-\eta wh}{c}}dw.$$

Proof: Refer to Appendix A.

Remark 3: Having obtained the closed form expression for $\mathcal{G}_f$ we can qualitatively study the sensitivity of $\mathcal{G}_f$ to changes in different parameters of the system. This can be done by evaluating expressions for the partial derivative of the expression for $\mathcal{G}_f$ with respect to different parameters. Further optimal value of parameters can be obtained.

[6] due to PASTA the queue length distribution at the arrival epochs of messages, which come as a Poisson stream, is same as the stationary queue length distribution and hence the last equality in (12).

We next write explicit expressions for R and $E[B_0^c] - E[A_0^c]$ from Eqs. (4), (5) and (8) which we shall need in further analysis of the expression for $\mathcal{G}_f$ from Proposition 2.

$$R = \left(\frac{h^2}{\eta c(h-c)}(1 - e^{\frac{-\eta(B-K)}{c}}) + \frac{1}{\eta(h-c)}\left(he^{\frac{-\eta(B-K)}{c}} - ce^{\frac{-\eta K}{c}}\right)\right)^{-1}$$

$$E[B_0^c] - E[A_0^c] = \frac{1}{\eta}\left(e^{\frac{-\eta(B-K)}{c}} - e^{\frac{-\eta(B-K)}{h-c}} + e^{\frac{-\eta K}{c}}\right)$$

After substituting the expressions for R, $E[B_0^c] - E[A_0^c]$, I_0, I_1 and I_2 in Eq. (14) we obtain $G_f = \frac{N_1+N_2}{D}$, where

$$\begin{aligned}
N_1 &= \left(1 - e^{\frac{-\eta B}{h-c}}\left(\frac{\eta B}{h-c}+1\right)\right)\left(e^{\frac{-\eta K}{c}}(1-\alpha c) + e^{\frac{-\eta(B-K)}{c}} - e^{\frac{-\eta(B-K)}{h-c}}\right) \\
N_2 &= \alpha c \left(1 - e^{\frac{-\eta(B-K)}{h-c}}\left(\frac{\eta(B-K)}{h-c}+1\right) + \frac{c}{h}e^{\frac{\eta(B-K)}{c}}\left(e^{\frac{-\eta(B-K)}{h-c}}\left(\frac{\eta(B-K)}{h-c}+\frac{c}{h}\right) - e^{\frac{-\eta B}{h-c}}\left(\frac{\eta B}{h-c}+\frac{c}{h}\right)\right)\right) \\
D &= \frac{\eta}{R}
\end{aligned}$$

We can now study some asymptotic properties of $\mathcal{G}_f$.

- Observe that,

$$\begin{aligned}
\lim_{\eta\to 0}\mathcal{G}_f &= \lim_{\eta\to 0}\frac{N_1+N_2}{D} \\
&= 0
\end{aligned}$$

This agrees with the fact that for $\eta \to 0$, mean message lengths $\to \infty$, and hence almost every accepted message will experience fluid loss (due to buffer overflow) and there shall be no good fluid leaving the node.

- Further

$$\lim_{\eta\to\infty}\mathcal{G}_f = \frac{\alpha c^2(h-c)}{h^2} = \frac{c}{h} \tag{15}$$

This is because when $\eta \to \infty$, the length of messages is very short and hence Goodput is equal to the relative amount of fluid that is lost, since a message corresponds to infinitesimal amount of fluid.

- *Large buffers:* for $B \to \infty$ with fixed K, $I_0 \to \frac{1}{\eta^2}$ and $I_1, I_2, I_3 \to 0$. Thus, from Eq. (14),

$$\lim_{B\to\infty}\mathcal{G}_f = \lim_{B\to\infty} R\left(E[B_0^c] - E[A_0^c] + \alpha\frac{c}{\eta}(1-e^{\frac{-\eta K}{c}})\right) \tag{16}$$

From Eqs. (4) and (8) we have,

$$\begin{aligned}
\lim_{B\to\infty} E[A_0^c] &= \frac{1}{\eta}\left(\frac{h}{c} - e^{\frac{-\eta K}{c}}\right), \\
\lim_{B\to\infty} E[B_0^c] &= \frac{h}{\eta c},
\end{aligned}$$

which with Eq. (16) gives

$$\lim_{B\to\infty}\mathcal{G}_f = \frac{\alpha c + e^{\frac{-\eta K}{c}}(1-\alpha c)}{\frac{h}{h-c}\left(\frac{h}{c} - \frac{c}{h}e^{\frac{-\eta K}{c}}\right)} \tag{17}$$

Thus we have a *non-trivial* limiting value of $\mathcal{G}_f$ as $B \to \infty$.

