STOCHASTIC NETWORKS

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SUMMARY

PART 1: NETWORKS VIA PETRI NETS

1. Bipartite Directed Graph with two types of nodes
   - $\mathcal{P}$ set of places reservoirs
   - $\mathcal{T}$ set of transitions synchronizers
2. Arcs
   - $p^{\bullet}$ $({}^{\bullet}p)$ set of transitions which follow (precede) p
   - $t^{\bullet}$ $({}^{\bullet}t)$ set of places which follow (precede) t
3. Initial Marking $\mu : \mathcal{P} \to I\!N$

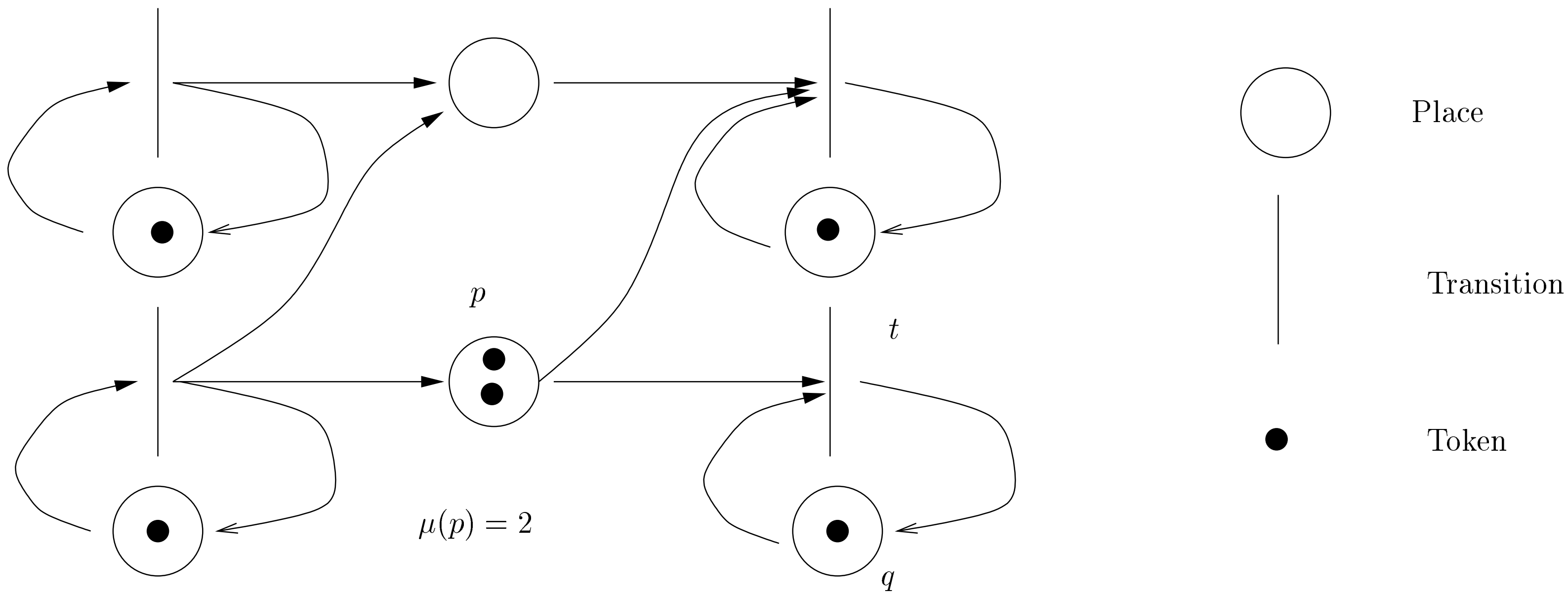


Figure 1: ${}^\bullet t = \{p, q\}$

1: LOCAL DYNAMICS OF SYNCHRONIZATION

1. Enabling of a transition $\forall p \in {}^{\bullet}t, \quad \mu(p) \geq 1$

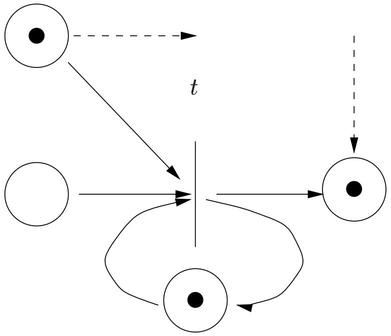


Figure 2: Transition t is not enabled

2. Firing of an enabled transition
   Tokens consumed by the firing of t
   $\forall p \in^{\bullet} t, \quad \mu(p) := \mu(p) - 1$
   Tokens produced by the firing of t
   $\forall p \in \; t^{\bullet}, \quad \mu(p) := \mu(p) + 1$

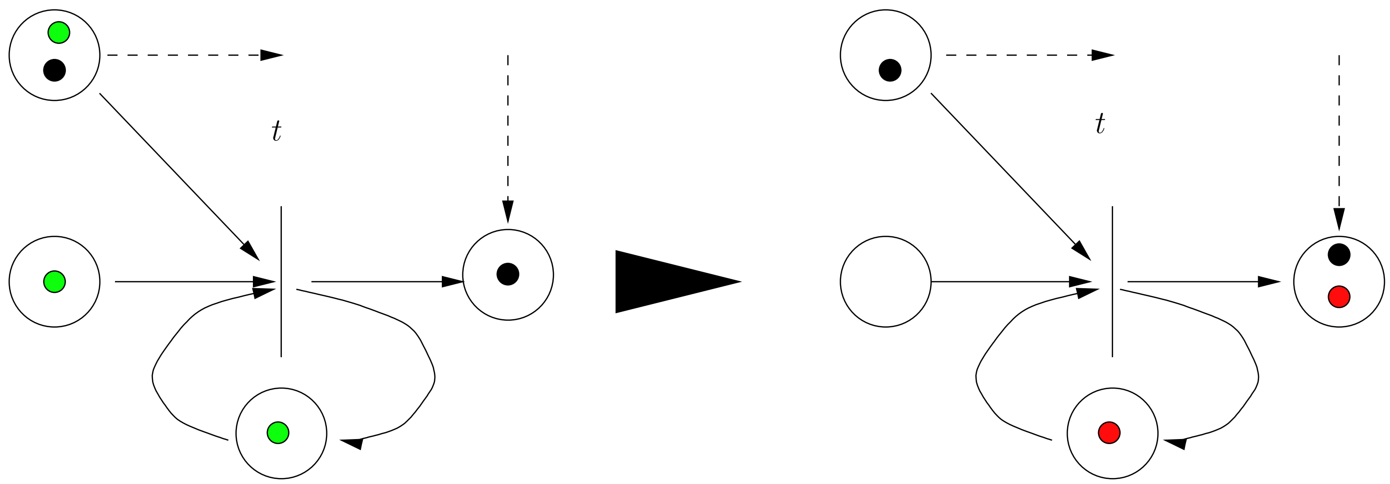
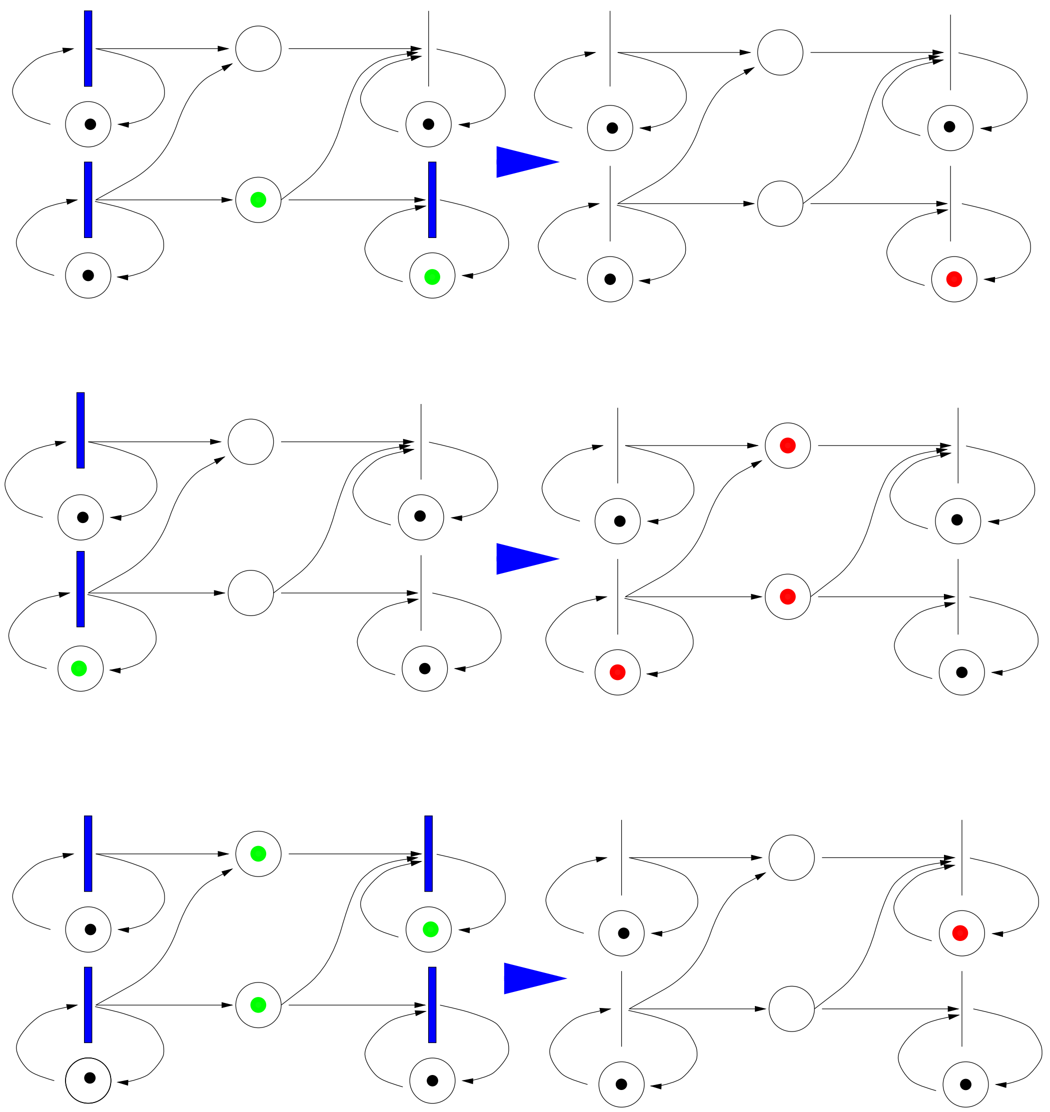


Figure 3: Local marking before and after the firing of t



1: BASIC ELEMENTS FOR SYNCHRONIZATION

On Transitions

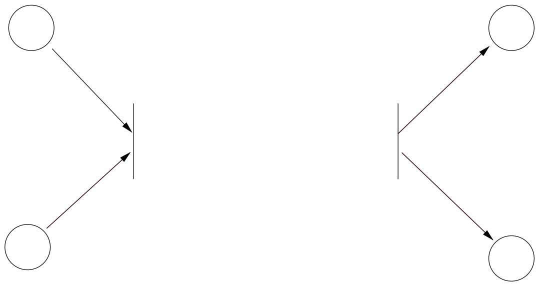


Figure 4: Convergence to a transition: In Synchronization – Divergence from a transition: Out Synchronization

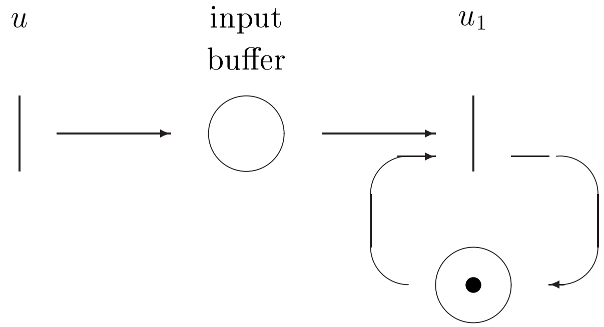
1: CLASSES OF NETS: EVENT GRAPHS

Pure Synchronization Each place has

– at most one input transition

– at most one output transition

Example: G/G/1/∞ queue



1: BASIC ELEMENTS FOR ROUTING

On Places

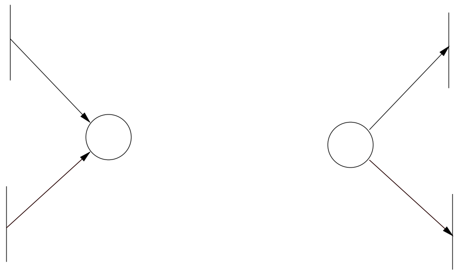


Figure 5: Convergence to a place: Superposition – Divergence from a place: Choice)

CLASSES OF NETS: FREE CHOICE NETS

In-Synchronization and Choice are separated

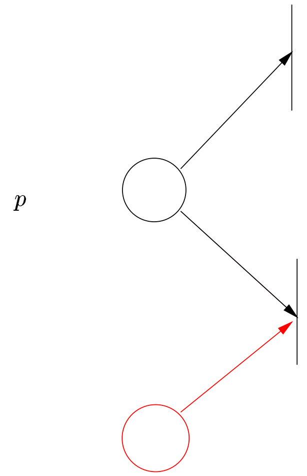
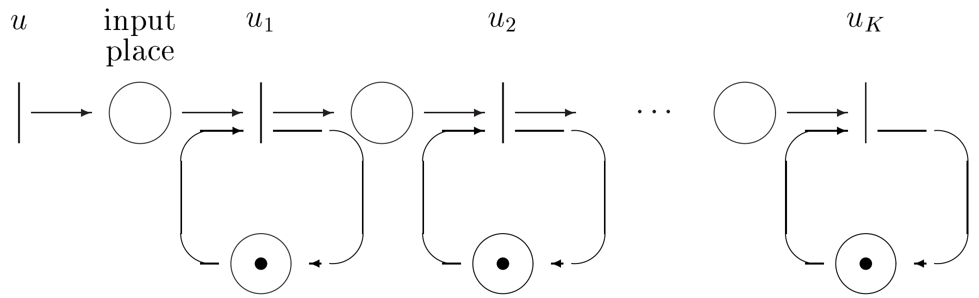


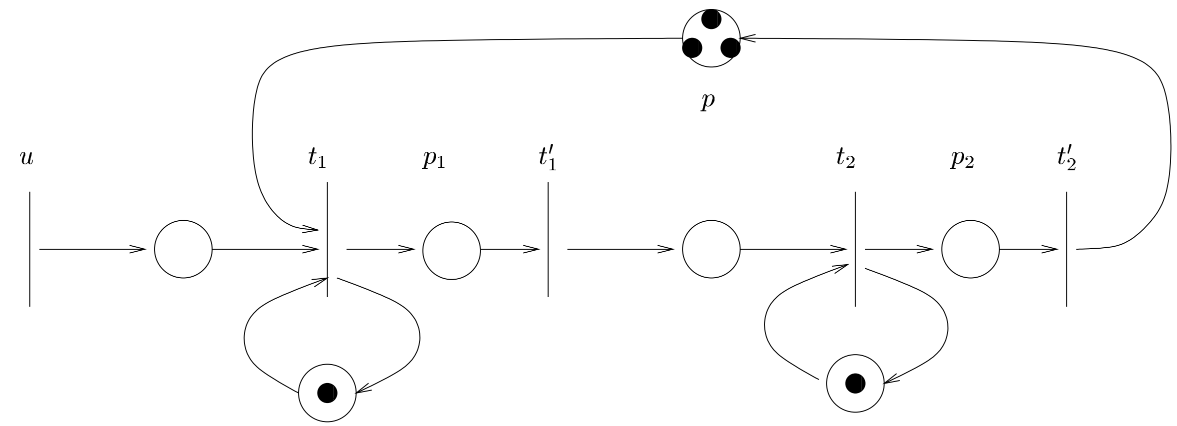
Figure 6: $Card(p^\bullet) > 1$ implies $Card(^\bullet t) = 1$ for all $t \in p^\bullet$

