# The Loss Process of Messages with Redundant Packets in an M/M/1/K Queue

Omar Ait Hellal, Eitan Altman and Alain Jean-Marie
INRIA Sophia Antipolis, B.P. 93
06902 Sophia Antipolis, France

**Abstract**

The purpose of this paper is to study the loss probabilities of messages in an M/M/1/k queue when redundant packets are added. The input to the queue is generated by several sessions, and is modeled by a superposition of Poisson inputs. We use multi-dimensional probability generating functions for solving the recursions introduced in Cidon, Khamisy and Sidi [2] for computing the loss probabilities and derive analytical formulas. Using asymptotic analysis (for large $n$ and small $\rho$), we obtain good approximations with very low complexity of computation and memory. Our analysis allows us to investigate when does adding redundancy decrease the loss probabilities.

## 1 Introduction

The loss probabilities of packets in queuing networks is an important performance measure in telecommunication networks and some other applications. Rapid progress in the development of fiber optics allows to achieve a bit error rate of $10^{-14}$; information loss is then essentially due to congestioned nodes and buffer overflow.

Often, when messages are divided in several packets, the loss of one packet results in the loss of the whole message. In order to reduce the loss probabilities, one may add redundant packets, so that lost packets can be often reconstructed. Indeed, there exist erasure recovery codes that, by adding $k$ redundant packets to a message, enable to reconstruct up to $k$ losses (see [2],[5],[6] and references therein). We note, however, that by adding redundant packets, the workload increases and thus the loss probability of a packet increases.

We analyse the tradeoff between the effects of increase of workload and the recovery of lost packets, and calculate the probability of no more losses than $k$ packets within $n$ consecutive ones in the presence of $k$ redundant packets. The computations are based on recursive formulas obtained by Cidon, Khamisy and Sidi [2].

In [5], the authors have used an approximation based on an assumption of independence between consecutive losses, and shown that redundancy results in decrease of loss probabilities by 10 to 100. Exact numerical methods based on recursions [2] led to an opposite conclusion, i.e. that redundancy causes increase in loss probabilities. One of the advantages of our analytical approach, together with the asymptotic approximations which we present, is that they enable to study both qualitative and quantitative behavior of the effect of redundancy in a systematic way. As was already shown in [3] for the case of a single source, we show that for both light traffic as well as heavy traffic conditions, redundancy decreases loss probabilities.

The paper is structured as follows: in section 2, we describe the model and calculate the probability generating function. The asymptotic analysis is presented in section 3. In section 4, we show that in light traffic, adding redundancy decreases the loss probabilities. Numerical examples which illustrate this improvement are given in section 5.

# 2 The Model and the Main Results

We consider an M/M/1 queue with a finite buffer of size $M$. We assume that packets arrive to the queue from $S$ independent sources, i.e. the inter-arrival times and the transmission times of packets from each source are mutually independent. The arrival process from source $s$, $s = 1, 2, \cdots, S$, is assumed to be Poisson with rate $\lambda_s$. The overall arrival process to the system is then Poisson with rate $\lambda \triangleq \sum_s^S \lambda_s$. Define $\rho = \lambda/\mu$, $\rho_s = \lambda_s/\mu = p_s \rho$. We summarize the recursive sheme introduced in [2] for computing $P^s(j, n), s = 1, 2, .., S$ which are the probabilities of $j$ losses among $n$ consecutive ones originating from source $s$. For the system with Poisson arrivals with rate $\lambda$ and exponential transmission rate $\mu$, in steady state, the probability of finding $i$ packets in the system at an arbitrary epoch is given by

$$\Pi(i) = \frac{\rho^i}{\sum_{l=0}^M \rho^l} \tag{1}$$

Define $Q_i(k)$ to be the probability that $k$ packets out of $i$ leave the system durant an inter-arrival epoch. We have

$$\begin{aligned} Q_i(k) &= \rho\alpha^{k+1} \qquad 0 \le k \le i-1 \\ Q_i(i) &= \alpha^i, \qquad \text{where} \quad \alpha := (1+\rho)^{-1} \end{aligned} \tag{2}$$

Denote by $P_i^{s,a}(j, n)$ *resp* $P_i^{\bar{s},a}(j, n)$, $i = 0, 1, ..., M$, $s = 1, 2, ..., S$, $n \ge 1$, $0 \le j \le n$, the probabilities of $j$ losses in a block of $n$ packets coming from source $s$, given that there are $i$ packets in the system just before the arrival of the first packet in the block, and just before the arrival of a packet from any other source (denoted by $\bar{s}$), respectively. Since the first packet in the block is arbitrary, we have

$$P^s(j, n) = \sum_{i=0}^M \Pi(i) P_i^{s,a}(j, n) \tag{3}$$

The probability that an arbitrary arrival is from source $s$ is equal to $\lambda_s/\lambda$. Define $p_s \triangleq \lambda_s/\lambda$ and $p_{\bar{s}} \triangleq 1 - p_s$, the recursive sheme is then

$$P_i^{s,a}(j,1) = \begin{cases} 1 & j = 0 \\ 0 & j \geq 1, \end{cases} \qquad i = 0, 1, ..., M-1 \tag{4}$$

$$P_M^{s,a}(j,1) = \begin{cases} 1 & j = 1 \\ 0 & j = 0, j \geq 2, \end{cases} \tag{5}$$