Observe that by substituting $K = B$ Propositions 10 and 2 we can obtain respectively expressions for the density function of the stationary workload and for the goodput of the PMD policy as a special case of our model. We thus have

Corollary 1: (Proposition 5 in [4]) The goodput for PMD policy is given by

$$\mathcal{G}_{f,PMD} = \frac{c}{(h - ce^{\frac{-B\eta}{c}})}\left(e^{\frac{-B\eta h}{(h-c)c}}(1-\frac{c}{h}) + (\frac{c}{h} - e^{\frac{-B\eta}{c}})\right)$$

And hence

- $\lim_{\eta\to 0}\mathcal{G}_{f,PMD} = 0$
- $\lim_{\eta\to\infty}\mathcal{G}_{f,PMD} = \frac{c^2}{h^2}$
- $\lim_{B\to\infty}\mathcal{G}_{f,PMD} = \frac{c^2}{h^2}$

While from Eq. (15) for EMD the limiting value of $\mathcal{G}_f$ is $\frac{c}{h}(> c^2/h^2)$ for $\eta \to \infty$. Thus for $\eta \to \infty$ the relative improvement in Goodput with EMD:

$$\frac{\mathcal{G}_{f,EMD} - \mathcal{G}_{f,PMD}}{\mathcal{G}_{f,EMD}} = \frac{\frac{c}{h} - \frac{c^2}{h^2}}{\frac{c}{h}} = 1 - \frac{c}{h}$$

We next provide as intuitive argument through an example that may justify the limit of c^2/h^2 for PMD for $\eta \to \infty$.

Consider $\eta = 1, h = 10$ and $c = 6$. We will show the approximate limit achievable for an expected behaviour of our queue and see that it is close to c^2/h^2. We look at the case when a message is discarded, then the expected length of the subsequent discarding period will be $1(= 1/\eta)$ (the expected remaining length of a $exp(1)$ distributed random variable). During this period there will be an approximate [7] expected reduction of $6(= c/\eta)$ in the fluid level. Then a new message, call it message a of expected length 1 starts arriving. The expected queue length at the end of the arrival of this message will approximately $B-6+(h-c) = B-2$. However the expected amount of good fluid that was injected in the queue by this message is approximately 10. Then message b starts arriving, whose expected remaining length is again 1. Thus had message b been completely accepted the expected amount of fluid injected into the queue would have been 10, but because of buffer overflow the expected amount of fluid that can be accepted is approximately

[7] Note that the reduction cannot be greater than B units of fluid, hence in fact the reduction is $\simeq min(B, Xc)$, where $X \simeq exp(\eta)$. The use of word *approximate/approximately* in subsequent discussion is to highlight the fact that we are approximating a restricted exponential distribution as an exponential distribution.

$\frac{h \times 2}{(h-c)} = 5$ units. Then the next discarding period starts , whose expected length will again be 1. And the expected amount of fluid that will be discarded in this discarding period will be approximately 10. Thus we have,

$$G_f = \frac{10 + 10 + \ldots}{10 + 5 + 10 + 10 + 5 + 10 + \ldots}$$

which gives,

$$G_f = \frac{10}{25} = 0.4 \approx c^2/h^2 = 0.36$$

We next characterise the discarding threshold at which $\mathcal{G}_f$ is maximum for fixed value of other parameters calling it the optimal discarding threshold $\mathcal{K}^*$. Taking $\frac{\partial \mathcal{G}_f}{\partial K} = 0$, we get the optimal discarding threshold as the root of the following equation

$$\frac{\frac{\partial N_1}{\partial K} + \frac{\partial N_2}{\partial K}}{\frac{\partial D}{\partial K}} = \frac{N_1 + N_2}{D} \tag{18}$$

The derivates in Eq. (18) are obtained in Appendix B. For a given set of parameters, the $\mathcal{K}^*$, can be obtained by solving [8] the nonlinear Eq. (18). For certain values of parameters one can work with approximations (like truncated Taylors expansion for exponentials) to solve Eq. (18) efficiently.