An Event Graph is a Free Choice net

PART 2: SYNCHRONIZATION & (max,plus) LINEARITY
2.1: EXAMPLES: TANDEM QUEUES

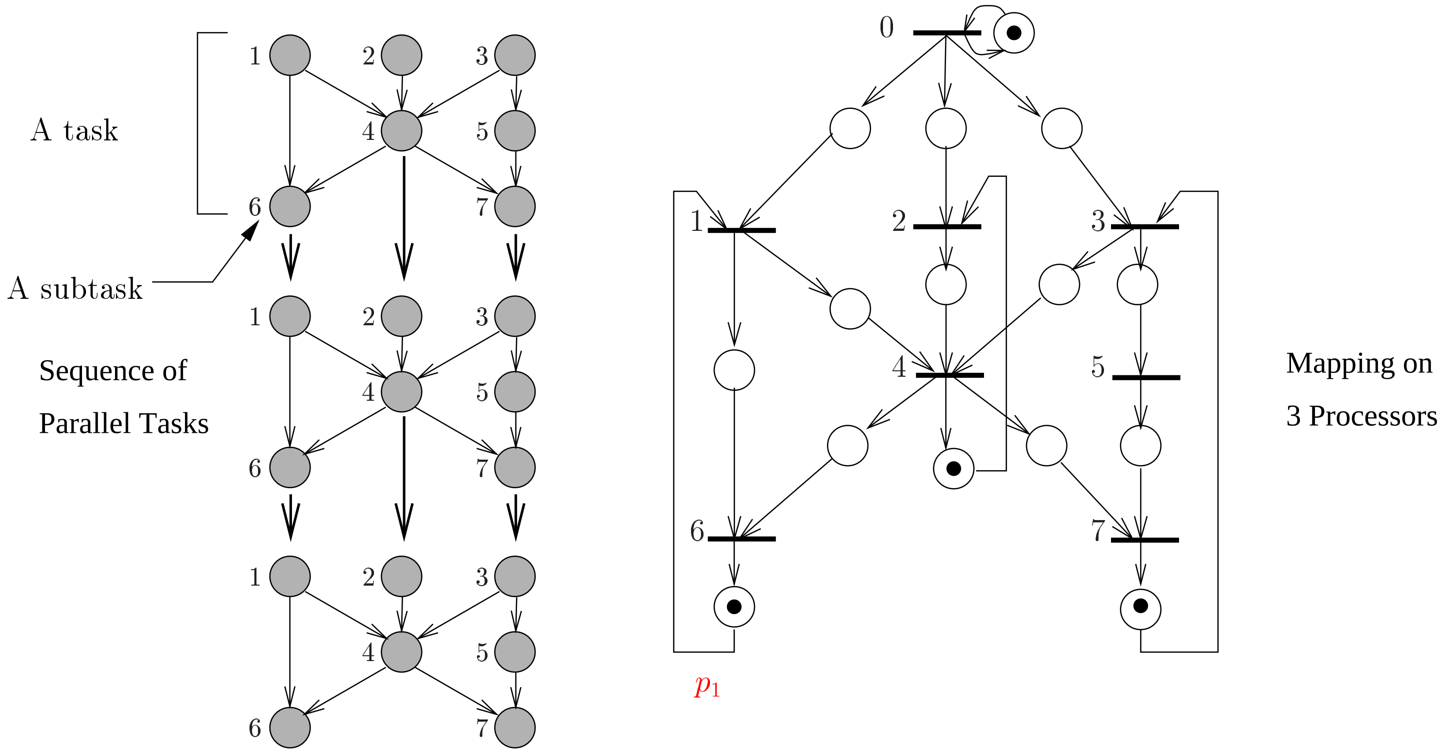


2.1: COMMUNICATION EXAMPLE: WINDOW FLOW CONTROL



u: Input Flow, t_i: Routers, t'_i: Transmission Delays, $\mu(p)$: Window Size.

2.1: PARALLEL PROCESSING EXAMPLE



0:Acquisition of data 1,6: Sequential subtasks to be performed by Processor 1

p_1: Processor 1

2.1: MANUFACTURING EXAMPLE

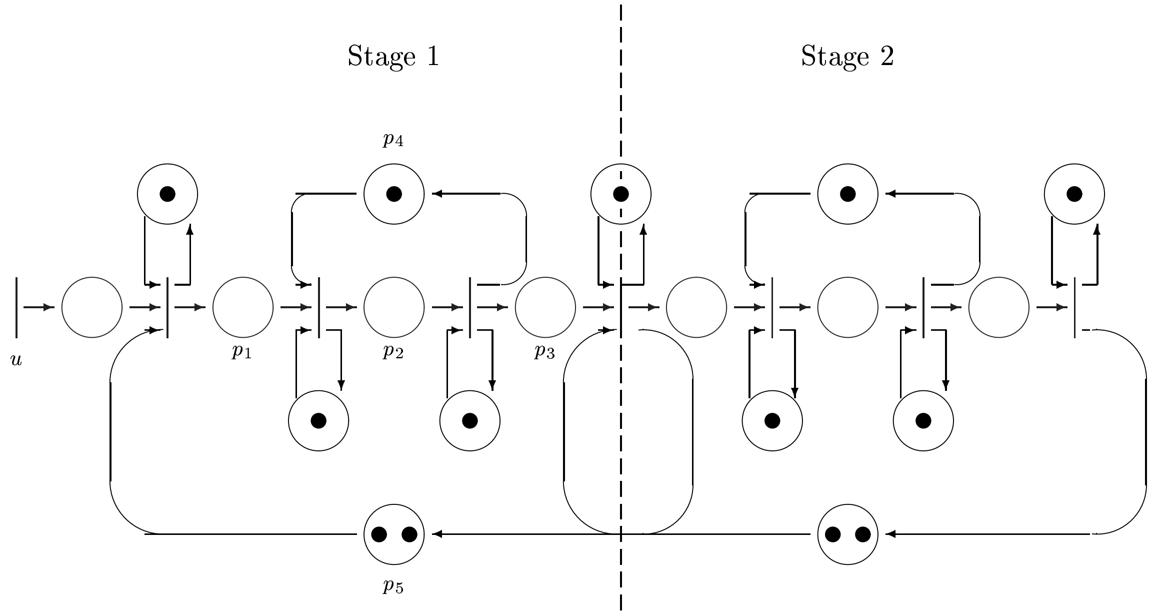


Figure 7: A Two Stage Kanban

u: orders, p_1: input buffer of machine 1, p_2: working buffer of machine 1, p_3: output buffer of machine 1, $\mu(p_5)$: number of kanbans of machine 1

2.1: MANUFACTURING EXAMPLE: FLOW SHOP

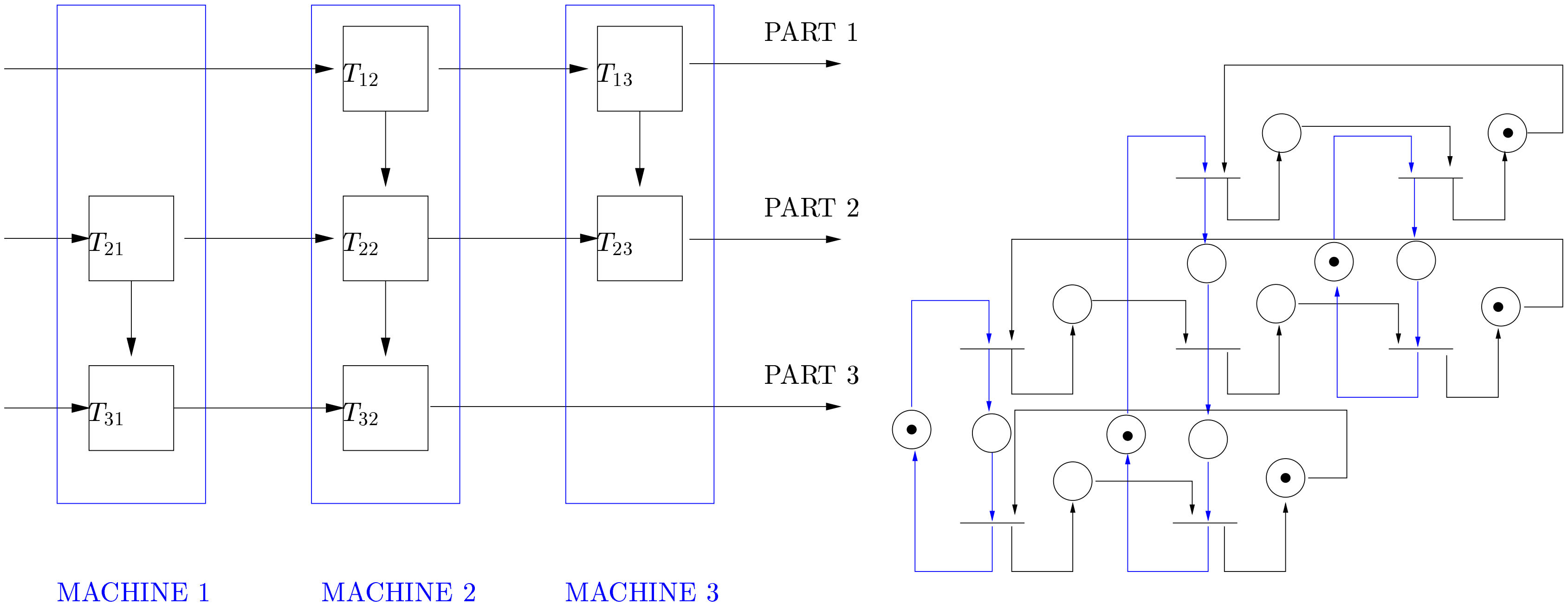


Figure 8: Process Plan and Associated Event Graph

2.2: NETWORKS AND ALGEBRA: THE (max,plus) ALGEBRA

Maslov 73, Gondran-Minoux 77, Cuninghame Green 79, Cohen-Quadrat-Viot 83, FB-Cohen-Olsder-Quadrat 92

1. Scalar Semi-Field $(\mathbb{R}_{max}, \oplus, \otimes)$

   - $\mathbb{R}_{max} = \mathbb{R} \cup \{\varepsilon = -\infty\}$, $\oplus = \max$, $\otimes = +$
   - $(\mathbb{R}_{max}, \oplus)$ commutative idempotent semi-group

$$a \oplus b = b \oplus a$$

$$a \oplus \varepsilon = \varepsilon \oplus a = a$$

$$a \oplus a = a$$

   - $(\mathbb{R}_{max}, \otimes)$ commutative group neutral element $e = 0$
   - $\otimes$ distributes over $\oplus$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

2. Matrix Semi-Ring $(\mathbb{R}_{max}^{K,K}, \oplus, \otimes)$

– Matrix addition Neutral $\mathcal{E}$, $\mathcal{E}_{ij} = \varepsilon$

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$$

– Matrix multiplication Neutral E, $E_{ij} = \varepsilon, \ i \neq j, \ E_{ii} = e$

$$(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj}$$

Example

$$\begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix} \otimes \begin{pmatrix} -7 & 2 \\ \varepsilon & 3 \end{pmatrix} = \begin{pmatrix} -6 & 7 \\ -10 & 3 \end{pmatrix}$$

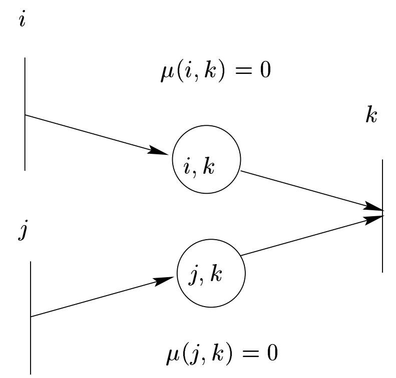
2.2: EVENT GRAPHS AS (max,plus) LINEAR SYSTEMS

- $M \in \mathbb{N}$: Maximal initial marking (data)
- $\sigma_n^k \in \mathbb{R}$: Duration of n-th firing of transition k (data)
- $X_n^k \in \mathbb{R}$: Date of n-th enabling of transition k

σ_n^k: Random variables on $(\Omega, \mathcal{F}, P)$

- G: the sequences $\{\sigma_n^k\}$ are jointly stationary and ergodic
- GI: the sequences $\{\sigma_n^k\}$ are independent and i.i.d.
- M: the sequences $\{\sigma_n^k\}$ are independent, i.i.d. and exponentially distributed.

X_n^k: State Variables



Evolution equation in FIFO case

$$X_n^k = \max_{l \in {}^{\bullet\bullet}k} X^l_{n-\mu(l,k)} + \sigma^l_{n-\mu(l,k)} = \bigoplus_p \bigoplus_{l \in S_{k,p}} X^l_{n-p} \otimes \sigma^l_{n-p}$$

Evolution equation in $I\!R^K_{max}$ $X_n = (X_n^k) \in I\!R^{K\times 1}_{max}$

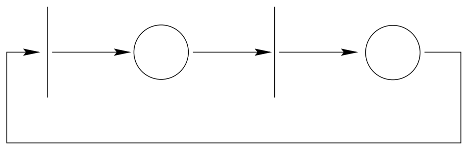
$$X_n = A(n,0) \otimes X_n \oplus A(n,1) \otimes X_{n-1} \ldots \oplus A(n,M) \otimes X_{n-M}$$

$A(n,p) \in I\!R^{K\times K}_{max}$

$$A(n,p)_{kl} = \begin{cases} \sigma^l_{n-p} & \text{if } l \in {}^{\bullet\bullet}k,\ \mu(l,k) = p \\ \varepsilon & \text{otherwise} \end{cases}$$

2.2 TRANSFORMATION 1: CONSTRUCTIVE RECURRENCES

1. Liveness all transitions can fire infinitely often. Example of non live net:



2. If the matrix $A(n,0)$ is strictly lower triangular up to a change of basis then the net is live and the following series converges

$$A(n,0)^\star = E \oplus A(n,0) \oplus A(n,0)^2 \oplus \ldots$$

3. Transformation $X_n = A(n,0)X_n \oplus A(n,1) \otimes X_{n-1}..A(n,M)X_{n-M} \to$

$$X_n = \overline{A}(n,1)X_{n-1} \oplus \ldots \oplus \overline{A}(n,M)X_{n-M}$$

with $\overline{A}(n,p) = A(n,0)^\star A(n,p)$

TRANSFORMATION 2: RECURRENCE OF ORDER 1

State extension $\rightarrow$ Standard form

$$\begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_{n-M+1} \end{pmatrix} = \begin{pmatrix} \overline{A}_{n,1} & \overline{A}_{n,2} & \ldots & \ldots & \overline{A}_{n,M} \\ E & \mathcal{E} & \ldots & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & E & \ddots & \vdots & \mathcal{E} \\ \vdots & \ddots & E & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & \ldots & \mathcal{E} & E & \mathcal{E} \end{pmatrix} \begin{pmatrix} X_{n-1} \\ X_{n-2} \\ \vdots \\ X_{n-M} \end{pmatrix}$$

2.2: CONCLUSION FROM TRANSFORMATIONS 1 & 2

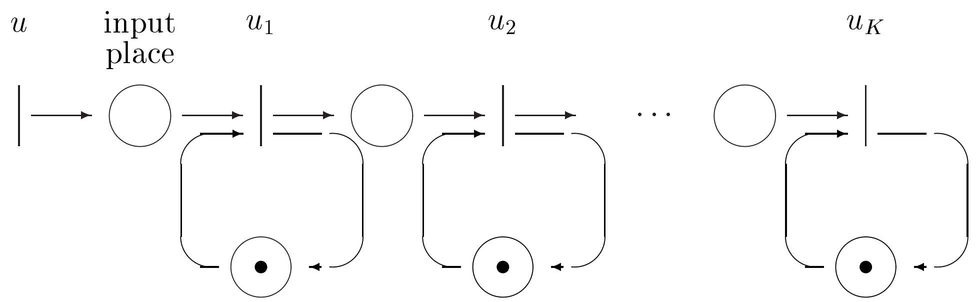
A live FIFO stochastic event graph admits a representation as a $(\max, +)$-linear recurrence of order 1 'in a random medium', i.e.

$$\widetilde{X}_{n+1} = \widetilde{A}_n \otimes \widetilde{X}_n$$

with the sequence of matrices $\{\widetilde{A}_n\}$ is stationary and ergodic
FB 92, Annals Prob.