For $n \geq 2$ we have

$$\begin{aligned} P_i^{s,a}(j,n) &= \sum_{k=0}^{i+1} Q_{i+1}(k) \left[ p_s P_{i+1-k}^{s,a}(j,n-1) + p_{\bar{s}} P_{i+1-k}^{\bar{s},a}(j,n-1) \right] \qquad 0 \leq i \leq M-1 \\ P_M^{s,a}(j,n) &= \sum_{k=0}^{M} Q_M(k) \left[ p_s P_{M-k}^{s,a}(j-1,n-1) + p_{\bar{s}} P_{M-k}^{\bar{s},a}(j-1,n-1) \right] \end{aligned} \tag{6}$$

where $P_i^{\bar{s},a}(j,n)$ for $n \geq 1$ is given by:

$$\begin{aligned} P_i^{\bar{s},a}(j,n) &= \sum_{k=0}^{i+1} Q_{i+1}(k) \left[ p_s P_{i+1-k}^{s,a}(j,n) + p_{\bar{s}} P_{i+1-k}^{\bar{s},a}(j,n) \right] \qquad 0 \leq i \leq M-1 \\ P_M^{\bar{s},a}(j,n) &= P_{M-1}^{\bar{s},a}(j,n) \end{aligned} \tag{7}$$

The complexity of these recursions is $O(M^2 nj)$ in arithmetic operations and $O(M^2)$ in memory space.

Next, we state the main results, whose detailled proofs are given in [1]. Define

$$q_s(y,z) \stackrel{def}{=} \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} y^j z^{n-1} P^s(j,n)$$

Let $x_1(z)$ and $x_2(z)$ be the solutions in $x$ of

$$x^2 - (1+\rho)x + \rho(p_{\bar{s}} + p_s z) = 0$$

i.e.

$$x_1(z) = \frac{1 + \rho + \sqrt{(1+\rho)^2 - 4\rho(p_{\bar{s}} + p_s z)}}{2}, \qquad x_2(z) = \frac{1 + \rho - \sqrt{(1+\rho)^2 - 4\rho(p_{\bar{s}} + p_s z)}}{2}$$

We shall often write simply $x_1$ and $x_2$ for $x_1(z)$ and $x_2(z)$. Both these functions are analytic in the disk $\{|z| < ((1+\rho)^2 - 4\rho_{\bar{s}})/4\rho_s\}$. Define, for all $k$, $\delta_k = x_1^k - x_2^k$, $\phi_k = (p_{\bar{s}} + p_s z)\delta_{k-1} - \delta_k$. Let $R_M = (\sum_{l=0}^{M} \rho^l)^{-1}$.

**Proposition 2.1.** *The probability generating function $q_s$ is given by*

$$q_s(y,z) = \frac{R_M}{(1-z)}\left(R_M^{-1} + \frac{\rho^{M-1}\delta_{M+1}}{z\phi_M} + \frac{\rho^{M-1}(1-z)\left[\delta_{M+1}\right]^2}{z\phi_M}\left[\frac{1}{z\delta_M - \delta_{M+1} - \rho z\phi_M y}\right]\right) \tag{8}$$

Once the probability generating function is obtained by Proposition 2.1, one can obtain the required probabilities by inverting $q_s$. We focus in the sequel on $P_\rho^s(> j, n)$, which is the probability of losing more than $j$ packets out of $n$ consecutive ones coming from source $s$. We investigate in particular the case of $j = 0, 1$, in order to be able to decide when does adding redundancy packet to each message result in decrease of the loss probability.

We shall use the notation $\left[z^k\right] f(z)$ to denote the coefficient of $z^k$ in the Taylor expansion of the function $f(z)$, i.e. if $f(z) = \sum_k f_k z^k$ then $\left[z^k\right] f(z) = f_k$.

**Corollary 2.1.** *The probability of losing more than $j$ packets out of $n$ consecutive packets that arrive from source $s$ is*

$$P_\rho^s(> j, n) = R_M \rho^{M+j}\left[z^{n-1-j}\right]\frac{1}{z-1}\left[\frac{\delta_{M+1}}{z\delta_M - \delta_{M+1}}\right]\left(\frac{\phi_M}{z\delta_M - \delta_{M+1}}\right)^j \tag{9}$$

In the following, we obtain a simple recursion on $n$, for computing the probabilities $P_\rho^s(> 0, n)$. Thus, we avoid a recursion on $j$ and a resolution of a set of linear equations of size $M$ for all $j$ and all $n$ required in [2].