V. Stability Analysis

In this section we comment on the stability of our fluid queue. This will ensure the existence of a stationary regime. In particular we show that the workload process is a renewal process. To that end we consider the Markov chain Z_n which is the workload process at the epoch of the arrival of the nth message, i.e., the nth increment in the arrival Poisson process. We shall show that Z_n is a Harris recurrent Markov chain and that the empty state is recurrent whenever $c > 0$. This will imply that the original workload process is a renewal process.

To that end, we recall the following sufficient condition for the Harris recurrence which follows from Theorem 14.0.1 in page 330 of [12].

Lemma 3: Assume that Z_n is $\Psi - irreducible$ (for some Ψ) and aperiodic. Let there be a function f, some $\epsilon > 0$, and some small set C such that

$$E[f(Z_{n+1}) - f(Z_n)|Z_n = z] < -\epsilon + I\{z \notin C\}. \tag{19}$$

Then the Markov chain Z is positive recurrent, it has a stationary probability π, and the n-step transition probabilities P^n converges to the π in total variation as $n \to \infty$.

In the above Lemma, a set C is said to be *small* if there exists some integer m, a constant $g > 0$ and a probability measure ϕ over the state space, denoted by R, such that

$$[P^m]_{zA} > g\phi(A) \tag{20}$$

for all $z \in C$ and measurable set $A \subset R$.

Note that in our case, the state space is the interval $R = [0, B])$.

[8]by using tools like matlab

Proposition 3: Assume that $c > 0$. Then the process Z_n is Harris recurrent.

Proof:

Define $C = [0, K]$ and $f(v) = v$. Let T be an exponentially distributed random variable with parameter η. Then for all $z \notin C$,

$$E[Z_{n+1} - Z_n|Z_n = z] < -cE[\min(T, B/c)] := -\epsilon.$$

Thus (19) holds. Next we check that C is indeed small. Let $Y_i, i = 1, 2$ be exponentially distributed random variable with parameter η. We have for any $z \in C$ and measurable $A \subset R$. Identifying Y_i with the length of two consecutive messages, the first of which started arriving when the workload is at z we have (with $0 \leq \Delta < \infty$).

$$\begin{aligned}
&[P^3]_{zA} \\
&\quad \geq P\left(\frac{K-z}{h-c} < Y_1 < \frac{K+\Delta}{h-c}, Y_2 > \frac{K+\Delta}{c}\right) P_{0A}. \\
&\quad \geq P\left(\frac{K}{h-c} < Y_1 < \frac{K+\Delta}{h-c}, Y_2 > \frac{K+\Delta}{c}\right) P_{0A}.
\end{aligned}$$

Thus, (20) holds with $\phi(A) = P_{0A}$ and $g = P(\frac{K}{h-c} < Y_1 < \frac{K+\Delta}{h-c}, Y_2 > \frac{K+\Delta}{c})$.

Observe that Z_n is Ψ-irreducible (p. 70,87 of [12]) since the probability of eventually reaching any measurable set $A \subset R$ from any state z is greater than $g\phi(A)$. Finally, the aperiodicity follows since the probability to go from state 0 to state 0 is strictly positive.

Remark 4: Observe that the we have proved the smallness of C without any condition on the value of B. Hence for large buffers ($B \to \infty$) and under heavy traffic ($h > c$) also our fluid queue is stable (of course with a finite K). For with K also tending to ∞ with B it is trivial that the queue is unstable under heavy traffic.