Builds on result by Cohen-Dubois-Quadrat-Viot 83, CDC on the deterministic case $(\sigma_n^k = \sigma^k, \forall n)$.

2.2: OPEN EXAMPLE: TANDEM QUEUES



$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$A(n,0) = \begin{pmatrix} \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma_n^2 & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \cdots & \sigma_n^{K-2} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_n^{K-1} & \varepsilon \end{pmatrix} A(n,1) = \begin{pmatrix} \sigma_{n-1}^1 & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \sigma_{n-1}^2 & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma_{n-1}^3 & \cdots & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \sigma_{n-1}^{K-1} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_{n-1}^K \end{pmatrix} B(n,0) = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$$

Transformation 1

$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$X_n = A(n,0)^\star A(n,1) \otimes X_{n-1} \oplus A(n,0)^\star \otimes B(n,0)U_n$$

Standard Representation for open systems

$$\widetilde{X}_{n+1} = \widetilde{A}_n \otimes \widetilde{X}_n \oplus \widetilde{B}_{n+1} U_{n+1}$$

Here

$$(\widetilde{A}_n)_{ij} = \begin{cases} \varepsilon & \text{if} \quad i < j \\ \sum_{k=j}^{i-1} \sigma_{n+1}^k + \sigma_n^j & \text{if} \quad i \geq j \end{cases}, \qquad \widetilde{B}_n = \begin{pmatrix} 0 \\ \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \sigma_n^1 + \sigma_n^2 + \sigma_n^3 \\ \vdots \\ \sum_{k=1}^{K-1} \sigma_n^k \end{pmatrix}.$$

2.3: PROBABILITY AND ALGEBRA
2.3.1: REPRESENTATION FORMULAS, ERGODIC THEOREMS

LYAPUNOV EXPONENT – STRICTLY POSITIVE CASE

$\{A_n\}$ stationary and ergodic sequence of $\mathbb{R}_{max}^{K\times K}$ matrices

If $\forall i, j$, $P((A_0)_{ij} = \varepsilon) = 0$ and $(A_0)_{ij}$ integrable, there exists a constant $\gamma \in \mathbb{R}$ / $\forall X_0, \forall i$

$$\lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \gamma, \quad P - a.s.$$

J.E. Cohen 88, Siam Review from Kingman's Subadditive Ergodic Theorem

$$|A \otimes B|_\oplus \le |A|_\oplus \otimes |B|_\oplus$$

$$|A_1 \otimes \ldots A_n \otimes \ldots A_p|_\oplus \le |A_1 \otimes \ldots A_n|_\oplus \otimes |A_{n+1} \otimes \ldots A_p|_\oplus$$

2.3.1: STRUCTURE OF LYAPUNOV EXPONENTS – FIXED SUPPORT CASE

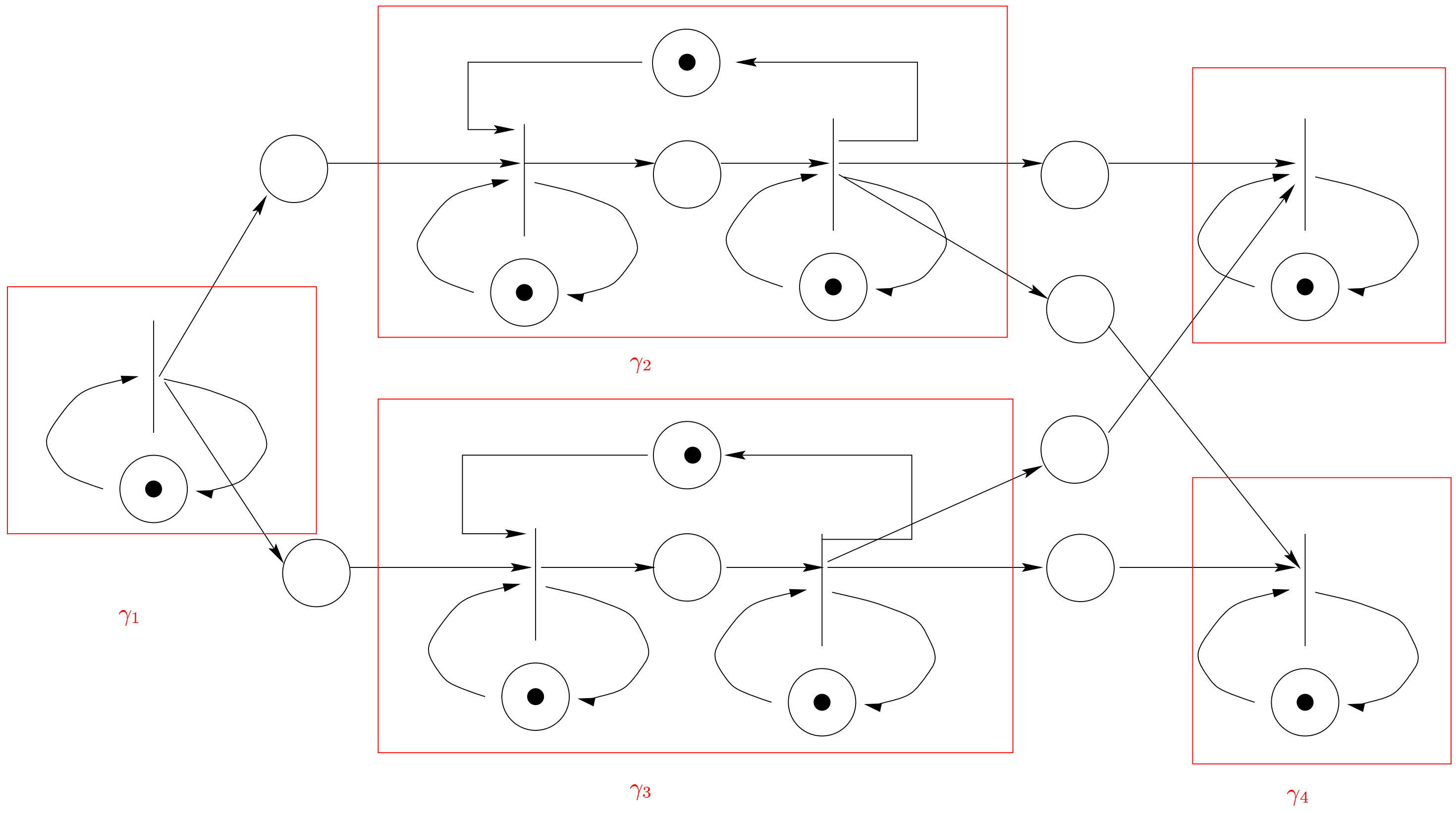
Fixed support: For all j, k, either $(A_0)_{kj} > \varepsilon$ a.s or $(A_0)_{kj} = \varepsilon$ a.s.
FIFO: $\forall j,\ (A_0)_{jj} \geq 0$ a.s.
Graph of A_0: Directed graph $j \to k$ if $(A_0)_{kj} > \varepsilon$ a.s
CC: Equivalence classes of $i\mathcal{R}j$ if $i \to^\star j$ and $j \to^\star i$

$$\forall i \in cc\ \tilde{u}, \lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \oplus_{\tilde{v} \in {}^\star\tilde{u}}\, \gamma_{\tilde{v}}$$

– ${}^\star\tilde{u}$: set of all ancestors of $\tilde{u}$ in reduced graph

– $\gamma_{\tilde{v}}$: Lyapunov exponent of the restriction of A_n to the coordinates of $\tilde{v}$

FB & Z. Liu 92, Annals Prob.
Various extensions to the non-fixed support case.



$$\lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4, \ \ i \in cc\ 4$$

2.3.1: REPRESENTATION FORMULAS: OPEN CASE

Reference equation – 1 dim input. Assumptions: FIFO + Fixed Support

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$$

Waiting Times

$$W_n^i = X_n^i - T_n$$

Equation for Waiting Times by subtracting T_{n+1} on both sides of (1)

$$W_{n+1} = A_n \otimes C(-\tau_n) \otimes W_n \oplus B_{n+1}$$

$C(x) = x \otimes E$, $\tau_n = T_{n+1} - T_n$

Particular cases:

- 1-dim, with $B_n = 0$: $W_{n+1} = W_n \otimes (\sigma_n) \otimes (-\tau_n) \oplus 0$
- Deterministic case: $W_{n+1} = A \otimes C(-\tau) \otimes W_n \oplus B$

2.3.1: REPRESENTATION FORMULAS: OPEN CASE

Backward Coupling

– $\{A_n, B_n, \tau_n\}$ is stationary and ergodic, with all entries non ε integrable

– γ top Lyapunov exponent of $\{A_n\}$

If $\rho = \lambda\gamma < 1$, with $\lambda^{-1} = E(\tau_0)$, $\{W_n\}$ converges in total variation to a unique stationary sequence. The stationary law of W_n is that of

(REP1) $$W = B_0 \oplus \bigoplus_{n \geq 1} C(T_{-n}) \otimes D_n, \qquad D_n = A_{-1} \otimes \ldots \otimes A_{-n} \otimes B_{-n}$$

If $\rho > 1$, $|W_n|_\oplus$ tends to ∞ a.s. FB 92, Annals Prob.

Particular cases:

– 1-dim, with $B_n = 0$: W: Loynes's variable for G/G/1∞ queue

– Deterministic case: $W = (A \otimes C)^* \otimes B$, unique solution of $W = A \otimes C \otimes W \oplus B$

EXTENSIONS: HIGHER DIMENSION INPUT

Division $u \in I\!R^m_{max}$, $v \in I\!R^n_{max}$

$$\frac{u}{v} \in I\!R^{n\times m}_{max}, \quad (\frac{u}{v})_{ij} = u_i - v_j$$

T dim J $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$

Extended Waiting Time Variables

$$W_n = \frac{X_n}{T_n} \in I\!R^{K\times J}_{max}$$

Waiting Time Equation $U^+_n = \frac{T_{n+1}}{T_n}$, $U^-_n = \frac{T_n}{T_{n+1}}$

$$W_{n+1} = A_n \otimes W_n \otimes U^-_n \oplus B_{n+1} \otimes U^+_n \otimes U^-_n$$

For extending the ergodic theorem, $\{U^+_n\}$ should be stationary ergodic with $EU^+_{ii} = \lambda^{-1}$

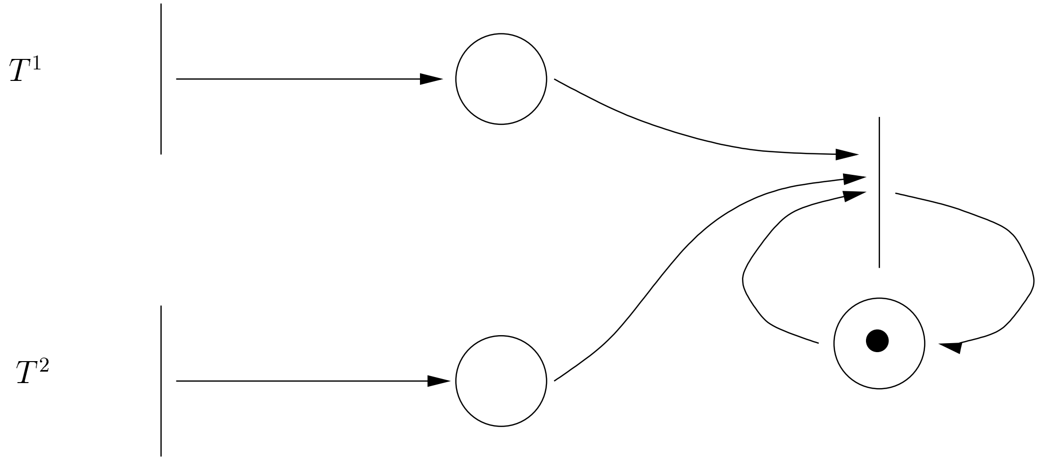


Figure 9: Solution to Harrison's stability problem in assembly lines

2.3.1: CYCLICITY, EIGENVALUE, EIGENVECTOR

A irreducible matrix of $I\!R_{max}^{K,K}$
$\rho \in I\!R$, $X \in I\!R_{max}^{K}$ eigenpair of A if $A \otimes X = \rho \otimes X.$
Main results of spectral theory of matrices

- Unique eigenvalue $\rho = \max_{i=1}^{K} \max_{k=1}^{K} \frac{A_{ii}^{k}}{k}.$
  Critical Circuit: $i_1, \ldots, i_k$ s.t. $A_{i_1 i_2} + \cdots + A_{i_k i_1} = k\rho.$
  Critical Matrix: $\tilde{A}_{ij} = A_{ij}$ if (i,j) belongs to a critical circuit, ε otherwise
- If there is a unique communication class COC 1 in $\tilde{A}$, A has a unique eigenvector
- If the GCD of the circuits in this class is 1, cyclicity of A is 1 CYC 1

$$\exists c \; s.t. \; \forall n \geq c, \;\; A^{n+1} = \rho \otimes A^{n}$$

2.3.1: REPRESENTATION FORMULAS: CLOSED CASE

$\{A_n\}$ stationary and ergodic, fixed support, irreducible.
Projective Space: $u\mathcal{P}v$ in $I\!R^K_{max}$ iff for some $\lambda \in I\!R$ $u^i = v^i + \lambda$
$\pi(u)$ equivalence class of u

Suppose there is an integer N such that within the interior of the support of $A_{N-1} \otimes \cdots \otimes A_0$, there exists an irreducible matrix $\mathcal{A}$ which is COC 1 CYC 1. Then $\pi(X_n)$ converges in total variation to a unique stationary distribution, independently of the initial condition.
J. Mairesse 97, JAP

IDEA OF PROOF & REPRESENTATION FORMULA

Backward coupling

Example: $A_n \in \{A, A(1), \ldots, A(m)\}$ with A COC 1–CYC 1.