Define $\beta$, $\gamma$ and $\theta$ as

$$\gamma = 1+\rho, \quad \beta = \sqrt{(1+\rho)^2 - 4\rho_{\bar{s}}}, \quad \theta = \frac{4\rho_s}{(1+\rho)^2 - 4\rho_{\bar{s}}} \tag{10}$$

Let

$$G = \begin{cases} \frac{M}{2} - 1 & \text{for M even} \\ \frac{M-1}{2} & \text{otherwise} \end{cases} \qquad a_n = \begin{cases} (-\theta)^n \sum_{k=n}^{G+1} \binom{M+1}{2k+1}\binom{k}{n}\beta\left(\frac{\beta}{\gamma}\right)^{2k} & n \le G+1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = \begin{cases} (-\theta)^{n-1} \sum_{k=n}^{G+1} \binom{M+1}{2k+1}\binom{k}{n}\left(\frac{4\gamma(2k+1)n}{\beta(M+1)(M+1-2k)} + \theta\beta\right)\left(\frac{\beta}{\gamma}\right)^{2k} & n \le G+1 \\ 0 & \text{otherwise} \end{cases}$$

**Corollary 2.2.** *For $n \geq 1$ we have*

$$P_\rho^s(> 0, n) = R_M\rho^M + \frac{1}{a_0}\sum_{k=1}^{n-1}\left(b_{n-k}P_\rho^s(> 0, k) + R_M\rho^M a_k\right) \tag{11}$$

We see that for $j = 0$, we obtain a procedure for computing the loss probabilities, of complexity $O(Mn+n^2)$ in operations and $O(M^2)$ in memory space if we consider that all the values $\left( \begin{array}{c} M \\ r \end{array} \right)$, $0 \leq r \leq M$ remain in memory. This is a substantial improvement over the complexity of $O(M^2 n)$ in operations and $O(M^2)$ in memory space required in [2].

In case $j > 0$, we obtain a procedure whose complexity is of the order $O(j^2 n^2 + Mnj^2)$ in operations and $O(M^2 j^2)$ in memory space [1].

A detailed proof of Proposition 2.1 and the proofs of Corollary 1 and 2 are given in [1].

# 3 Asymptotic behavior

In order to reduce the complexity of calculations of $P_\rho^s(> j, n)$, we make the following approximations, based on an asymptotic expression for these probabilities for $n$ large. From (9) we have

$$\begin{aligned} P_\rho^s(> j, n) &= -R_M \rho^M \left[z^{n-1}\right] \frac{1}{1-z} \frac{x_1^{M+1}}{x_1^M (z - x_1)} \left( \frac{z x_1^M (1 - x_1)}{x_1^M (z - x_1)} \right)^j + \left[z^{n-1}\right] f_j(z) \\ &= -R_M \rho^M \left[z^{n-1}\right] \frac{1}{1-z} \left( \frac{\rho_{\bar{s}}}{x_1 - \rho_{\bar{s}}} + \frac{1}{1 - x_1} \right) \left( x_2 - \frac{\rho_{\bar{s}}(x_1 - \rho)}{x_1 - \rho_{\bar{s}}} \right)^j + \left[z^{n-1}\right] f_j(z) \\ &= \tilde{p}_\rho^s(> j, n) + \left[z^{n-1}\right] f_j(z). \end{aligned}$$

It can be shown that the term $[z^{n-1}] f_j(z)$ can be neglected for large $n$ and $n < M + 1$ and hence $P_\rho^s(> j, n) \cong \tilde{P}_\rho^s(> j, n)$, which we compute next.

**Proposition 3.1.** *For $n \geq 1$ we have*

$$\tilde{P}_\rho^s(> 0, n) = \begin{cases} \dfrac{R_M \rho^M}{1 - \rho_{\bar{s}}} \left[ (1-\rho) n + \left( \dfrac{\rho_s}{1-\rho} - \dfrac{\rho_s \rho_{\bar{s}}}{1 - \rho_{\bar{s}}} \right) + \theta^n O\left(n^{-3/2}\right) \right] & \text{if } \rho < 1 \\ \dfrac{1}{M+1} \dfrac{1}{p_s} \left[ \left( 2\sqrt{p_s} + \dfrac{p_{\bar{s}}}{\sqrt{p_s} n} \right) \dfrac{\sqrt{n}}{\sqrt{\pi}} \left( 1 + O\left(\dfrac{1}{n}\right) \right) - p_{\bar{s}} \right] & \text{if } \rho = 1 \\ 1 - \left( \dfrac{4\rho_s^2}{\rho(\rho-1)^3} - \dfrac{\rho_s \rho_{\bar{s}}}{\rho(1-\rho_{\bar{s}})} \left( \dfrac{1}{\rho - 1} - \dfrac{\rho - 1}{(1 + \rho_s - \rho_{\bar{s}})^2} \right) \right) & \\ \theta^{n-1} \dfrac{\beta\ n^{-3/2}}{(1 - \rho_{\bar{s}}) \sqrt{\pi}} \left(1 + o(1)\right) & \text{if } \rho > 1 \end{cases} \tag{12}$$