VI. Numerical Examples

We shall next plot the density of the stationary workload process $\rho(v)$ and the goodput, $\mathcal{G}_f$ using our analytical expressions for an example. We take $h = 200$, $c = 150$, $B = 100$, $K = 50$ and $\eta = 10$. To compare the behavior of the fluid approximation with the packet model we also plot the queue length distribution and the goodput ratio for the packet model in [16] calculated using the recursions provided there. For the packet model we took $\lambda = h = 200, \mu = c = 150, q = \eta/\lambda = 0.05, B = 100, K = 50$. The plot for $\rho(v)$ is given in Fig. 3 together with the queue-length distribution for the packet model [9]. Note that the density has a peak at K. This means that the EMD policy maintains the workload about the discarding threshold. We next plot $\mathcal{G}_f$ as a function of K in Fig. 4 for our example. We also plot the goodput ratio for the packet model as a function of K. Clearly there exists value(s) of K which results in maximum goodput. In fact with appropriate choice of K a goodput ratio of as high as 0.75 can be achieved. We also observe that over a range of values of K, the goodput remains high. We next study the asymptotic

[9]The queue-length distribution is only defined at integer points for the packet model

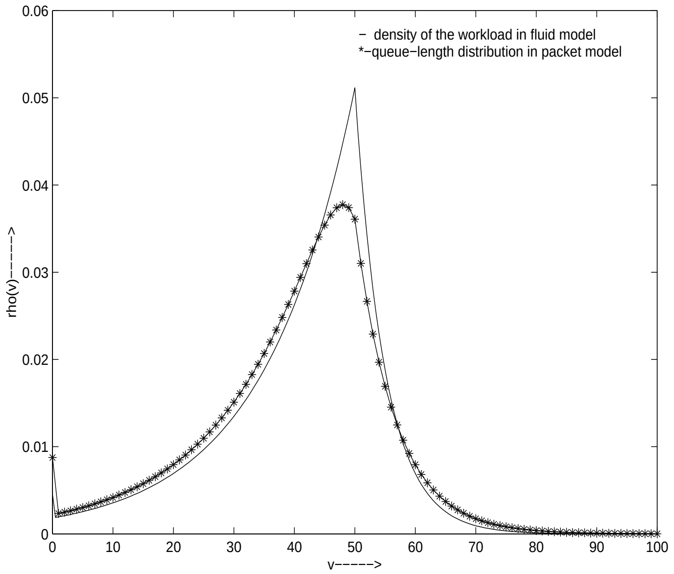

Fig. 3. The probability density function $\rho(v)$ of the stationary workload process and the queue length distribution for the packet model.

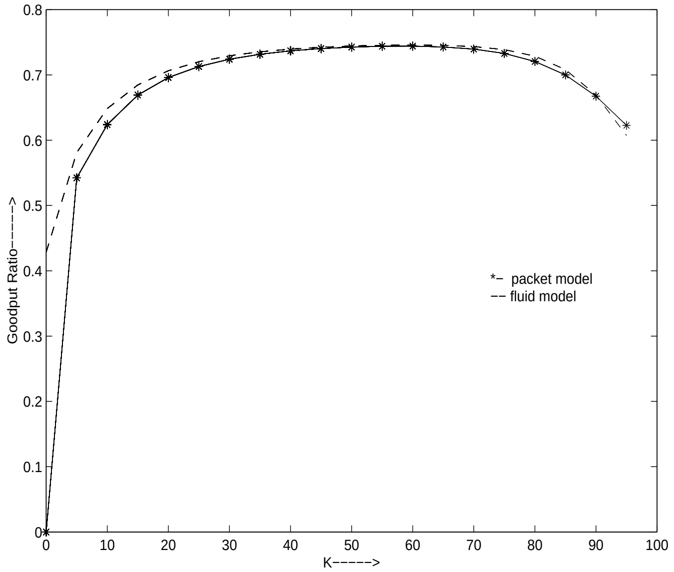

Fig. 4. The goodput as a function of K for the EMD policy for the packet model and its fluid approximation.

behavior of $\mathcal{G}_f$. We plot $\mathcal{G}_f$ as a function of η in Fig. 5. We observe that $\mathcal{G}_f$ reaches the limiting value of $\frac{c}{h} = 0.75$ (Eq. (15)) even at low values of η. In Fig. (6) we plot $\mathcal{G}_f$ as a function of B (with $K = 50$). We observe $\mathcal{G}_f$ to converge to a value around 0.74 for B as low as 80. This is exactly the limiting value of $\mathcal{G}_f$ from Eq. (17):

$$\frac{4 + e^{\frac{-50}{15}}(1-4)}{4(\frac{4}{3} - \frac{3}{4}e^{\frac{-50}{15}})} = 0.744$$