Backward Scheme

$$X^{0,0} = X, \quad X^{-1,0} = A_{-1}X, \quad X^{-2,0} = A_{-1}A_{-2}X, \ldots$$

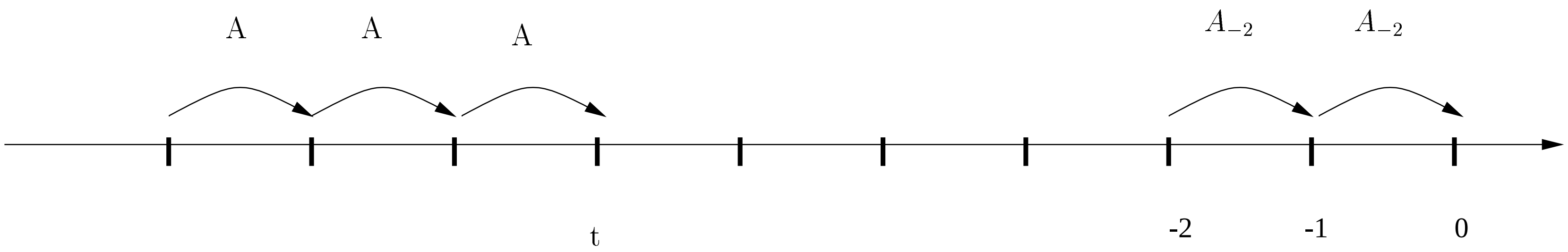


Figure 10: c=3, t: first backward time that A comes 3 times in a row

$$\text{(REP2)} \qquad \pi(X^{-n,0}) \to_{n\to\infty} \pi(X^{-\infty,0}) = \pi(X^{-t,0}).$$

2.3.2: LIGHT TRAFFIC EXPANSIONS: MEAN RESPONSE TIMES

Open event graph with 1-dim Poisson input $W_n^i = X_n^i - T_n$

– $\{T_n\}$ Poisson process of intensity λ

– $\{\sigma_n^k,\ k = 1, \ldots, K\}$ i.i.d. sequence, for all k, $E[(\sigma_n^k)^{m+3}] < \infty$

From (REP1)

$$W = B_0 \oplus \bigoplus_{n \geq 1} C(T_{-n}) \otimes D_n,\ D_n = A_{-1} \otimes \ldots \otimes A_{-n} \otimes B_{-n}$$

$$\Downarrow$$

$$EW^i = \sum_{k=0}^{m} \lambda^k E p_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) \ + \ \mathcal{O}(\lambda^{m+1})$$

If λ small enough, with $p_k(x_0, \ldots, x_{k-1})$ class of combinatorial polynomials

FB & V. Schmidt 96, Annals Appl. Prob.

POLYNOMIALS

$$p_k(x_0,\ldots,x_{k-1}) = \sum_{i=(i_0,\ldots,i_{k-1})\in S_k} (-1)^{q_k(i)} \frac{x_0^{i_0}}{i_0!}\frac{x_1^{i_1}}{i_1!}\cdots\frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!}$$

$$S_k = \{i \in I\!N^k : i_0 + .. + i_{k-1} = k;\ \text{if } i_s = l > 1,\ i_{[s-1]_k} = .. = i_{[s-l+1]_k} = 0\}$$

$$q_k(i_0,i_1,\ldots,i_{k-1}) = 1 + \sum_{s=0}^{k-1} \mathbf{1}(i_s > 0)$$

$$\begin{aligned}
p_1(x_0) &= x_0 \\
p_2(x_0,x_1) &= \frac{1}{2}\left[x_0^2 + x_1^2 - 2x_0x_1\right] \\
p_3(x_0,x_1,x_2) &= \frac{1}{6}\left[x_0^3 + x_1^3 + x_2^3 - 3(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) + 6x_0x_1x_2\right] \\
p_4(x_0..x_3) &= \frac{1}{24}\left[x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4(x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0)\right. \\
&\quad -6(x_0^2x_2^2 + x_1^2x_3^2) + 12(x_0^2x_1x_2 + x_1^2x_2x_3 + x_2^2x_3x_0 + x_3^2x_0x_1) \\
&\quad \left. -24x_0x_1x_2x_3\right]
\end{aligned}$$

POLYNOMIALS – FIRST PROPERTIES

1. special values For all $k \geq 2$

$$p_k(0, \ldots, 0) = 0, \qquad p_k(1, \ldots, k) = \frac{1}{2}.$$

2. invariance by circular permutation: $k \geq 1$

$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_1, x_2 \ldots, x_{k-1}, x_0)$$

3. translation-invariance for all $t \in I\!R, k \geq 2$,

$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_0 + t, x_1 + t, \ldots, x_{k-1} + t)$$

4. multiplicative-invariance for all $t \in I\!R$,

$$p_k(tx_0, tx_1, \ldots, tx_{k-1}) = t^k p_k(x_0, x_1 \ldots, x_{k-1}).$$

5. differences

$$p_k(x_0, x_1, \ldots, x_{k-1}) - p_k(0, x_1, \ldots, x_{k-1})$$

$$= \frac{x_0^k}{k!} - \sum_{j=1}^{k-1} \frac{x_0^{k-j}}{(k-j)!} \left\{ p_j(x_1, x_2, \ldots, x_j) - p_j(0, x_1, \ldots, x_{j-1}) \right\}$$

$$p_{k+1}(x_0, \ldots, x_{k-1}, x_k) - p_k(x_0, \ldots, x_{k-1}, 0)$$

$$= \frac{x_k^{k+1}}{(k+1)!} - \sum_{j=1}^{k} \frac{(x_{k-j})^{k+1-j}}{(k+1-j)!} \{p_j(x_{k+1-j} .. x_k) - p_j(x_{k+1-j} .. x_{k-1}, 0)\}$$

POLYNOMIALS – COMBINATORIAL PROPERTIES

FB, S. Hasenfuss & V. Schmidt 98, Stoch. Models

$$p_1(x_0) = x_0$$

and Integral recurrence

$$p_{k+1}(x_0,\ldots,x_k) = \sum_{n=0}^{k-1} \int_{x_n-x_0}^{x_{n+1}-x_0} \left[p_k(\underbrace{x_0,\ldots,x_0}_{n}, x_{n+1}-u,\ldots,x_k-u) \right.$$
$$\left. -p_k(\underbrace{x_0,\ldots,x_0}_{n+1}, x_{n+1}-u,\ldots,x_{k-1}-u) \right] du.$$

Direct recurrence

$$p_{k+1}(x_0,x_1,\ldots,x_k) = \frac{1}{k+1} \sum_{m=0}^{k} \{x_m - x_{(m-1 \bmod k+1)}\} p_k(x_{([m]_{k+1})},\ldots,x_{([m+k-1]_{k+1})})$$

EXAMPLE: SINGLE SERVER QUEUE

Lindley's equation $W_{n+1} = (\sigma_n \otimes (-\tau_n) \otimes W_n) \oplus e$

$D_0 = 0$ and $D_k = \Sigma_{n=1}^{k} \sigma_{-n}$

$$p_2(D_0, D_1) = \frac{\sigma_{-1}^2}{2}, p_3(D_0..D_2) = \frac{\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2}{6}$$

$$\begin{aligned} p_4(D_0, \ldots, D_3) = \frac{1}{24}(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 \\ +4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3)) \\ -6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) \\ +12\left(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2\right) \end{aligned}$$

Assume $\{\sigma_n\}$ i.i.d.

$$Ep_2(D_0, D_1) = \frac{E[\sigma_n^2]}{2}, \quad Ep_3(D_0, D_1, D_2) = \frac{E[\sigma_n^2]E[\sigma_n]}{2}$$

$$Ep_4(D_0, D_1, D_2, D_3) = \frac{E[\sigma_n^2]E[\sigma_n]^2}{2}$$

COROLLARIES

Assume moment and stability conditions

1. Pollaczek-Khinchine Formula

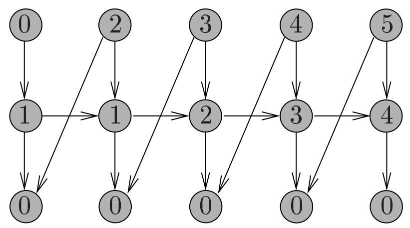
$$EW = \lambda \frac{E[\sigma_n^2]}{2} + \lambda^2 \frac{E[\sigma_n^2]E[\sigma_n]}{2} + \lambda^3 \frac{E[\sigma_n^2]E[\sigma_n]^2}{2} + \mathcal{O}(\lambda^4).$$

$$EW = \frac{\lambda E[\sigma_n^2]}{2(1-\lambda E[\sigma_n])} = \frac{E[\sigma_n^2]}{2} \sum_{k=1}^{\infty} \lambda^k E[\sigma_n]^{k-1}$$

2. Conditional expectation formula $\mathcal{F}_\sigma$: σ–algebra generated by $\{\sigma_n\}$

$$\begin{aligned} E[W|\mathcal{F}_\sigma] &= \frac{\lambda}{2}\sigma_{-1}^2 + \frac{\lambda^2}{6}\left(\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2\right) \\ &+ \frac{\lambda^3}{24}(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 + 4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3) \\ &- 6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) + 12(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2)) + \ldots \end{aligned}$$

EXAMPLE 1: M/D-TYPE PARALLEL PROCESSING MODEL



A task with 15 subtasks
to be mapped on 5 processors

State reduction X_n^i time when first subtask of task n starts on machine i

$$A = \begin{pmatrix} 1 & 2 & \varepsilon & \varepsilon & \varepsilon \\ 2 & 3 & 3 & \varepsilon & \varepsilon \\ 4 & 5 & 5 & 4 & \varepsilon \\ 7 & 8 & 8 & 7 & 5 \\ 11 & 12 & 12 & 11 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Evolution equation for waiting times

$$W_{n+1} = A \otimes C(\tau_n) \otimes W_n \oplus B, \quad W_n^i = X_n^i - T_n$$

NUMERICAL EXPANSION

$$D_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \\ 12 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 5 \\ 8 \\ 12 \\ 17 \\ 21 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 10 \\ 15 \\ 21 \\ 26 \\ 30 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 18 \\ 24 \\ 30 \\ 35 \\ 39 \end{pmatrix}, \quad D_5 = \begin{pmatrix} 26 \\ 33 \\ 39 \\ 44 \\ 48 \end{pmatrix}$$

$D^k_{n+1} = D^k_n + 9$ for all k and for all $n \geq 5$

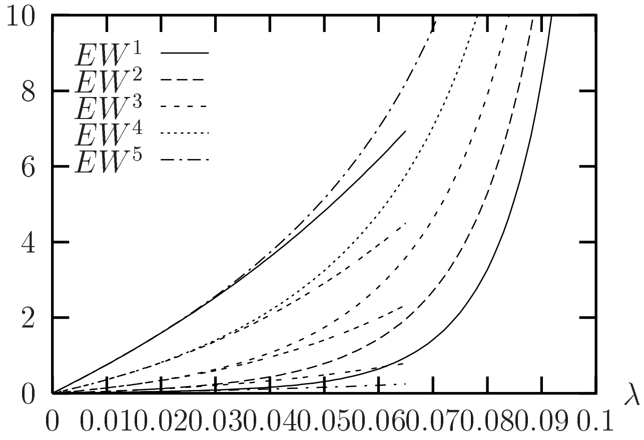


Figure 11: Expected waiting times for a task

M/D TYPE : FORMULA FOR STATIONARY WAITING TIMES

Case D_n ultimately 1-pseudo-periodic (A CYC 1):

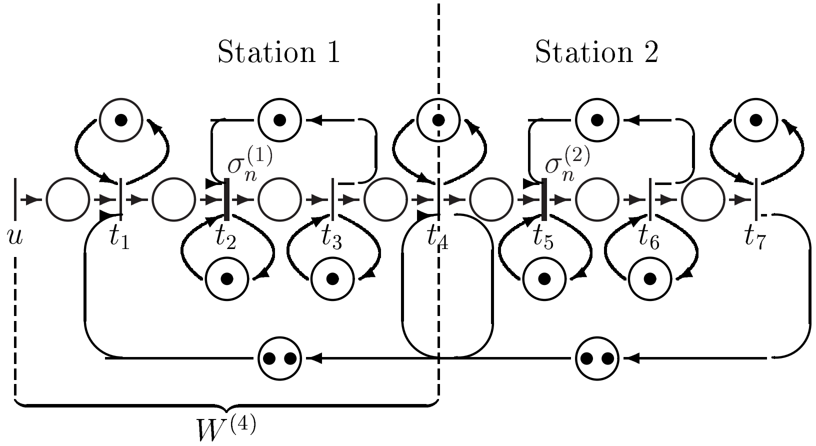
$$D_k = \begin{cases} \eta_k, & \text{for} \quad k = 0, \ldots, \xi - 1, \\ \eta_\xi + (k-\xi)\gamma, & \text{for} \quad k \geq \xi, \end{cases} .$$

$$E\left[W^{\{\xi+1\}}\right] = \frac{\gamma\rho}{2(1-\rho)} + \eta_\xi - \frac{\xi}{\lambda} + \frac{(1-\rho)}{\lambda}\left\{\sum_{m=0}^{\xi-1} e^{-\lambda(\eta_\xi+(m-\xi)\gamma-\eta_m)} + \sum_{m=0}^{\xi-2} e^{-\lambda(\eta_\xi+(m-\xi)\gamma-\eta_m)}\right.$$
$$\left. + \sum_{l=1}^{\xi-1-m} e^{-\rho l}\Big\{p_l(\lambda(\eta_\xi+(l+m-\xi)\gamma), \lambda\eta_{m+1}, \ldots, \lambda\eta_{m+l-1}) - p_l(\lambda\eta_{m+1}, \ldots, \lambda\eta_{m+l})\Big\}\right\}.$$

$$E\left[W^{\{1\}}\right] = \frac{\gamma\rho}{2(1-\rho)} + \eta_0$$

$$E\left[W^{\{3\}}\right] = \frac{\gamma\rho}{2(1-\rho)} + \eta_2 - \frac{2}{\lambda} + \frac{(1-\rho)}{\lambda}$$
$$\left\{ e^{-\lambda(\eta_2-2\gamma-\eta_0)} + e^{-\lambda(\eta_2-\gamma-\eta_1)} + \lambda(\eta_2-\gamma-\eta_1)e^{-\lambda(\eta_2-\gamma-\eta_0)} \right\}$$