$$\tilde{P}^s_\rho(>1,n)=\begin{cases}\frac{R_M\rho^M\rho_s}{(1-\rho_{\bar{s}})^2}\left[(1-\rho)n+\frac{\rho_s+\rho-1}{1-\rho}-\frac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}+\theta^n O\left(n^{-3/2}\right)\right] & \text{if } \rho<1\\ \frac{1}{M+1}\frac{1}{p_s}\left[\left(2\sqrt{p_s}+\frac{2p_{\bar{s}}}{\sqrt{p_s}n}\right)\frac{\sqrt{n}}{\sqrt{\pi}}\left(1+O\left(\frac{1}{n}\right)\right)-(1+p_{\bar{s}})\right] & \text{if } \rho=1\\ 1-\frac{\rho_s}{1-\rho_{\bar{s}}}\left(\frac{4\rho_s^2}{\rho(\rho-1)^3}+\frac{\rho_{\bar{s}}(\rho-1)}{\rho(1-\rho_{\bar{s}})(1+\rho_s-\rho_{\bar{s}})}\left(\frac{(1+\rho_s-\rho_{\bar{s}})^2}{(\rho-1)^2}\right.\right. & \\ \left.\left.-\frac{\rho_s}{(1+\rho_s-\rho_{\bar{s}})}+\rho_{\bar{s}}^2+\frac{4\rho_s\rho_{\bar{s}}^2(1-\rho_{\bar{s}})}{(1+\rho_s-\rho_{\bar{s}})^2}\right)\right)\frac{\theta^{n-1}\beta\ n^{-3/2}}{(1-\rho_{\bar{s}})\sqrt{\pi}}(1+o(1)) & \text{if } \rho>1\end{cases} \quad (13)$$

The proof of this proposition is given in [1] and is based on [4].

**Proposition 3.2.** *For $n\geq 1$, $0\leq j<n$, we have for small $\rho$*

$$P^s_\rho(>j,n)=\rho^M\rho_s^j(n-j+O(\rho_s)).$$

# 4 When is it better to add redundancy

In this section we compare the loss probabilities of messages of size $n$, with and without an additional redundancy of a single packet. Without loss of generality, we may rescale time so that the service rate is one: $\mu=1$. We assume that the rate at which messages arrive is the same for the two cases and is given by $x=p_{\bar{s}}x+p_sx=x'+x''$. We ditinguish two cases:

1. adding redundancy for all sources. Hence, the rate at which packets arrive is $\lambda=\rho=(n+1)x$.
2. adding redundancy only for source $s$; the workload is then $\lambda=\rho=nx+x''$ and $\rho_{\bar{s}}$ still constant.

The message is lost, in both last cases, if and only if more than one packet is lost out of $n+1$ consecutive ones coming from source $s$.

We are thus interested in the difference $\Delta=P^s_{nx}(>0,n)-P^s_{(n+1)x}(>1,n+1)$. If $\Delta>0$ then the redundancy decreases the loss probabilities of messages.

**Proposition 4.1.** *For any $n$ and $M$, adding redundancy results in a decrease of the loss probabilities for all $x$ small enough (light traffic regime).*

**Proof:** We consider case 1. Case 2 follows similarly. From Proposition 3.2 we have $P^s_{nx}(>0,n)=(nx)^M(n+O(nx''))$ and

$$P^s_{(n+1)x}(>1,n+1)=((n+1)x)^{M+1}p_s(n+O(nx'')).$$

The proof now follows by noting that

$$\lim_{x\to 0}\frac{P^s_{(n+1)x}(>1,n+1)}{P^s_{nx}(>0,n)}=\lim_{x\to 0}\left(\frac{n+1}{n}\right)^M(n+1)p_sx=0<1. \quad (14)$$

# 5 Numerical examples

We have shown that adding redundancy is profitable in light traffic. A natural question is how small should the traffic be in order for this conclusion to hold in practice.

Below, we fix $\rho$, $p_s$ and obtain a set of $n$ and $M$ for which redundancy will lead to better performance and for which the loss probability of messages is of a given order (ex:$10^{-8}$). We shall restrict to a family of $n$ and $M$ that are inter-related by $n \cong \eta M$, where $\eta$ is a constant to be determined, and we shall consider $n >> 1$. In fact, the approximations turn out to be quite accurate even for moderate values of $n$ and $M$. In Figure 1, we display the values of $P^s_\rho(>0,n)$ and its approximations (from Proposition 3.1):

$$A_0(>0,n) \stackrel{\text{def}}{=} \frac{R_M\rho^M}{(1-\rho_{\bar{s}})}\left((1-\rho)n+\rho_s\left(\frac{1}{1-\rho}-\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right)\right),\ B_0(>0,n) \stackrel{\text{def}}{=} \frac{R_M\rho^M}{(1-\rho_{\bar{s}})}(1-\rho)n$$

in the case $M = 10$, $n \leq 10$, $\rho = 0.4$ and $p_s = 0.6$. In Figure 2, we make the same comparison for $P^s_\rho(>1,n)$ and

$$A_1(>1,n) \stackrel{\text{def}}{=} \frac{R_M\rho^M\rho_s}{(1-\rho_{\bar{s}})^2}\left((1-\rho)n+\frac{\rho+\rho_s-1}{1-\rho}-\frac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right),\ B_1(>1,n) \stackrel{\text{def}}{=} \frac{R_M\rho^M\rho_s}{(1-\rho_{\bar{s}})^2}(1-\rho)n$$