VII. Conclusion

We provided explicit expressions for the stationary distribution of the workload process and the goodput for the EMD policy. We also provided some asymptotic results and established

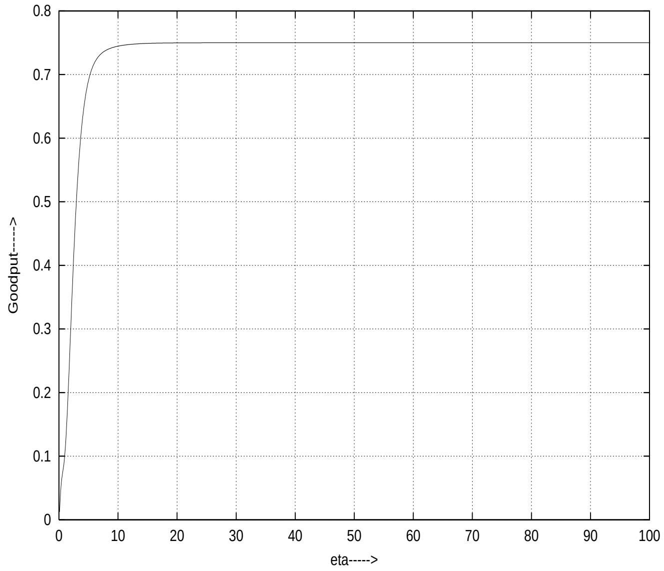

Fig. 5. The goodput G_f as a function of η for the EMD policy. Observe the convergence to $c/h = 0.75$ even for low values of η.

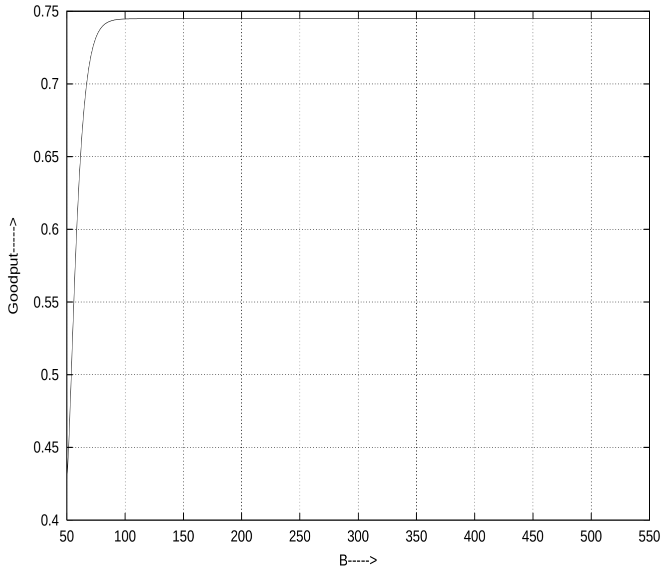

Fig. 6. The goodput G_f as a function of B for the EMD policy with $K = 50$. Observe the converge to 0.744 for B as low as 80

the stability of the fluid queue in our model. The fluid approximation is valid for heavy traffic conditions and gives explicit expressions for performance metrics. The expressions are helpful in a qualitative study of the system. Having the closed form expression for goodput, one can analytically study the sensitivity of the goodput to different parameters and even obtain the (*approximately*) [10] optimal value of the the discarding threshold for some optimality criteria (for e.g., optimal threshold being the one which maximizes the goodput). We also established the stability of the fluid queue. This model is a first step towards ob-

[10] In Knowledge Engineering community an *approximately optimal solution* is defined as "one whose measure of quality is close to the best theoretically possible".

taining closed form expressions for performance metrics (especially the "goodput") of message discarding policies. Our simple fluid model seems to capture the features of the EMD policy for packet model (see for e.g., plots in [16], where goodput is described via some recursions). Our next step is towards obtaining goodput expression for bursty fluids (for e.g., Markov modulated fluid process) and on experimental verifications (by working with real-world packet networks) of our model/findings.