EXAMPLE: TWO STAGE KANBAN



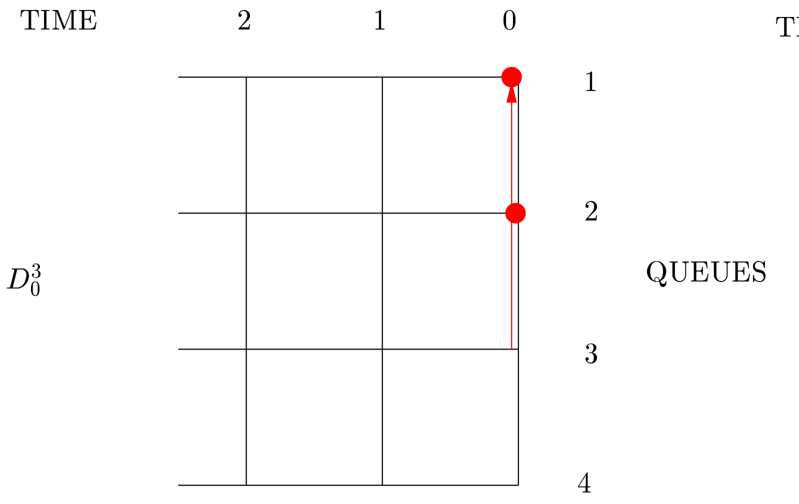
If $\sigma_n^{(1)} = 3,\ \sigma_n^{(2)} = 5$: $D_0^4 = 3,\ D_1^4 = 6,\ D_2^4 = 9,\ D_n^4 = 13 + 5\,(n-3),\ n \geq 3$

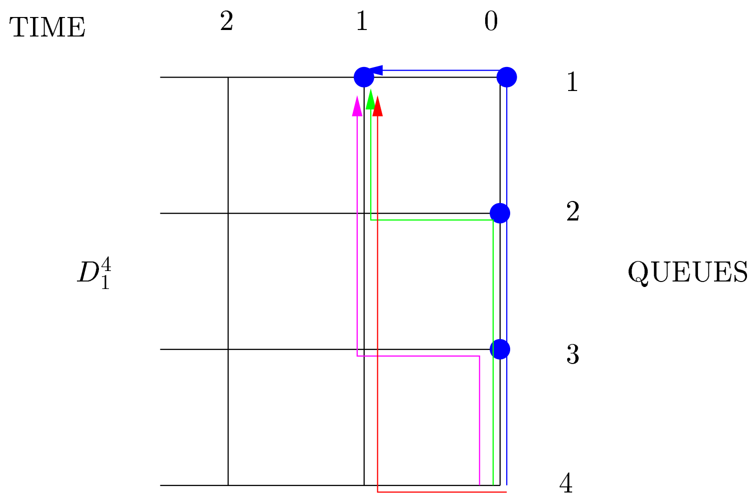
For $\lambda \in [0, 0.2)$

$$E\left[W^4\right] = \frac{25\,\lambda}{2 - 10\,\lambda} + 13 - \frac{3}{\lambda} + \frac{(1 - 5\,\lambda)\left(e^{5\,\lambda} + e^{3\,\lambda} + e^{\lambda} - 3\,\lambda - \lambda\,e^{-2\,\lambda} - \frac{5}{2}\,\lambda^2\,e^{-5\,\lambda}\right)}{\lambda}$$

EXAMPLE M/GI-TYPE: QUEUES IN TANDEM: GENERAL CASE

$$D_n^i = \max_{1\le l_n\le\ldots\le l_1\le i}\left\{\sum_{k=l_1}^{i-1}\sigma_0^k + \sum_{k=l_2}^{l_1}\sigma_{-1}^k + \ldots + \sum_{k=1}^{l_n}\sigma_{-n}^k\right\}$$





EXAMPLE M/GI-TYPE

Case with identical random service times on all nodes

$$
\begin{aligned}
D_0^3 &= 2\sigma(0), \quad D_1^3 = \sigma(1) + 2\max\{\sigma(0), \sigma(1)\} \\
D_2^3 &= \sigma(1) + \sigma(2) + 2\max\{\sigma(0), \sigma(1), \sigma(2)\},
\end{aligned}
$$

$$
\begin{aligned}
E[W^3] \;=\;& 2E[\sigma(0)] + \frac{\lambda}{2} E[(2(\sigma(1) - \sigma(0))^+ + \sigma(1))^2] \\
+\;& \frac{\lambda^2}{6} \Big\{ E\Big[\Big(2(\sigma(1) - \sigma(0))^+ + \sigma(1)\Big)^3 \\
& \quad + \Big(2\max(\sigma(1) - \sigma(0), \sigma(2) - \sigma(0), 0) + \sigma(1) + \sigma(2)\Big)^3 \\
& \quad - 3\Big(2(\sigma(1) - \sigma(0))^+ + \sigma(1)\Big)^2 \\
& \qquad \times \Big(2\max(\sigma(1) - \sigma(0), \sigma(2) - \sigma(0), 0) + \sigma(1) + \sigma(2)\Big)\Big] \Big\} \\
+\;& \mathcal{O}(\lambda^3)
\end{aligned}
$$

GENERIC EXPANSION THEOREM

Open event graph with 1-dim Poisson input

– $f : I\!R^+ \rightarrow I\!R^+$ be a Riemann-integrable, non-negative function, $f(x) \leq c\ x^\nu$ for all $x \geq 0$. $f(x) = x^p \rightarrow$ moments, $f(x) = e^{-sx} \rightarrow$ Laplace Transform ...

– $F^{[0]}(x) = f(x)$,

$$F^{[n]}(x) = \int F^{[n-1]}(x)\, dx.$$

– $q_1(x_0) = f(x_0)$, and

$$q_{k+1}(x_0, x_1, \ldots, x_k) = \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} F^{[k]}(x_n) - \sum_{n=0}^{k-1} \sum_{j=n}^{k-1} \binom{j}{n} (-1)^{j-n} F^{[j]}(x_n)$$
$$\{p_{k-j}(x_{n+1}, \ldots, x_{k-j+n}) - p_{k-j}(x_n, \ldots, x_{k-j+n-1})\}$$

EXPANSION THEOREM

$$G_n = \bigoplus_i \left\{ (A_{-1} \otimes A_{-2} \otimes \ldots \otimes A_{-n} \otimes (B_{-n} \oplus O))^i \right\}$$

$$H_{n,\ell} = \bigoplus_i \left\{ (A_{-(n\ell+1)} \otimes A_{-(n\ell+2)} \otimes \ldots \otimes A_{-(n+1)\ell} \otimes (B_{-(n+1)\ell} \oplus O))^i \right\}$$

1. If the $(\nu + m + 2)$-th moment of H_n is finite,

$$E[f(W^i)] = \sum_{k=0}^{m} \lambda^k \, E[\, q_{k+1}(D_0^i, D_1^i, \ldots, D_k^i)\,] + \mathcal{O}(\lambda^{m+1}).$$

2. If $\exists\ \theta \in [0, \theta^*),\ E[\, e^{\theta G_n}] \leq L_\theta \, (\phi(\theta))^n$, for finite functions $\phi(\theta) > 1$ and L_θ, then, $E[f(W^i)]$ is infinitely differentiable in 0^+, the expansion holds for all m, and

$$\lim_{\lambda \downarrow 0} \frac{d^n}{d\lambda^n} E[f(W^i)](\lambda) = E[\, q_{n+1}(D_0^i, D_1^i, \ldots, D_n^i)\,]$$

FB, S. Hasenfuss & V. Schmidt 97, Questa

APPLICATION TO LAPLACE TRANSFORMS

$$E[e^{-sW^i}] = \sum_{k=0}^{m} \lambda^k \, E[\, q_{k+1}(s; D_0^i, D_1^i, \ldots, D_k^i) \,] + \mathcal{O}(\lambda^{m+1}),$$

where $q_1(s; x_0) = e^{-sx_0}$,

$$\begin{aligned} & q_{k+1}(s; x_0, x_1, \ldots, x_k) \\ = \; & \frac{1}{s^k} \sum_{n=0}^{k} \binom{k}{n} (-1)^n e^{-sx_n} - \sum_{j=0}^{k-1} \frac{1}{s^j} \sum_{n=0}^{j} \binom{j}{n} (-1)^n e^{-sx_n} \\ & \{p_{k-j}(x_{n+1}, \ldots, x_{k-j+n}) - p_{k-j}(x_n, \ldots, x_{k-j+n-1})\} \\ = \; & \frac{q_k(s; x_0, x_1, \ldots, x_{k-1}) - q_k(s; x_1, x_2, \ldots, x_k)}{s} \\ & - e^{-sx_0}[p_k(x_1, x_2, \ldots, x_k) - p_k(x_0, x_1, \ldots, x_{k-1})], \end{aligned}$$

p_k polynomials defined before

APPLICATION: SECOND MOMENT, VARIANCE

$$E(W^i)^2 = \sum_{k=0}^{m} \lambda^k \, E q_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) + \mathcal{O}(\lambda^{m+1}),$$

with

$$\begin{aligned} q_1(x_0) &= x_0^2 \\ q_2(x_0, x_1) &= \frac{1}{3}\left(2x_0^3 + x_1^3 - 3x_0^2 x_1\right) \\ q_3(x_0, x_1, x2) &= \frac{1}{12}\left(3x_0^4 + 2x_1^4 + x_2^4 - 8x_0^3 x_1\right. \\ &\quad \left. -4x_1^3 x_2 - 6x_0^2 x_2^2 + 12 x_0^2 x_1 x_2\right). \end{aligned}$$

$$\mathrm{Var}\left(W^i\right) = \sum_{k=0}^{m-1} \lambda^k \, q_{k+1}^\star(D_0^i, D_1^i, \ldots, D_k^i) + \mathcal{O}(\lambda^m),$$

with

$$q_{k+1}^\star(\ldots) = E\, q_{k+1}(\ldots) - \sum_{j=0}^{k} E\, p_{j+1}(\ldots)\, E\, p_{k+1-j}(\ldots)$$

IDEA OF PROOF FOR EXPANSIONS

Reiman & Simon light traffic formulas for functionals of Poisson PP

– $D = \{D_n\}$, $I\!R^K$-valued random variables

– $T = \{T_n\}$, Poisson point process of intensity λ

Under some regularity conditions, $f(\lambda) = E\,\psi(T, D)$ is infinitely differentiable in λ in 0^+ and

$$\lim_{\lambda \downarrow 0} \frac{d^k}{d\lambda^k} E\,\psi(T, D) = E \int_{-\infty}^{0} \ldots \int_{-\infty}^{0} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell}\, \psi\Big(\sum_{j=1}^{\ell} \delta_{s_j}, D\Big)\, ds_1 \ldots ds_k$$

Take

$$\psi(T, D) = W(T, D) \quad = \quad \Big(D_0 \vee \bigvee_{n \geq 1} (D_n + T_{-n})\Big).$$

and you find the polynomials (thanks to Maple).

IDEA OF PROOF FOR EXPANSIONS: REGULARITY

Finite Horizon: T^t, restriction of T on $[t, 0)$

$$E\,\psi(T^t, D) \;=\; e^{\lambda t} \sum_{k=0}^{\infty} \frac{(-\lambda t)^k}{k!}\, \beta_k(t),$$

where

$$\beta_k(t) \;=\; \frac{1}{(-t)^k} \int_t^0 \ldots \int_t^0 \; E\,\psi(\sum_{j=1}^{k} \delta_{u_j}, D)\, du_1 \ldots du_k.$$

Interchange of limits in this series

2.3.2: LIGHT TRAFFIC EXPANSIONS: LYAPUNOV EXPONENTS

The simplest possible case: $\{A_n\}$ i.i.d. $A_n \in \{A, A'\}$, $P(A_n = A) = p$

– A COC 1, CYC 1, irreducible, with eigenvector X_o and coupling time c;

– $\gamma(p)$: top and bottom Lyapunov exponent of $\{A_n\}$.

From (REP2), $\gamma(p)$ analytic in p in 0, with radius of convergence at least $\frac{1}{2c}$

$$\frac{1}{l!}\frac{d^l}{dp^l}\left[\Gamma(p)\right]_{p=0} = \begin{bmatrix} \pi(l) \\ \vdots \\ \pi(l) \end{bmatrix} = (-1)^l \left\{ \binom{c}{l-1} X_o + \binom{c+1}{l} \Gamma(0) \right\} + \sum_{k=1}^{l}$$

$$\left\{ (-1)^{l-k} \sum_{j_1,\ldots,j_{k-1}=0}^{c-1} \binom{2c+j_1+..+j_{k-1}}{l-k} A^c \otimes A' \otimes A^{j_1} \otimes A' \cdots A^{j_{k-1}} \otimes A' \otimes X_o \right\}.$$

For $p < \delta < \frac{1}{2c}$, error term in Taylor expansion decreases like $(2c\delta)^n$

FB & D. Hong 98 $\rightarrow$ Annals Appl. Prob.

FIRST COEFFICIENTS

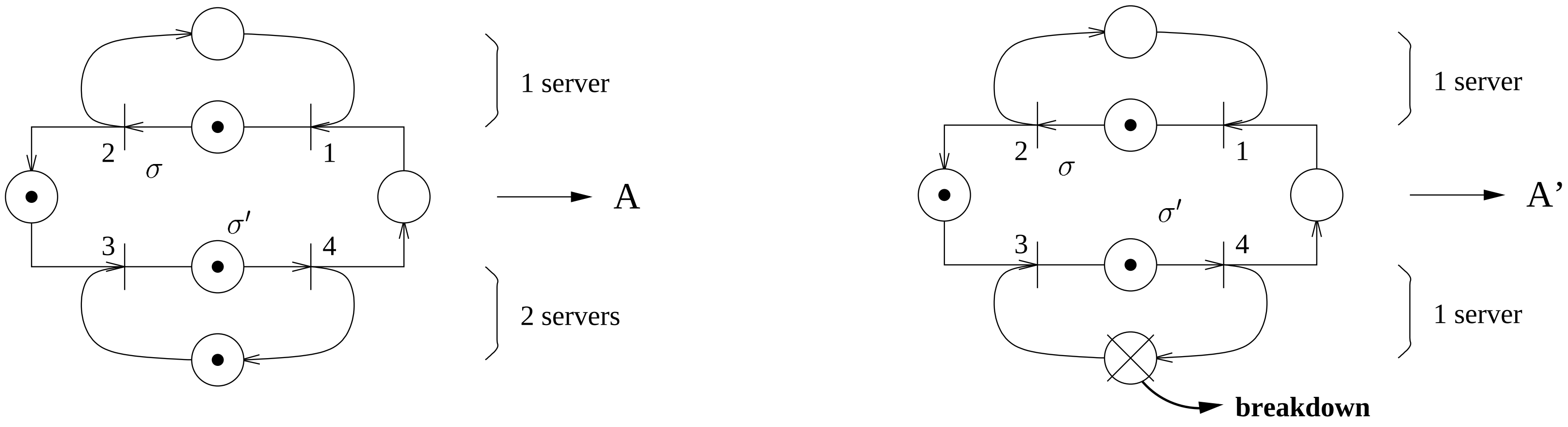
$$\begin{bmatrix} \pi(0) \\ \vdots \\ \pi(0) \end{bmatrix} = \begin{bmatrix} \gamma(0) \\ \vdots \\ \gamma(0) \end{bmatrix} = \Gamma(0),$$

$$\begin{bmatrix} \pi(1) \\ \vdots \\ \pi(1) \end{bmatrix} = A^c \otimes A' \otimes X_o - X_o - (c+1)\Gamma(0),$$

$$\begin{bmatrix} \pi(2) \\ \vdots \\ \pi(2) \end{bmatrix} = \sum_{j=0}^{c-1} \{A^c \otimes A' \otimes A^j \otimes A' \otimes X_o\}$$

$$-2cA^c \otimes A' \otimes X_o + cX_o + \binom{c+1}{2} \Gamma(0).$$

EXAMPLE: NETWORK WITH BREAKDOWNS



$$A = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}, \quad A' = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \sigma' & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$