These approximations are obtained from Proposition 3.1($\rho < 1$), by taking the two first and the first term, respectively, in the asymptotic expansion (in $n$) of $\tilde{P}^s$.

## 5.1 Adding redundancy for all sources

We wish to determine $x^*$ for which $P^s_{(n+1)x}(>1,n+1) - x^*P^s_{nx}(>0,n) = 0$. $[0,x^*]$ will be the interval for which we should use redundancy. From (14), we have for large $M$

$$x^* = \frac{1}{p_s(n+1)}\left(\frac{n}{n+1}\right)^M \cong \frac{1}{p_s\eta M}\frac{n}{n+1}\left(1-\frac{1}{\eta M}\right)^M \cong \frac{n}{n+1}\frac{\exp(-1/\eta)}{p_s\eta M}$$

Since $\rho^*_s = p_s x^* \eta M$, we have

$$\rho^*_s \cong \frac{n}{n+1}\exp(-1/\eta) \cong \exp(-1/\eta) \tag{15}$$

More generaly, for $\rho_j = \frac{n}{n+j}\rho^*$ adding $j$ packets leads to better performance. This heuristic is quite optimistic and we obtain this experimental result: We fix $\rho = 0.6$ and for $p_s \in ]0, p^*_s]$ where $p^*_s = exp(-1/\eta)$, adding one redundant packet decreases the loss probabilities of messages and for the same values of $p_s$, for $\rho = \frac{n}{n+j}0.6$, adding $j$ packets decreases these probabilities.

**Example 1:** Let $\rho = 0.6$, $p_s = 0.1$, we wish to determine $n$ and $M$ for which the loss probability is of order $10^{-8}$ and redundancy leads to better performance. It folows from (15) that $\eta \geq 0.36$, from Proposition 3.1, we have

$$P^s_{0.6}(>0,n) \cong \frac{1-0.6}{1-0.6^{M+1}}\frac{0.6^M}{1-0.54}(1-0.6)0.36M \cong 10^{-8} \quad \Rightarrow M \cong 39,\ n = \eta M \cong 15$$

The exact calculation for $A(\rho) \stackrel{\text{def}}{=} P^s_{nx}(>0,n)$ and $S(\rho) \stackrel{\text{def}}{=} P^s_{(n+1)x}(>1,n+1)$ yields that for $n=15$, $A(0.6)=1.17*10^{-8}$ and $S(0.6)=1.88*10^{-8}$. For $n=19$, $S(0.6)=1.42*10^{-8}$ et $A(0,6)=1.48*10^{-8}$. We have to take $\eta$ greater than the value $\eta=0.36$ obained above. If we choose $\eta=0.5$ we obtain $M=40$ and $n=20$; $A(0.6)=9.39*10^{-9}$ and $S(0.6)=8.59*10^{-9}$. In Figure 3, we display the probability with and without redundancy as a function of $p_s$ for $\rho=0.6$, $M=39$ and $n=19$. We note that for the sources whose proportion in the overall arrival stream does not exceed 10% ($p_s \leq 0.1$), redundancy decreases their loss probabilities, but not for the others.

We note also, that the loss probability without redundancy decreases in $p_s$ contrary to the loss probability with redundancy which increases in $p_s$. Figure 4 illustrates the loss probabilities as function of the number of redundant packets for $\rho=(19/(19+4))0.6 \cong 0.5$, $p_s=0.1, n=19, M=39$.

## 5.2 Adding redundancy only for source $s$

We are interested by case when redundancy is added only for the source $s$. We assume that the rate at which messages arrive is the same with and without redundancy. When one redundant packet is added for source $s$, we have $\lambda=\rho=nx+p_sx$. We proceed similarly as above, we get for large $M$

$$\rho_s^* = \exp(-p_s/\eta). \tag{16}$$

**Remark 5.1.** *In all the paper, by substituting 1 for $p_s$, we obtain all the equations and formulas of [3], the case of single source.*