Appendix A: Proof of Lemma 2 and Proposition 2

Observe that in (12), the integral can be taken upto K and not B as,

$$P(\mathcal{V}_f = 1 | W \leq w, V_a = v) = 0 \quad \text{for } v \geq K$$

Observe that, for $v < K$,

$$P(\mathcal{V}_f = 1 | W \leq w, V_a = v) = \begin{cases} 1 & \text{if } w \in [0, \frac{B-v}{h-c}] \\ P(W < \frac{B-v}{h-c} | W < w) & \text{if } w > \frac{B-v}{h-c}. \end{cases}$$

Or in other words,
for $w \in [0, \frac{B-K}{h-c}]$, $P(\mathcal{V}_f = 1 | W \leq w, V_a = v)$ is

$$= \begin{cases} 1 & \text{if } v < K \\ 0 & \text{if } v \geq K \end{cases}$$

for $w \in (\frac{B-K}{h-c}, \frac{B}{h-c}]$, $P(\mathcal{V}_f = 1 | W \leq w, V_a = v)$ is

$$= \begin{cases} 1 & \text{if } v \in [0, B - w(h-c)] \\ \frac{P(W < \frac{B-v}{h-c})}{P(W<w)} & \text{if } v \in (B - w(h-c), K) \\ 0 & \text{if } v \geq K. \end{cases}$$

And for $w > \frac{B}{h-c}$, $P(\mathcal{V}_f = 1 | W \leq w, V_a = v)$ is

$$= \begin{cases} \frac{P(W < \frac{B-v}{h-c})}{P(W<w)} & \text{if } v \in [0, K) \\ 0 & \text{if } v \geq K \end{cases}$$

Thus we write,
for $w \in [0, \frac{B-K}{h-c}]$,

$$\begin{aligned} F(w,1) &= (1 - e^{-\eta w}) \int_0^K \rho(v) dv \\ &= R(1 - e^{-\eta w}) \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \alpha \int_0^K e^{\frac{\eta(v-K)}{c}} dv \right] \\ &= R(1 - e^{-\eta w}) \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \frac{\alpha c}{\eta}(1 - e^{\frac{-\eta K}{c}}) \right] \end{aligned} \tag{21}$$

for $w \in (\frac{B-K}{h-c}, \frac{B}{h-c}]$, $F(w,1) = F_1(w) + F_2(w)$, where,

$$\begin{aligned} F_1(w) &= (1 - e^{-\eta w}) \int_0^{B - w(h-c)} \rho(v) dv \\ &= R(1 - e^{-\eta w}) \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \alpha \int_0^{B-w(h-c)} e^{\frac{\eta(v-K)}{c}} dv \right] \\ &= R(1 - e^{-\eta w}) \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \frac{\alpha c}{\eta} e^{\frac{-\eta K}{c}} (e^{\frac{\eta(B - w(h-c))}{c}} - 1) \right] \end{aligned} \tag{22}$$

And,

$$\begin{aligned} F_2(w) &= \int_{B-w(h-c)^+}^K (1 - e^{-\frac{B-v}{h-c}\eta}) R\alpha e^{\frac{\eta(v-K)}{c}} dv \\ &= \frac{R\alpha c}{\eta} \left[1 - e^{\frac{-\eta K}{c}} e^{\frac{\eta(B-w(h-c))}{c}} \right. \\ &\quad \left. + \frac{h-c}{h} (e^{\frac{\eta(B-K-wh)}{c}} - e^{\frac{-\eta(B-K)}{h-c}}) \right] \end{aligned} \tag{23}$$

and for $w > \frac{B}{h-c}$

$$F(w) = \int_0^K (1 - e^{-\frac{(B-v)\eta}{h-c}}) \rho(v) dv \tag{24}$$

Observe that for $w > \frac{B}{h-c}$, from Equation (24)

$$dF(w,1) = 0$$

Thus $\mathcal{G}_f$ from (11) can be written as in Lemma 2.
From Equations (21), (22) and (23) for $w \in [0, \frac{B-K}{h-c}]$

$$\begin{aligned} &\frac{dF(w,1)}{dw} = \\ &\quad R\eta e^{-\eta w} \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \frac{\alpha c}{\eta}(1 - e^{\frac{-\eta K}{c}}) \right] \end{aligned}$$