EXPANSION

1. For $\sigma' < 2\sigma$, there are two regimes :

   – if $2\sigma' < 3\sigma$, $\gamma(p) = \sigma$

   – if $2\sigma' > 3\sigma$:

$$\begin{aligned}\gamma(p) \;=\; & \sigma + (2\sigma' - 3\sigma)p + (5\sigma - 3\sigma')p^2 + (4\sigma' - 6\sigma)p^3 + (7\sigma - 5\sigma')p^4 \\ & + (9\sigma' - 13\sigma)p^5 + (26\sigma - 17\sigma')p^6 + \cdots\end{aligned}$$

2. For $\sigma' > 2\sigma$, A becomes 2-periodic.

2.3.2: CURRENT DIRECTIONS OF RESEARCH ON LIGHT TRAFFIC EXPANSIONS

1. Region of analyticity of expansions
   1.1. less than stability region A. Jean-Marie 97, INRIA Report
   1.2. via contraction FB & D. Hong 99, $\rightarrow$ JAP
   1.3. via weak derivatives B. Heidergott 99, INFORMS
   1.4. via rational languages S. Gaubert & D. Hong 99, INFORMS
2. Further functionals or variables
   2.1. Transient FB, S. Hasenfuss & V. Schmidt 97, Questa and B. Heidergott 99
   2.2. Joint Laplace transforms of two waiting times H. Ahyan & FB 99, INFORMS
   2.3. Complexity V. Blondel, S. Gaubert, J. Tsitisklis 99

2.3.3: SUBEXPONENTIAL ASYMPTOTIC TAIL BEHAVIOR

1-D Problem: $GI/GI/1/\infty$ Queue

J.W. Cohen 73, A.G. Pakes 75, N. Veraverbeke 77

– $\{\tau_n\}$ i.i.d., integrable

– $\{\sigma_n\}$ i.i.d., integrable, such that $F_\sigma(x) = P(\sigma < x)$ and

$$F_\sigma^s(x) = \frac{\int_0^x 1 - F_\sigma(u)du}{\int_0^\infty 1 - F_\sigma(u)du}$$

both subexponential: $\mathcal{S}^*$ (e.g. Pareto, Log Normal, Weibull)

$$W_{n+1} = (W_n + \sigma_n - \tau_n)^+ \rightarrow_{n\to\infty} W$$

If $\rho = \frac{E(\sigma)}{E(\tau)} < 1$

$$P(W > x) \sim \frac{\rho}{1-\rho} F_\sigma^s(x)$$

2.3.3: SUBEXPONENTIAL ASYMPTOTIC TAIL BEHAVIOR

General Dimension

Technical Assumptions $W_{n+1} = A_n \otimes C(-\tau_n) \otimes W_n \oplus B_{n+1}$

– A_n, B_n i.i.d., fixed Support, D_n non decreasing, stability;

– $\max_i(A_{-1} \otimes \cdots \otimes A_{-n} \otimes B_{-n})_i \leq \max_{ij}(A_{-1} \otimes \cdots \otimes A_{-n} \otimes A_{-n-1})_{ij}$

– There exists a coordinate i_0 such that

 * $F_{i_o}(x) = P[(A_0)_{i_0 i_0} \leq x] \in \mathcal{S}^*$.

 * $\max_{ij}(A_{-1} \otimes \cdots \otimes A_{-n})_{ij} \leq \Sigma_{l=1}^{n}(A_{-l})_{i_0 i_0} + H_n$
 with

  · H_n independent of $\Sigma_{l=1}^{n}(A_{-l})_{i_0 i_0}$

  · $P[H_n > x] = o(1 - F_{i_0}(x))$

2.3.3: MAIN RESULT – IRREDUCIBLE CASE

Under the technical assumptions, $W_n \to W$ in total variation and

$$P(W > x) \sim \frac{\lambda \mu_{i_0}}{1 - \lambda\gamma} F^s_{i_0}(x)$$

with

– $\lambda = E(\tau)^{-1}$

– γ Lyapunov exponent of $\{A_n\}$

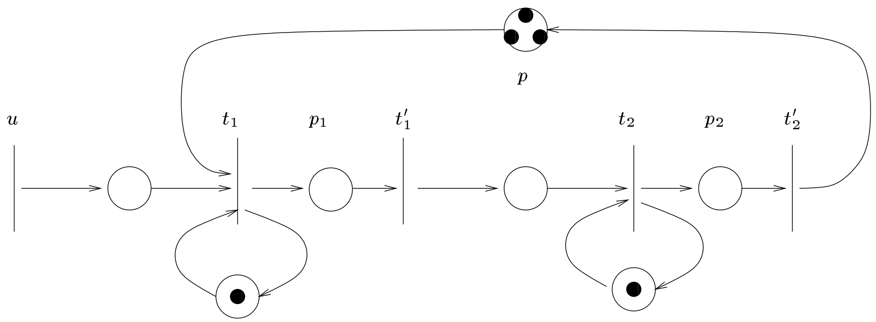
– μ_{i_0} mean value of F_{i_0}.

– $F^s_{i_0}$ excess distribution of F_{i_0}.

FB, S. Schlegel & V. Schmidt 99, Questa

2.3.3: EXAMPLE: WINDOW FLOW CONTROL

Two Routers, Window equal to K, σ_n^2 subexponential



For appropriate state variables (after state extension and reduction)

$$A_{n-1} \in I\!R^{K+1,K+1} = \begin{pmatrix} \sigma_n^1 & \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 & \sigma_n^2 & \varepsilon & \cdots & \varepsilon & \sigma_n^1 + \sigma_n^2 \\ \varepsilon & 0 & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & & \vdots & \varepsilon \\ \vdots & \vdots & & \ddots & \vdots & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & 0 & \varepsilon \end{pmatrix}, \quad B_n = \begin{pmatrix} \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix}$$

IDEA OF PROOF

Two key sequences Fix i, with $B_n^i \geq 0$, fix $p \geq q$

- $g_{p,q} = \max_i (A_{p-1} \otimes \cdots \otimes A_{q-1})_{ii}$: super additive;
- $G_{p,q} = \max_{jj'} (A_{p-1} \otimes \cdots \otimes A_{q-1})_{jj'}$: sub additive;

From ergodic theorems, both satisfy a SLLN with slope γ.

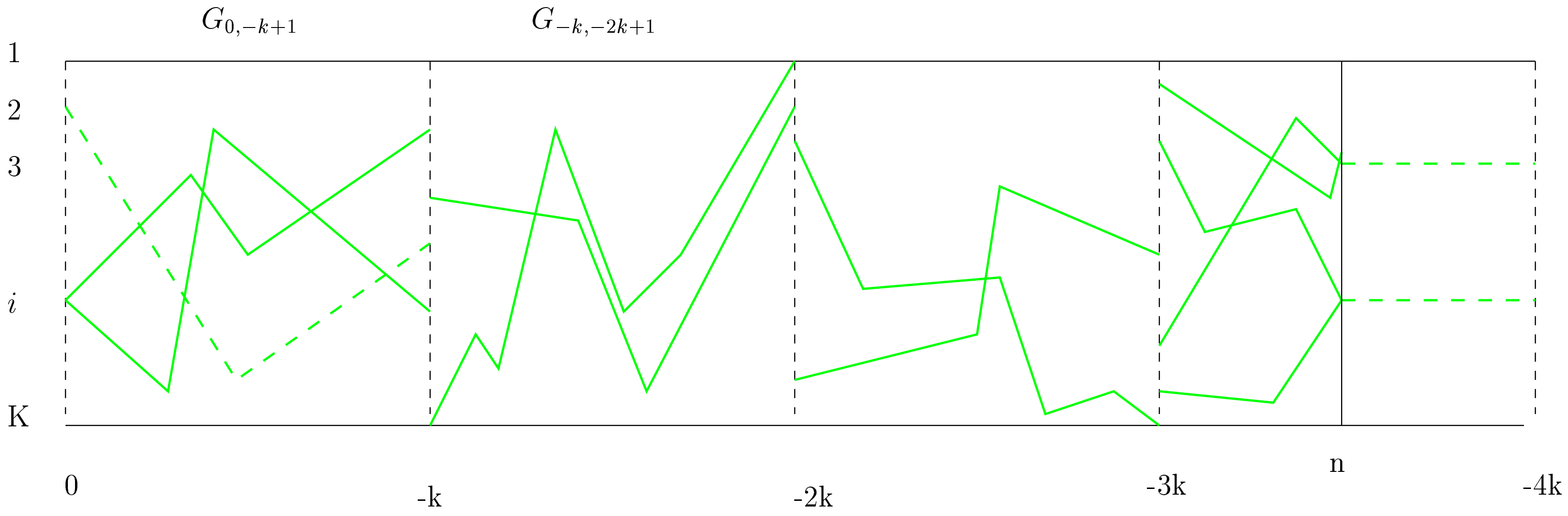
Upper Bound:

From Assumptions $\quad D_n^i \leq \Sigma_{l=0}^{\lfloor \frac{n+1}{k} \rfloor} G_{-lk,-(l+1)k+1}$

From (REP1) $\quad W^i \leq \sup_{p \geq 0} \Sigma_{l=0}^{p} G_{-lk,-(l+1)k+1} + T_{-pk}$

Max of a random walk again!

$$\limsup \frac{P[W^i > x]}{\overline{G}_k^s(x)} \leq \frac{\frac{\lambda}{k} E G_{1,k}}{1 - \frac{\lambda}{k} E G_{1,k}}, \qquad \overline{G}_k^s(x) \sim \frac{k \mu_{i_0}}{E G_{1,k}} \overline{F}_{i_0}^s(x)$$



$G_{0,-k+1}$
$G_{-k,-2k+1}$
1
2
3
i
K
0
-k
-2k
-3k
n
-4k

2.3.3: TAILS, LARGE DEVIATIONS

1. Subexponential Tails
   1.1. non irreducible case and several subexponential distributions FB, S. Schlegel & V. Schmidt 99, Questa
   1.2. via Harris recurrence T. Huang & K. Sigman 99, Questa for tandem queues
2. Cramer Tails
   2.1. Large buffers in tandem queues Ganesh & Anantharam 96, Questa
   2.2. LDP for products of (max,plus) random matrices F. Toomey 97, PhD Dissertation

2.4: GENERAL (MAX,PLUS) NETWORKS

1. Beyond Fixed Support
   1.1. Tetris models Gaubert & Mairesse 98, Idempotency
2. Inf Convolutions
   2.1. Deterministic service guarantees C.S. Chang 97, Infocom; R. Agrawal & R. Rajan 96, Allerton; C.S. Chang 98, Preprint
   2.2. Stochastic case R. Agrawal & FB 98, INRIA report
3. Infinite Dimensional Systems
   3.1. Relations with literature on percolation
   3.2. Tandem case Glynn and Whitt 91 Annals Appl. Prob.; FB, A. Borovkov & J. Mairesse 99

2.4: DETERMINISTIC SETTING: TRAFFIC REGULATION

Deterministic Bounds in Stochastic Networks

Yet another semi-ring: Inf Convolutions

- F set of non-decreasing functions $\mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$
- $a \oplus b(t) = \min(a(t), b(t))$
- $a \otimes b(t) = \inf_{0 \leq u \leq t} a(u) + b(t-u)$ with neutral e

Regulation

- a process to be regulated
- b enveloppe process (e.g. $b(t) = \sigma + \rho t$)

What is the largest process c such that

$$c(t) \leq a(t), \qquad c(t) - c(u) \leq b(t-u), \ \forall u \leq t?$$

Answer: $c \leq a, \quad b \leq c \otimes b \quad \Rightarrow c \leq a \oplus c \otimes b$

$$c_0 = a \otimes b^* \qquad \text{with} \quad b^* = e \oplus b \oplus b^2 \oplus \cdots$$

MOTIVATIONS FOR STOCHASTIC FRAMEWORK

– F set of functions $(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^{+} \cup \{\infty\}$: $a(s,t),\ s \le t$

– $a \oplus b(s,t) = \min(a(s,t), b(s,t))$

– $a \otimes b(s,t) = \inf_{s \le u \le t} a(s,u) + b(u,t)$

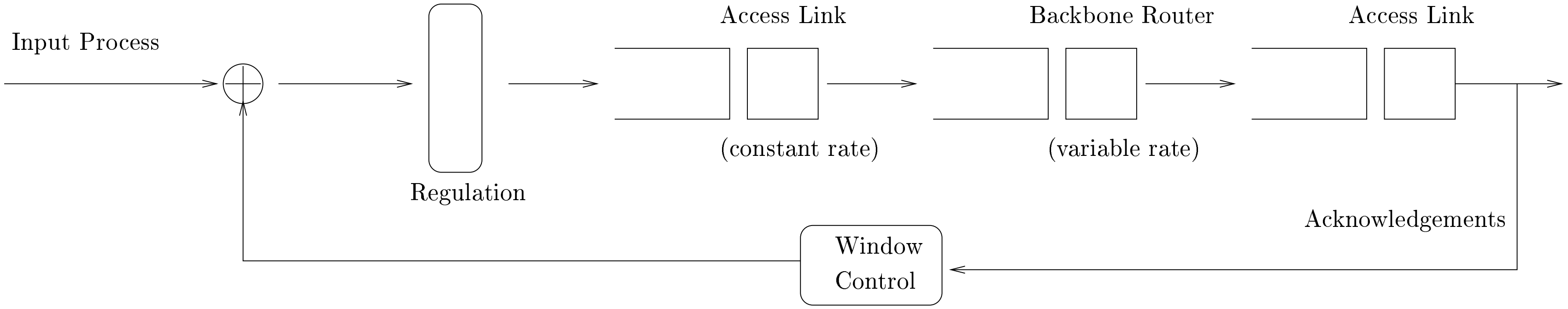
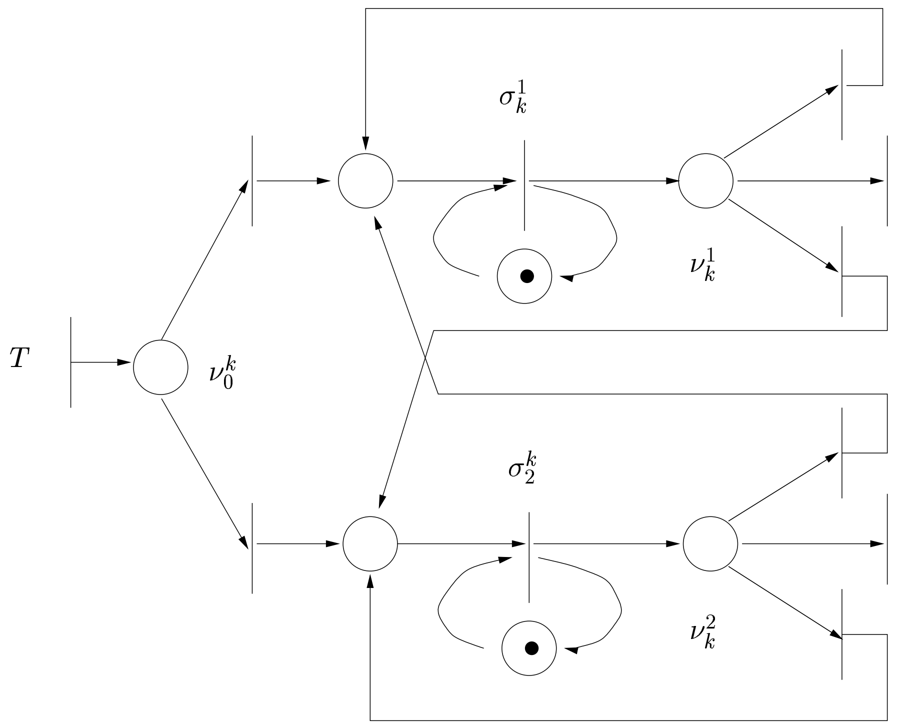


Figure 12: Results in progess: Lyapunov Exponents, Representation Formulas, Tails e.t.c.