**Exemple 2:** Let $\rho=0.8$ and $p_s=0.1$, we wish to determine $n$ and $M$ for which the loss probability is of order $10^{-9}$ and redundancy leads to better performance. From (16) we have $\eta=0.04$ and from Proposition 3.1 we have

$$P^s_{0.8}(>0,n) \cong \frac{1-0.8}{1-0.8^{M+1}} \frac{0.8^M}{1-0.72}(1-0.8)0.04M \cong 10^{-9} \quad \Rightarrow M \cong 90,\ n=\eta M \cong 4$$

exact calculation for $A(\rho) \stackrel{\text{def}}{=} P^s_{nx}(>0,n)$ and $S'(\rho) \stackrel{\text{def}}{=} P^s_{nx+p_sx}(>1,n+1)$ yields that for $n=4$ and $M=90$, $S(0.8)=2.67*10^{-9}$ and $A(\rho)=1.28*10^{-9}$ and for $n=7$, $M=90$ we have $S(0.8)=1.86*10^{-9}$ and $A(\rho)=2.12*10^{-9}$, thus for $n\geq 7$, $S(\rho)<A(\rho)$. We display in Figure 5 the loss probabilities as function of the number of redundant packets for $n=8$, $M=90$, $\rho=0.8$ and $p_s=0.1$.

We note that the larger size of messages leads to better perfomance in the presence of redundancy. For the same example, by taking $n=14$, we reduce the loss probabilities of messages by an order of 10 when we add 4 redundant packets only for source $s$ (Figure 6). (Note that this improvement has negative effect on the other sources.)

For the example above, if we consider only two surces $s$ and $\bar{s}$, we have $p_{\bar{s}}=0.9$ and $P^{\bar{s}}_{0.8}=2.114\ 10^{-9}$. When we add 4 redundant packets for $s$ the workload becomes $\rho=0.8+\frac{4}{14}0.08=0.8228$, $p_s=\frac{18}{14}0.1=0.1285$ and $p_{\bar{s}}=1-0,1285=0.8715$. For these values, we obtain $P^{\bar{s}}_{0.8228}=2.305\ 10^{-8}$. Finaly, we see that the redundancy in source $s$ has an effect of increasing the loss probability for the other source (by the same order of magnitude as the decrease in the loss probabilities of $s$).

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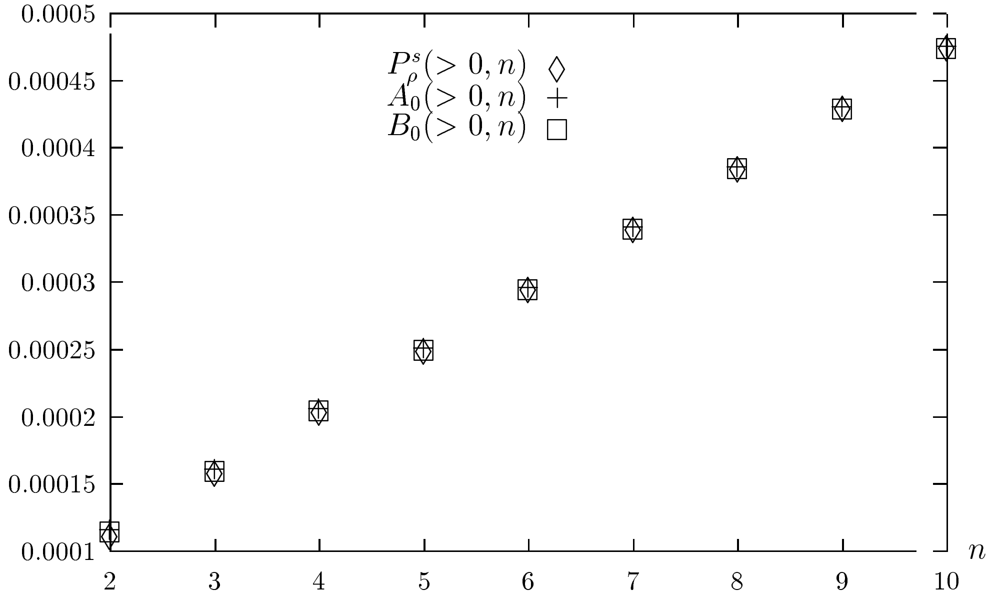


Figure 1: $P^s_\rho(>0,n)$ and its approximations $\rho = 0.4$, $p_s = 0.6$ and $M = 10$.