and

$$\begin{aligned} &\frac{dF_1(w)}{dw} = \\ &\quad R\alpha(1 - e^{-\eta w})(h-c) \\ &\quad \times \left(-e^{-\eta K/c} e^{\frac{\eta(B-w(h-c))}{c}} \right) \\ &\quad + R\eta e^{-\eta w} \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \frac{\alpha c}{\eta} e^{\frac{-\eta K}{c}} (e^{\frac{\eta(B-w(h-c))}{c}} - 1) \right] \end{aligned} \tag{25}$$

$$\begin{aligned} &\frac{dF_2(w)}{dw} = \\ &\quad R\alpha e^{\frac{-\eta K}{c}} (h-c) \left(e^{\frac{\eta(B-w(h-c))}{c}} - e^{\frac{\eta(B-wh)}{c}} \right) \end{aligned} \tag{26}$$

Thus from Eqs. (25) and (26) we have, for $w \in (\frac{B-K}{h-c}, \frac{B}{h-c}]$,

$$\begin{aligned} \frac{dF(w,1)}{dw} &= \frac{dF_1(w)}{dw} + \frac{dF_2(w)}{dw} \\ &= R\eta \left((E[B_0^c] - E[A_0^c]) e^{-\eta w} \right. \\ &\quad \left. + \frac{\alpha c}{\eta} e^{\frac{-\eta K}{c}} (e^{\frac{\eta(B-wh)}{c}} - e^{-\eta w}) \right) \end{aligned} \tag{27}$$

Now, from Eq. (13) we have

$$\mathcal{G}_f = \eta \left(\int_0^{\frac{B-K}{h-c}} w dF(w,1) + \int_{\frac{B-K}{h-c}}^{\frac{B}{h-c}} w dF(w,1) \right) \tag{28}$$

We shall next evaluate the two integrals in the last equation. Defining I_0, I_1, I_2 as in Proposition 2 and I_3 as

$$I_3 = \int_{\frac{B-K}{h-c}}^{\frac{B}{h-c}} w e^{-\eta w} dw$$

we can write, from Eq. (25)

$$\begin{aligned} &\int_0^{\frac{B-K}{h-c}} w dF(w,1) = \\ &\quad R\eta I_1 \left[E[B_0^c] - E[A_0^c] \right. \\ &\quad \left. + \frac{\alpha c}{\eta}(1 - e^{\frac{-\eta K}{c}}) \right] \end{aligned} \tag{29}$$

and from Eq. (27)

$$\begin{aligned} &\int_{\frac{B-K}{h-c}}^{\frac{B}{h-c}} w dF(w,1) = \\ &\quad R\eta \left(E[B_0^c] - E[A_0^c] \right) I_3 \\ &\quad \left. + \frac{\alpha c}{\eta} e^{\frac{-\eta K}{c}} (e^{\frac{\eta B}{c}} I_2 - I_3) \right) \end{aligned} \tag{30}$$

Thus from Eqs. $(28), (29)$, and (30) we have

$$\begin{aligned} \mathcal{G}_f &= \eta^2 R \left[I_0 \left(E[B_0^c] - E[A_0^c] \right. \right. \\ & \left. \left. - \frac{\alpha c}{\eta} e^{\frac{-\eta K}{c}} \right) + \frac{\alpha c}{\eta} (I_1 + e^{\frac{\eta(B-K)}{c}} I_2) \right] \end{aligned} \tag{31}$$

Finally Eq. (14) follows from Eq. (31).

Appendix B: Sensitivity of Goodput

$$\begin{aligned} &\frac{\partial N_1}{\partial K} = \\ &\quad \left(\frac{-\eta}{c} e^{\frac{-\eta K}{c}} (1 - \alpha c) + \frac{\eta}{c} e^{\frac{-\eta(B-K)}{c}} \right. \\ &\quad \left. - \frac{\eta}{h-c} e^{\frac{-\eta(B-K)}{h-c}} \right) \\ &\quad \times \left(1 - e^{\frac{-\eta B}{h-c}} \left(\frac{\eta B}{h-c} + 1 \right) \right) \end{aligned}$$