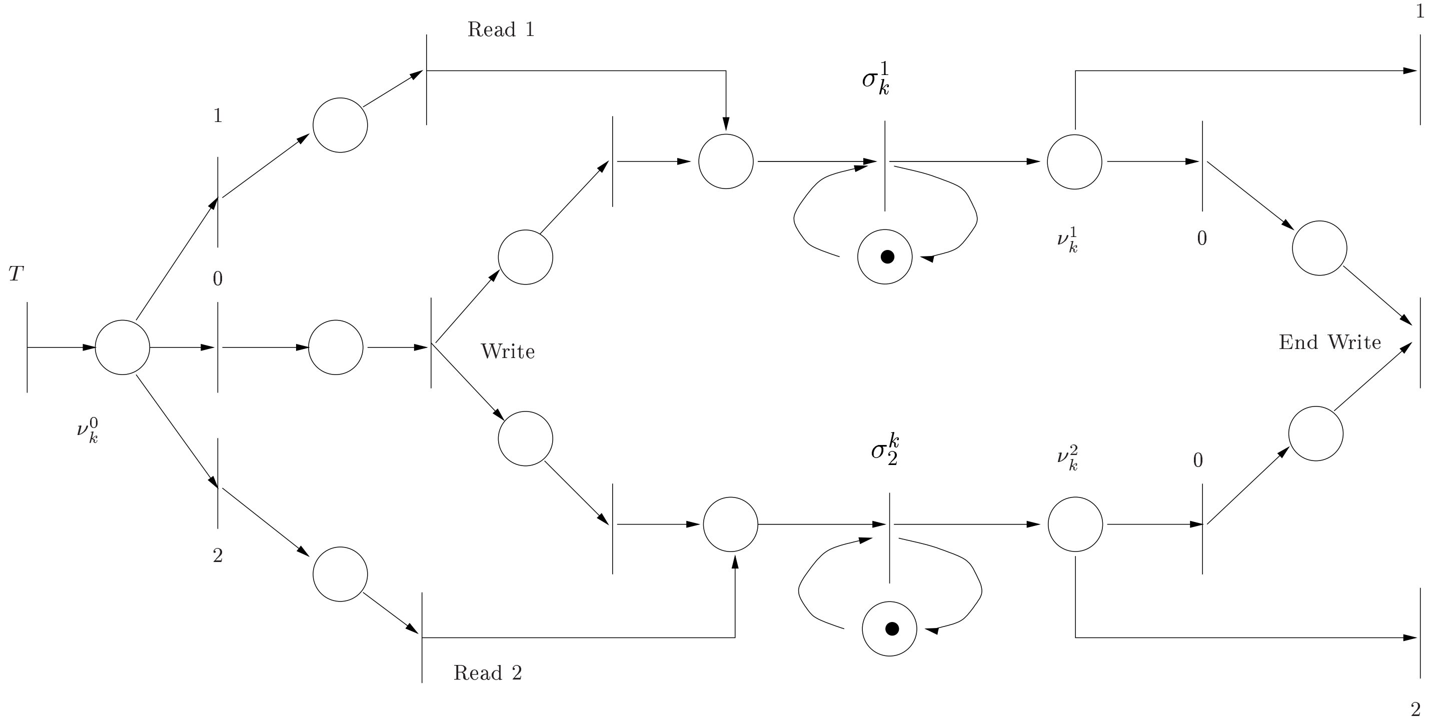
PART3: ROUTING & SYNCHRONIZATION

Example 1: Jackson Network as a Free Choice Net



Data: For all $q \in \mathcal{T}$, $\sigma_n^q \in I\!R$ For all routing place p, $\nu_n^p \in p^\bullet$

3.1: EXAMPLE 2
Read-Write on Replicated Data



3.1: COUNTERS AND DATERS

Counters – Daters

- X_k^q daters of transition q
- Y_t^q # of firings initiated by q before time t:

$$Y_t^q = \sum_{k \geq 1} 1_{X_k^q \leq t}$$

State vectors

$$\begin{aligned} Y_t &= (Y_t^q), \quad q \in \mathcal{S} \qquad \textit{transitions preceded by serial places only} \\ \overline{Y}_t &= (Y_t^q), \quad q \in \overline{\mathcal{S}} = \mathcal{T} - \mathcal{S} \qquad \textit{preceded by a non-serial place} \end{aligned}$$

Routing functions For all $q \in \overline{\mathcal{S}}$: $\mathcal{H}^q : I\!N \to I\!N$

$$\mathcal{H}^q(m) = \sum_{l=1}^{m} 1_{\{\nu_l^{\bullet q} = q\}}$$

3.1: COUNTER EQUATIONS FOR FREE CHOICE NETS

Evolution equations: case with constant integer service times

Reference algebras $(\mathbb{R}^K_{min}, \oplus, \otimes)$: $\oplus = \min$, $\otimes = +$ $\qquad (\mathbb{R}^K, +, \times)$

$$
\begin{aligned}
Y_t &= \bigoplus_{m=1}^{M} \left[A_m \otimes Y_{t-m} \oplus B_m \otimes \overline{Y}_{t-m}\right] \\
\overline{Y}_t &= \mathcal{H}\left(\sum_{m=1}^{M} \left[P_m \times \overline{Y}_{t-m} + Q_m \times Y_{t-m}\right] + R_t\right)
\end{aligned}
$$

- $M \in \mathbb{N}$: maximal firing time
- $\mathcal{H}(Z) = (\mathcal{H}^q(Z^q))$
- R^q_t: # arrivals of T_n in non-serial place ${}^\bullet q$ by time t
- $(A_m)_{ij} = c$ if $j \in {}^{\bullet\bullet}i$, $i, j \in \mathcal{S}$, $\mu(j,i) = c$, $\sigma^j = m$, ε otherw.
- $(P_m)_{ij} = 1$ if $j \in {}^{\bullet\bullet}i$, $i, j \in \overline{\mathcal{S}}$ and $\sigma^j = m$, 0 otherw.

FB, & B. Gaujal 94, LNCIS

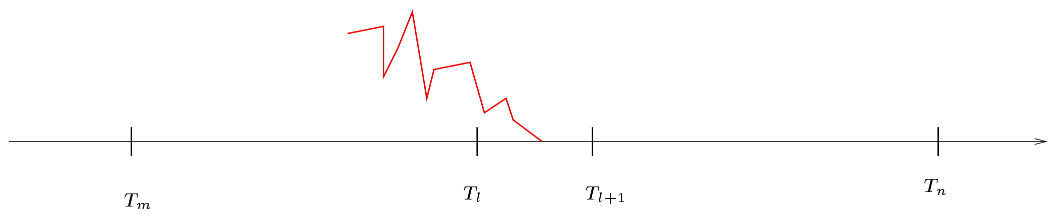
3.2: ERGODIC THEOREM–JACKSON NETWORK

Saturation Law

$N = \{T_n\}$ Exogenous arrival process

$Z_{[m,n]}$ $m \leq n$ time of last activity if the net starts empty at T_m and is fed by the $[m,n]$ restriction of N

1. Monotonicity: $Z_{[m,n]}(N) \geq Z_{[m,n]}(N')$ if $T_p \geq T'_p$ for all p
2. Homogeneity: $Z_{[m,n]}(N + c) = c + Z_{[m,n]}(N)$
3. Separability: $Z_{[m,n]}(N) = Z_{[l+1,n]}(N)$ for $Z_{[m,l]}(N) \leq T_{l+1}$



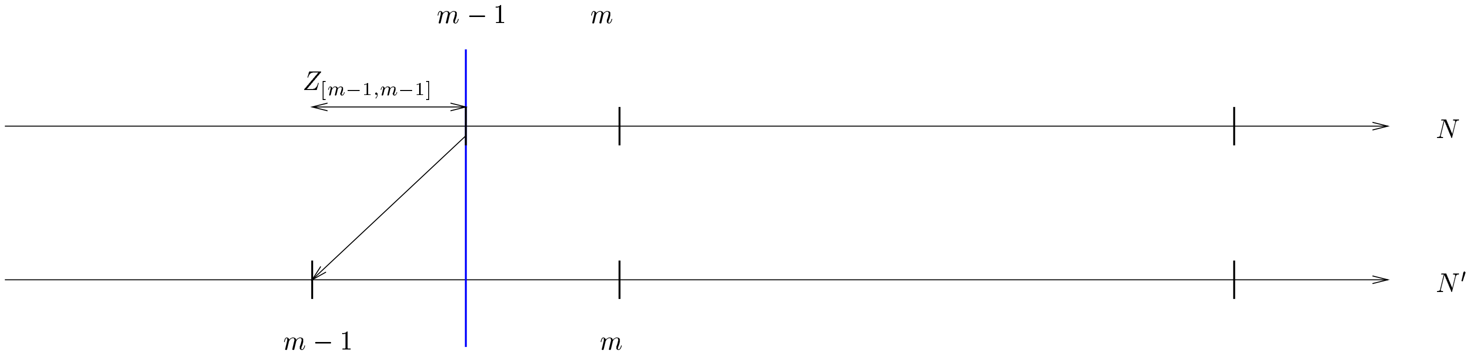
FB & S. Foss 94, Questa

3.2: PATHWISE PROPERTIES–JACKSON NETWORK

Internal Monotonicity

$$W_{[m,n]} = Z_{[m,n]} - T_n$$

$$Z_{[m-1,n]} \geq Z_{[m,n]} \qquad W_{[m-1,n]} \geq W_{[m,n]}$$

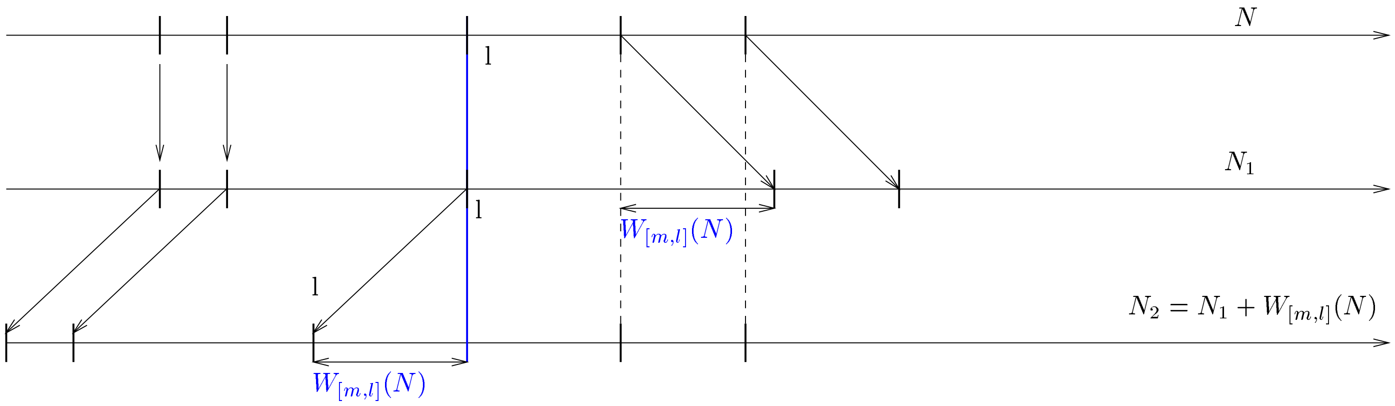


$$\begin{aligned} Z_{[m,n]}(N) &= Z_{[m,n]}(N') \\ Z_{[m,n]}(N') &= Z_{[m-1,n]}(N') \quad (3) \\ Z_{[m-1,n]}(N') &\leq Z_{[m-1,n]}(N) \quad (1) \end{aligned}$$

3.2: PATHWISE PROPERTIES–JACKSON NETWORK

Subadditivity

$$m \leq l < n \qquad W_{[m,n]} \leq W_{[m,l]} + W_{[l+1,n]}$$



$$\begin{aligned} Z_{[m,n]}(N) &\leq Z_{[m,n]}(N_1) \quad (1) \\ &= Z_{[l+1,n]}(N_1) \quad (3) \\ &= W_{[m,l]}(N) + Z_{[l+1,n]}(N_2) \quad (2) \\ &= W_{[m,l]}(N) + Z_{[l+1,n]}(N) \end{aligned}$$

3.2: PATHWISE PROPERTIES–JACKSON NETWORK

Non-Expansiveness

Theorem: each dater X_p^i is Lipschitz continuous w.r.t. T:

$$|X_p^i(N) - X_p^i(N')| \leq \sup_n |T_n - T'_n|$$

Same property holds true w.r.t. σ's

FB & S. Foss 94, Questa

3.2: MAIN ERGODIC THEOREM
Backward Construction

Under appropriate stationary ergodic assumptions on input, routing and service times, for all $c \geq 0$

$$\exists \lim_{n\to\infty} \frac{W_{[-n,0]}(cN)}{n} = \gamma(c) \quad \text{a.s} \quad \text{and in} \quad L_1.$$

Saturation Rule FB & S. Foss 95, JAP

1. If $\lambda\gamma(0) < 1$ then $\lim_{n\to\infty} W_{[-n,0]}(N) < \infty$.
2. If $\lambda\gamma(0) > 1$ then $\lim_{n\to\infty} W_{[-n,0]}(N) = \infty$.

