Optimal Choice of Threshold in Two Level Processor Sharing

Konstantin Avrachenkov[a], Patrick Brown[b], Natalia Osipova[a, 1]

[a] INRIA Sophia Antipolis, France, e-mail: {K.Avrachenkov, N.Osipova}@sophia.inria.fr
[b] France Telecom R&D, France, e-mail: Patrick.Brown@orange-ft.com

Abstract

We analyze the Two Level Processor Sharing (TLPS) scheduling discipline with hyper-exponential distribution for the job size and with Poisson arrival process. TLPS is a convenient model to represent the file size based differentiation in TCP/IP networks. We find a closed form analytic expression for the expected sojourn time and an approximation for the optimal value of the threshold that minimizes the expected sojourn time.

1 Introduction and Model Formulation

We study the Two Level Processor Sharing (TLPS) scheduling discipline with hyper-exponential distribution for the job size. The TLPS discipline was first introduced in [5]. TLPS is a convenient model to represent the file size based differentiation in TCP/IP networks [2]. It is assumed that jobs arrive to the system according to a Poisson process with rate λ. Let θ be a given threshold. If there are jobs in the system that attained a service less than θ, they are assigned to the high priority queue which is served according to Processor Sharing (PS) discipline. If in addition there are jobs with attained service greater than θ, those jobs wait in the lower priority queue. When the high priority queue is empty, the jobs that received more than θ are served according to the PS discipline.

In the present work we assume that the job size distribution is hyper-exponential with two phases. Namely, the cumulative distribution function $F(x)$ is given by

$$F(x) = 1 - p_1\, e^{-\lambda_1 x} - p_2\, e^{-\lambda_2 x}, \tag{1}$$

where $p_1 + p_2 = 1$, and $p_1, p_2 > 0$.The hyper-exponential distribution represents well the file size distribution in the Internet [4]. Then, the mean job size is $m = p_1/\lambda_1 + p_2/\lambda_2$ and the system load is $\rho = \lambda m$. We assume that the system is stable ($\rho < 1$) and is in steady state. Since the hyper-exponential distribution has a decreasing hazard rate, there exists a value of the threshold that minimizes the expected sojourn time in the TLPS system [1, 2]. The main goal of the present work is to determine this optimal value of the threshold.

2 The expected sojourn time in TLPS system

Let us denote by $\overline{X_\theta^n} = \int_0^\theta y^n \frac{d}{dy} F\left(y\right) dy + \theta^n \left(1 - F\left(\theta\right)\right)$ the n-th moment of the truncated distribution at θ. The utilization factor for the truncated distribution is $\rho_\theta = \lambda\, \overline{X_\theta^1}$.

For flows with a size $x < \theta$ the system behaves as a standard PS system where the service time distribution is truncated at θ. Then in this case the expected conditional sojourn time is given by

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$$\overline{T}^{TLPS}(x) = \frac{x}{1-\rho_\theta}, \qquad x \in [0,\theta]. \tag{2}$$

For flow size $x > \theta$ the expected conditional sojourn time is given by

$$\overline{T}^{TLPS}(x) = \frac{\overline{W}(\theta) + \theta + \alpha(x-\theta)}{1-\rho_\theta}, \qquad x \in (\theta, \infty) \tag{3}$$

where $(\overline{W}(\theta)+\theta)/(1-\rho_\theta)$ is the expression of the time needed to reach the low priority queue. This time consists of the time spent waiting in the high priority queue, where the flow is serviced up to the the threshold θ, plus the time waiting for the high priority queue to empty. Here $\overline{W}(\theta) = \frac{\lambda \overline{X_\theta^2}}{2(1-\rho_\theta)}$. The remaining term $\alpha(x-\theta)/(1-\rho_\theta)$ is the time spent in the low priority queue.

To find $\alpha(x)$ we can use the interpretation of lower priority queue as a PS system with batch arrivals. As it was shown in [5], $\alpha'(x) = d\alpha/dx$ is a solution of the following integral equation

$$\alpha'(x) = \lambda\overline{n}\int_0^\infty \alpha'(y)B(x+y)dy + \lambda\overline{n}\int_0^x \alpha'(y)B(x-y)dy + bB(x) + 1. \tag{4}$$

Here $\overline{n} = \frac{1-F(\theta)}{1-\rho_\theta}$ is the average batch size, $B(x) = \frac{1-F(\theta+x)}{1-F(\theta)}$ is the complementary truncated distribution and $b = \frac{2\lambda\,(1-F(\theta))(\overline{W}(\theta)+\theta)}{(1-\rho_\theta)}$ is the average number of jobs that arrive to the low priority queue in addition to the tagged job.

To solve the integral equation (4) we use the method described in [3].