$$\begin{aligned} &\frac{\partial N_2}{\partial K} = \\ &\quad \alpha c \left(\frac{-\eta^2 (B-K)}{(h-c)^2} e^{\frac{-\eta(B-K)}{h-c}} \right. \\ &\quad + \frac{\eta c}{h(h-c)} e^{\frac{\eta(B-K)}{c}} \left(e^{\frac{-\eta(B-K)}{h-c}} \left(\frac{\eta(B-K)}{h-c} + \frac{c}{h} \right) \right. \\ &\quad \left. - e^{\frac{-\eta(B-K)}{h-c}} \right) - \left(e^{\frac{-\eta(B-K)}{h-c}} \left(\frac{\eta(B-K)}{h-c} + \frac{c}{h} \right) \right. \\ &\quad \left. \left. - e^{\frac{-\eta B}{h-c}} \left(\frac{\eta B}{h-c} + \frac{c}{h} \right) \right) \frac{\eta}{h} e^{\frac{\eta(B-K)}{c}} \right) \end{aligned}$$

$$\begin{aligned} &\frac{\partial D}{\partial K} = \\ &\quad \frac{\eta h}{(h-c)c} e^{\frac{-\eta(B-K)}{c}} (1 - \frac{h}{c}) + \frac{\eta e^{\frac{-\eta K}{c}}}{(h-c)} \end{aligned}$$

References

[1] *http://www.ieng.com/warp/public/cc/pd/si/bp8600/tech/*.

[2] F. Baccelli and P. Bremaud. *Elements of queueing theory: Palm-Martingale Calculus and Stochastic Recurrence*. Springer-Verlag, 1994.

[3] P. Dube and E. Altman. On the Workload Process in a Fluid Queue with Bursty Input and Selective Discarding. *in the proceedings of International Teletraffic Congress ITC 17*, Dec 2001.

[4] P. Dube and E. Altman. Queueing and Fluid Analysis of Partial Message Discard Policy. *in the proceedings of 9th IFIP Working Conference on Performance Modeling and Evaluation of ATM and IP Networks*, June 2001.

[5] P. Dube and E. Altman. Queueing Analysis of Early Message Discard Policy. *in the proceedings of IEEE ICC 2002*, May 2002.

[6] S. Floyd and V. Jacobson. Random Early Detection Gateways for Conjestion Avoidance. *IEEE/ACM Transactions on Networking*, 1(4):25–39, 1993.

[7] S. Floyd and A. Romanow. Dynamics of TCP traffic over ATM networks. *in Proc. of ACM SIGCOMM'94*, pages 79–88, September 1994.

[8] J. Carlson, P. Langer, E. H. Valencia, J. Manchester. RFC 2823: PPP over Simple Data Link (SDL) using SONET/SDH with ATM-like framing. *available at* http://www.faqs.org/rfcs/rfc2823.html, May 2000.

[9] K. Kawahara, K. Kitajima, T. Takine and Y. Oie. Packet Loss Performance of Selective Cell Discard Schemes in ATM Switches. *IEEE Journal of Selected Areas in Communications*, 15(5):903–913, June 1997.

[10] Y. H. Kim and S. Q. Li. Performance Analysis of Data Packet Discarding in ATM Networks. *IEEE/ACM Transactions on Networking*, 7(2):216–227, 1999.

[11] D. Kofman and M. Gagnaire. Réseaux Haut Débit: réseaux ATM, réseaux locaux et réseaux tout-optiques. InterEditions (groupe Masson), May 1996.

[12] S. P. Meyn and R. L. Tweedie. *Markov Chain and Stochastic Stability*. Springer-Verlag, 1993.

[13] Papers of J. Dai on fluid limits. *available at http://www.isye.gatech.edu/faculty/dai/reprints*.

[14] J. S. Turner. Maintaining high throughput during overload in ATM switches . *in Proc. of INFOCOM '96*, pages 287–295, April 1996.

[15] W. Whitt. *Stochastic-Process Limits*. Springer-Verlag, 2002.

[16] Y. Lapid, R. Rom and M. Sidi. Analysis of Discarding Policies in High-Speed Networks. *IEEE Journal on Selected Areas in Communications*, 16(5):764–777, June 1998.