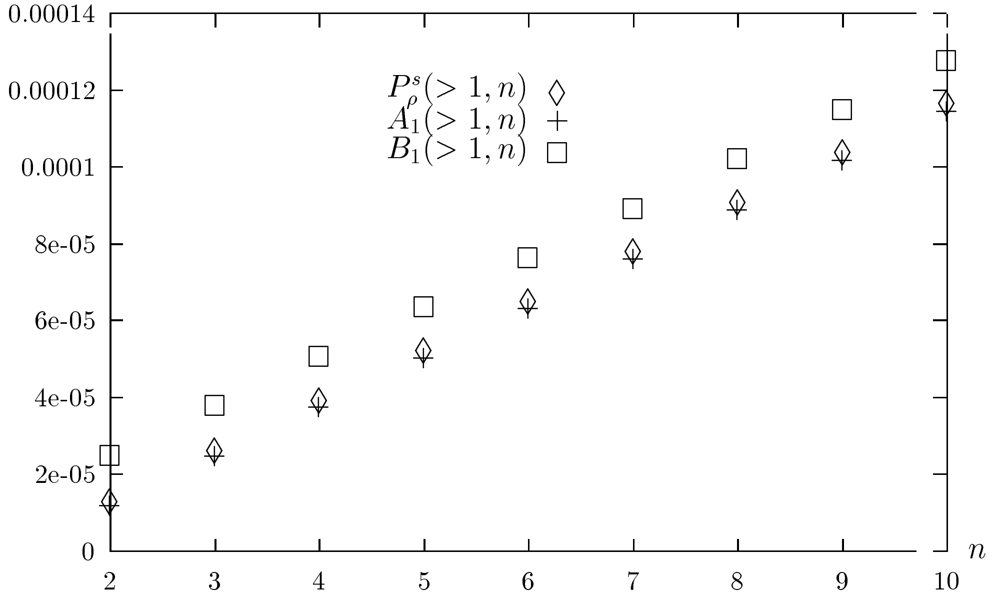


Figure 2: $P^s_\rho(>1,n)$ and its approximations $\rho = 0.4$, $p_s = 0.6$ and $M = 10$.

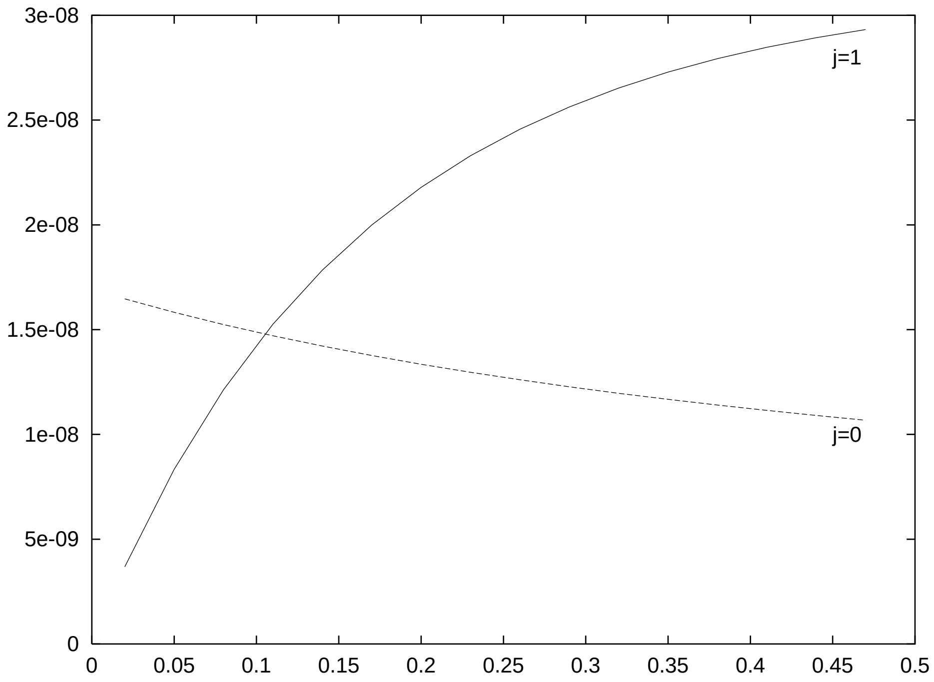


Figure 3: $P^s_\rho(> j, n+j)$, $j = 0, 1$ as function of $p_s$ for $\rho = 0.6$, $n = 19$ and $M = 39$.

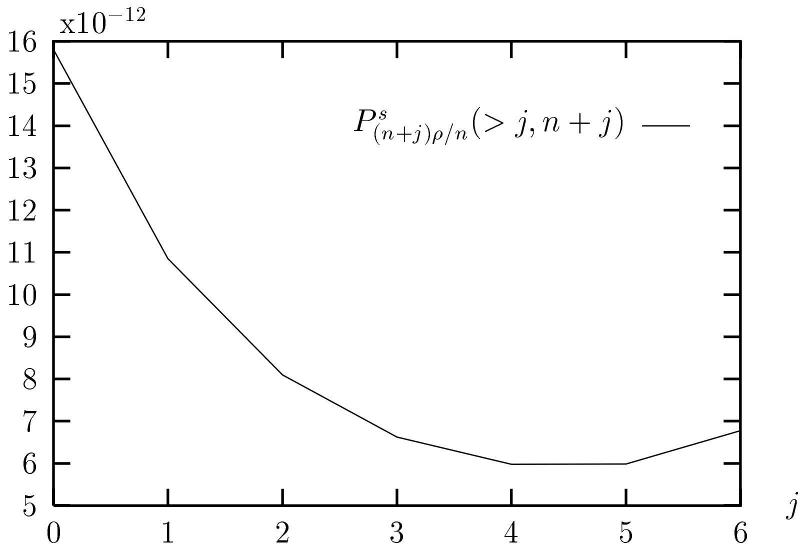


Figure 4: $P^s_{(n+j)\rho/n}(> j, n+j)$ as function of $j$ for $\rho = 0.5$, $p_s = 0.1$, $n = 19$ and $M = 39$