Theorem 1 *Consider TLPS system with hyper-exponential job size distribution with two phases. Then, the expected sojourn time in the system is given by*

$$\overline{T}(\theta) = \int_0^\infty \overline{T}^{TLPS}(x)\,dF(x) = \frac{\mathcal{T}_1 + \mathcal{T}_2(\theta) + \mathcal{T}_3(\theta) + \mathcal{T}_4(\theta) + \mathcal{T}_5(\theta)}{\mathcal{T}_z(\theta)}, \tag{5}$$

where

$$\mathcal{T}_1 = {\lambda_1}^2{\lambda_2}^2(\lambda_1+\lambda_2)\,m\,(1-\rho)^2,$$
$$\mathcal{T}_2(\theta) = {\lambda_1}^2\,{\lambda_2}^2\,\rho\,(1-\rho)(\lambda_1+\lambda_2)\left(\frac{A_1(\theta)}{\lambda_1} + \frac{A_2(\theta)}{\lambda_2}\right),$$
$$\mathcal{T}_3(\theta) = {\lambda_1}^2\,{\lambda_2}^2\,\lambda^2\left(\frac{A_1(\theta)}{\lambda_1} + \frac{A_2(\theta)}{\lambda_2}\right)^2\left(\frac{\alpha_1}{\lambda_1}\lambda_2 + \frac{\alpha_2}{\lambda_2}\lambda_1\right),$$
$$\mathcal{T}_4(\theta) = -A_1(\theta)\,A_2(\theta)\,\lambda\,(\lambda_1-\lambda_2)^2(1-\rho)(\lambda_1+\lambda_2+\theta\lambda_1\lambda_2),$$
$$\mathcal{T}_5(\theta) = \lambda\,(1-\rho)\left(\alpha_1\,{\lambda_2}^3 + \alpha_2\,{\lambda_1}^3\right)(A_1(\theta)+A_2(\theta)),$$
$$\mathcal{T}_z(\theta) = -{\lambda_1}^2{\lambda_2}^2(1-\rho)\left((\lambda_1+\lambda_2)(1-\rho) + \lambda\left(\frac{A_1(\theta)}{\lambda_1}\lambda_2 + \frac{A_2(\theta)}{\lambda_2}\lambda_1\right)\right)\times$$
$$\times\left(1-\rho+\lambda\left(\frac{A_1(\theta)}{\lambda_1} + \frac{A_2(\theta)}{\lambda_2}\right)\right),$$

where $A_i(x) = p_i\,e^{-\lambda_i x},\ i = 1,2.$

3 Optimal threshold

Our main goal is to minimize the expected sojourn time $\overline{T}(\theta)$ with respect to θ. Of course, one can differentiate the exact analytic expression provided in Theorem 1 and set the result of differentiation to zero. However, this will give a transcendental equation for the optimal value of the threshold. To obtain some explicit expression for the optimal value of the threshold, we consider a particular case when $\lambda_2 << \lambda_1$. This is motivated by the fact that in the Internet connections belong to two distinct classes with very different size of transfer. The first class is composed of short HTTP connections and P2P signaling connections. The second class corresponds to downloads (PDF files, MP3 files, MPEG files, etc.).

Theorem 2 *Let θ_{opt} denote the optimal value of the threshold. Namely, $\theta_{opt} = \arg\min \overline{T}(\theta)$. The value $\tilde{\theta}_{opt}$ given by*

$$\tilde{\theta}_{opt} = -\frac{1}{\lambda_1 - \lambda_2} \ln\left(\frac{\lambda_2(1-\rho)}{\lambda_1 - \lambda}\right)$$

approximates θ_{opt} so that $\overline{T}'(\tilde{\theta}_{opt}) = o(\lambda_2/\lambda_1)$.

Now let us study the gain that we obtain using TLPS with nearly optimal threshold in comparison with the standard Processor Sharing. Towards this goal, we plot the ratio $\Delta(\rho) = \frac{\overline{T}(0)-\overline{T}(\tilde{\theta}_{opt})}{\overline{T}(0)}$ in Figure 1. We note that the point $\theta = 0$ corresponds to the PS scheduling discipline. To study the sensitivity of the TLPS system with respect to θ we plot on the same Figure 1 $\Delta_1(\rho) = \frac{\overline{T}(0)-\overline{T}(\frac{3}{2}\tilde{\theta}_{opt})}{\overline{T}(0)}$ and $\Delta_2(\rho) = \frac{\overline{T}(0)-\overline{T}(\frac{1}{2}\tilde{\theta}_{opt})}{\overline{T}(0)}$. Here we fix the following values: $\lambda_2 = \frac{1}{10}\lambda_1, \lambda_1 = 1, m = \frac{17}{11}$. One can see that it is beneficial to use TLPS instead of PS in the case of heavy and moderately heavy loads. We also observe that the optimal TLPS system is not too sensitive to the value of the threshold. Nevertheless, it is better to choose larger rather than smaller values of the threshold.

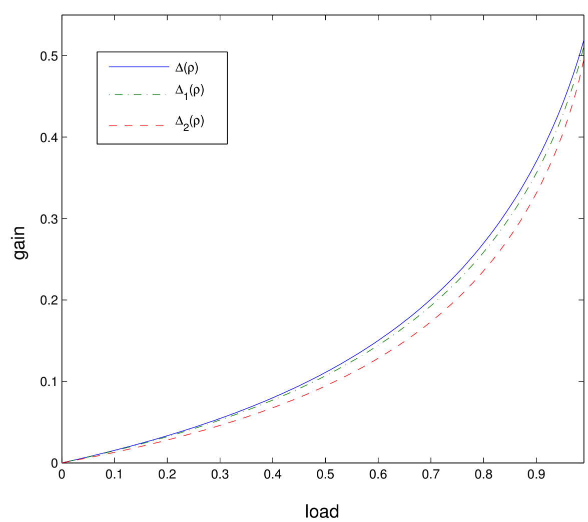

Figure 1: The gain in using TLPS in comparison with PS.

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