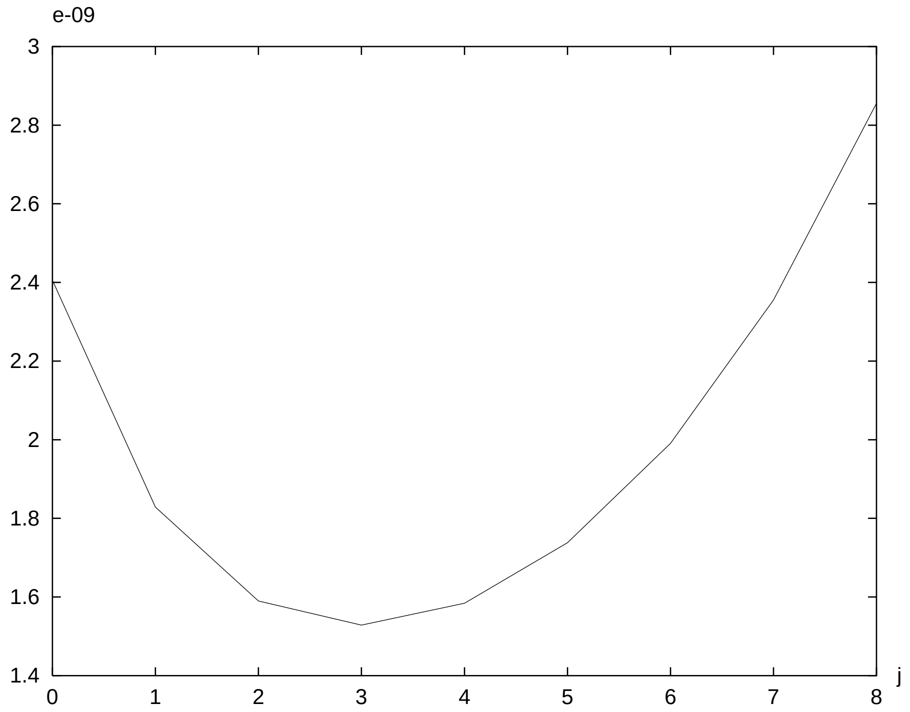


Figure 5: $P^s_{\rho+(j/n)\rho_s}(> j, n+j)$ as function of $j$ for $\rho = 0.8$, $p_s = 0.1$, $n = 8$ and $M = 90$

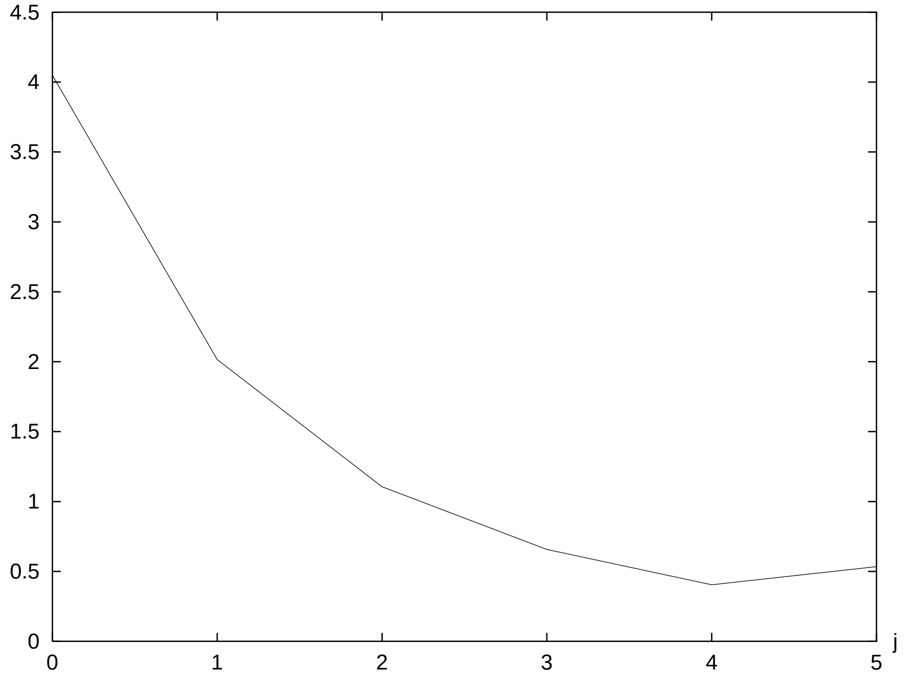


Figure 6: $P^s_{(\rho+(j/n)\rho_s)}(> j, n+j)$ as function of $j$ for $\rho = 0.8$, $p_s = 0.1$, $n = 14$ and $M = 90$