ERGODIC THEOREMS: FREE CHOICE

Assumptions for SLLN in a free choice net

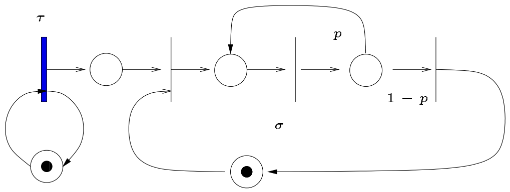
– A1 – Net is live

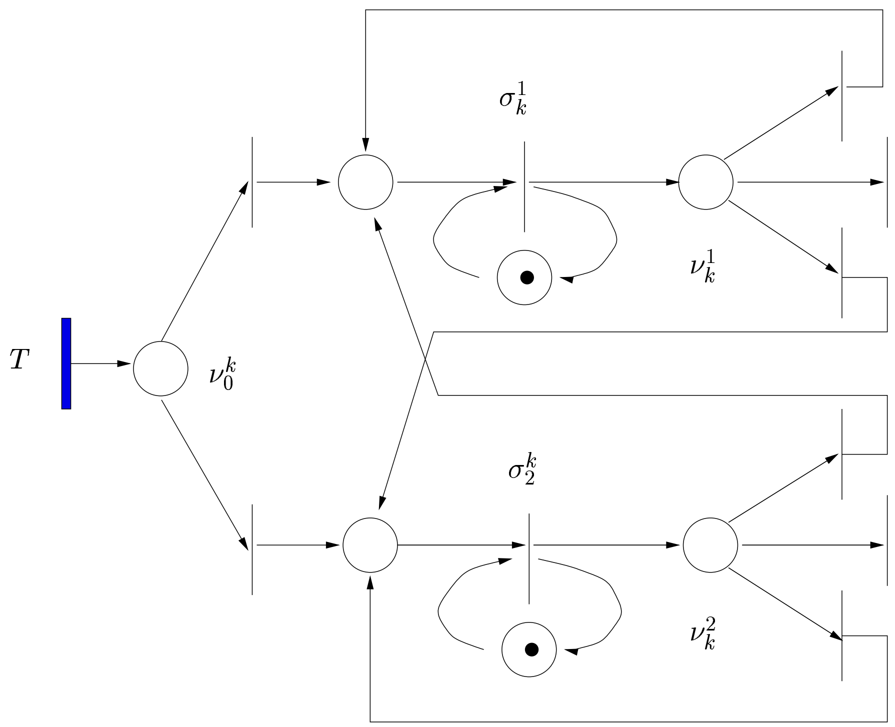
– A2 – Net contains an a.s. prompt transition q_0

– A3 – All deadlocks associated with q_0 are the same reference deadlock

For SI-FCnets (all event graph components have a simple input place) there is a unique deadlock, whatever the routing $\mathcal{H}$-free property

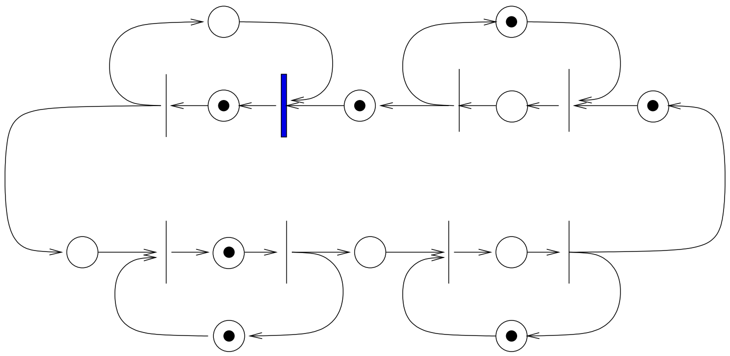
EXAMPLES – $\mathcal{H}$-FREE, OPEN

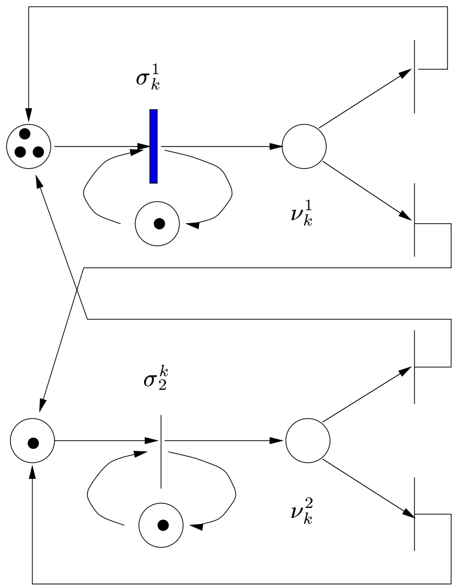




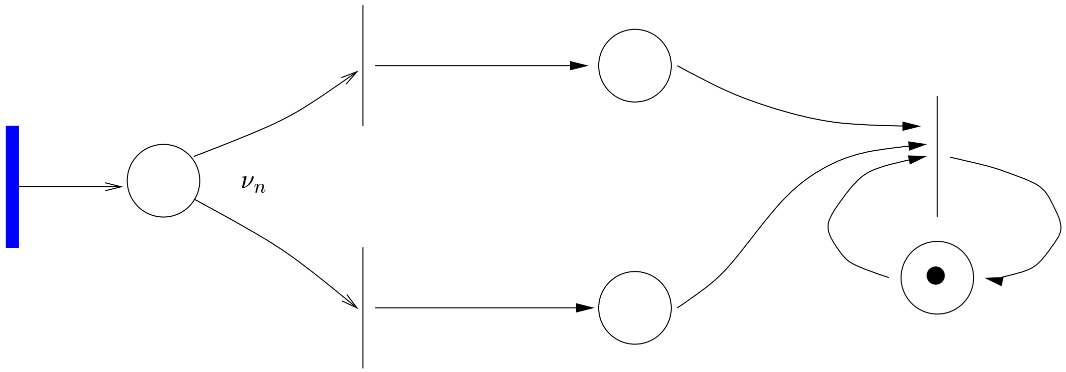
σ_k^1
ν_k^1
T
ν_0^k
σ_2^k
ν_k^2

EXAMPLES – $\mathcal{H}$-FREE, CLOSED





EXAMPLE – $\mathcal{H}$-BOUND



3.2: JACKSON & FREE CHOICE: FURTHER RESULTS

– Deterministic case with fluid routing, relations with stochastic control equations G. Cohen, S. Gaubert, J.P. Quadrat 95, CDC

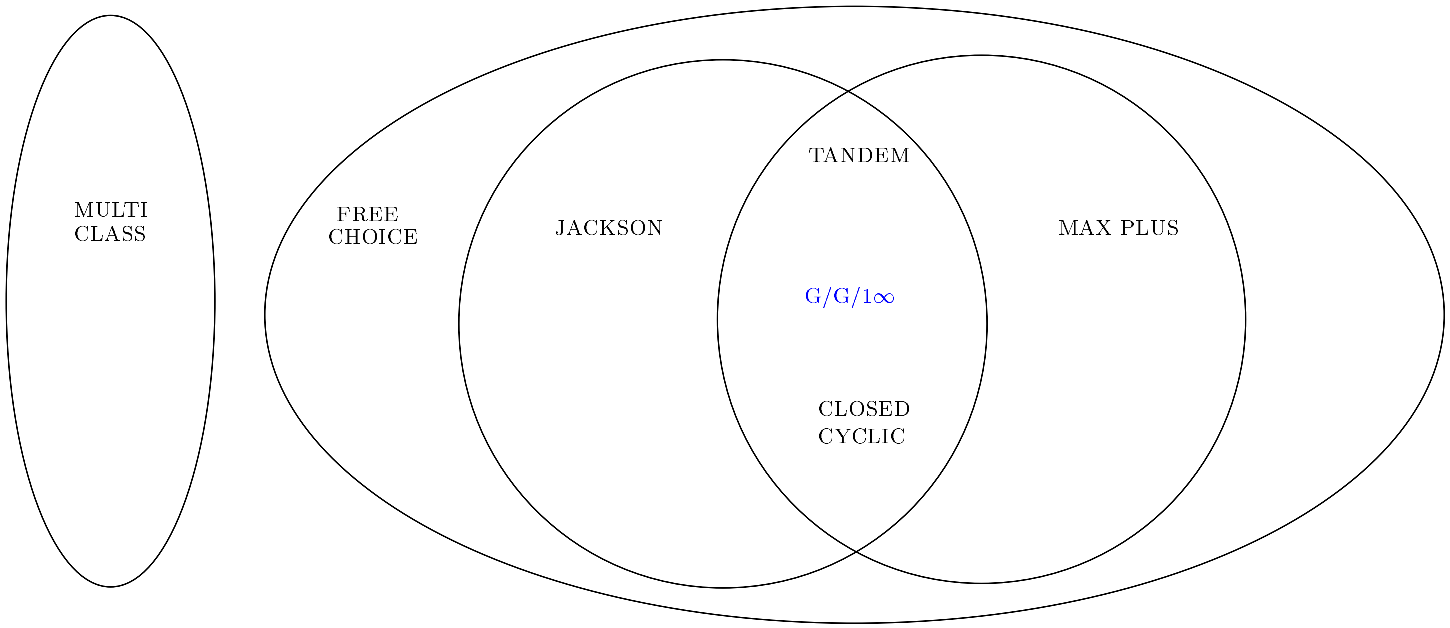
– Computation of γ_0 FB, S. Foss & B. Gaujal 95, IEEE Aut. Cont.

– Representation of stationary workload FB, S. Foss, J. Mairesse 96, Stochastic Networks

– Multiple regimes

– Logical properties

PART 4: BEYOND FREE CHOICE



MULTICLASS AND SYNCHRONIZATION

Window flow control & cross traffic

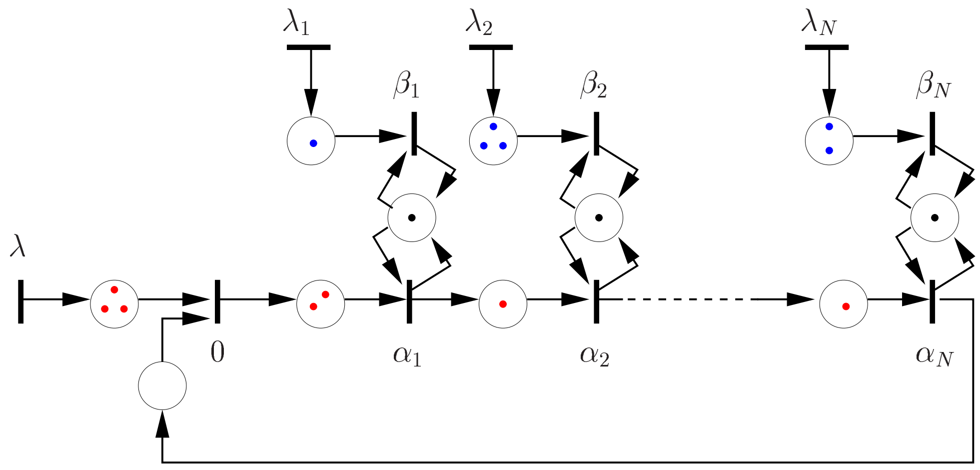


Figure 13: Controlled flow Cross flows Questions: Saturation rule, Maximum throughput for controlled flow?

FB & T. Bonald 99, Questa

MAXIMUM THROUGHPUT

Monotonicity Let N_1 and N_2 be two arrival point processes with the same finite number of points. Then, $N_1 \leq N_2 \quad \Rightarrow \quad D_1 \leq D_2$.

Remark 1 *no* homogeneity, the non–expansiveness is *not* satisfied.

Remark 2 *no* monotonicity w.r.t. the cross flows.

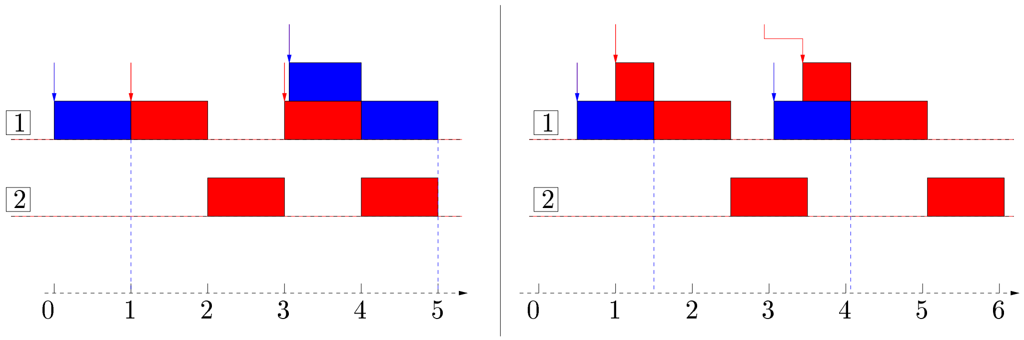


Figure 14: Two routers, one cross flow, $K = 1$

EXTENSION OF SATURATION RULE

Deterministic service times for controlled customers

$\bar{X}(s,t)$, $s \leq t$, number of controlled customers entering the network in the interval $[s,t)$, when the input is saturated at time s.

Sub-additivity

$$\forall s \leq t \leq u, \quad \bar{X}(s,u) \leq \bar{X}(s,t) + \bar{X}(t,u).$$

By Kingman's sub-additive theorem, under appropriate ergocicity and integrability assumptions,

$$\forall s \in I\!R, \quad \frac{\bar{X}(s,t)}{t} \xrightarrow{p.s.} \bar{\lambda} \quad \text{when} \quad t \to +\infty.$$

λ intensity of controlled flow:

– If $\lambda < \bar{\lambda}$, the system is stable (backward scheme)

– If $\lambda > \bar{\lambda}$, the system is unstable.

BOUNDS ON MAXIMUM THROUGHPUT

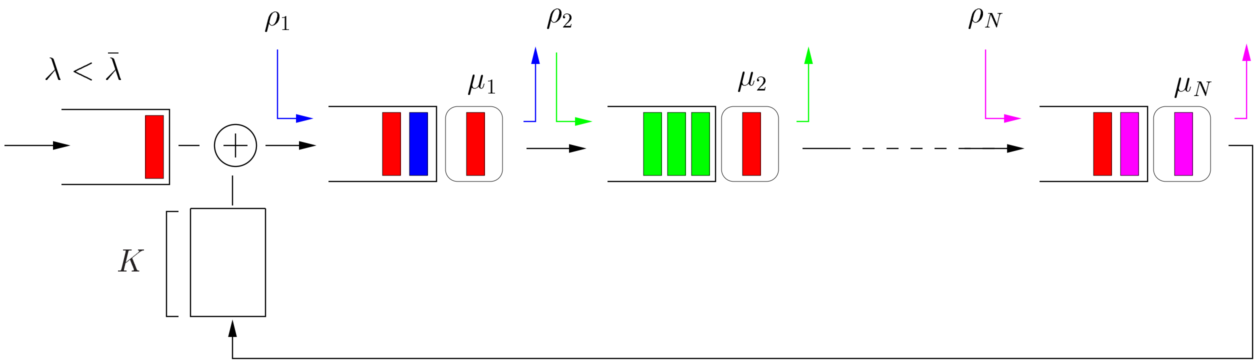


Figure 15: N routers, window: K, service time $\sigma_i = \mu_i^{-1}$ in router i

Upper Bounds: $\gamma = \max(\frac{\sum_i \sigma_i}{K}, \max_i \sigma_i) \quad \mu = \gamma^{-1}$

$$\bar{\lambda} \leq \mu \quad \text{and} \quad \bar{\lambda} \leq \min_i \sigma_i^{-1}(1 - \rho_i).$$

Lower Bound:

$$\bar{\lambda} \geq \mu\, p$$

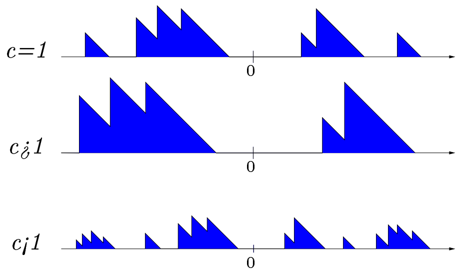
$$p = P\{\text{system without controlled flow empty}\}.$$

TIGHTNESS OF THE BOUNDS

$$\underbrace{\mu\ p}_{\lambda_{\min}} \leq \bar{\lambda} \leq \underbrace{\min\left\{\mu, \min_i \mu_i(1-\rho_i)\right\}}_{\lambda_{\max}}.$$

Time–Space Scaling For any $c > 0$, let $\bar{\lambda}(c)$ be the maximum throughput obtained when the cross flows are scaled by c. Then,

$$\lim_{c \to +\infty} \bar{\lambda}(c) = \lambda_{\min} \quad \text{and} \quad \lim_{c \to 0} \bar{\lambda}(c) = \lambda_{\max}.$$



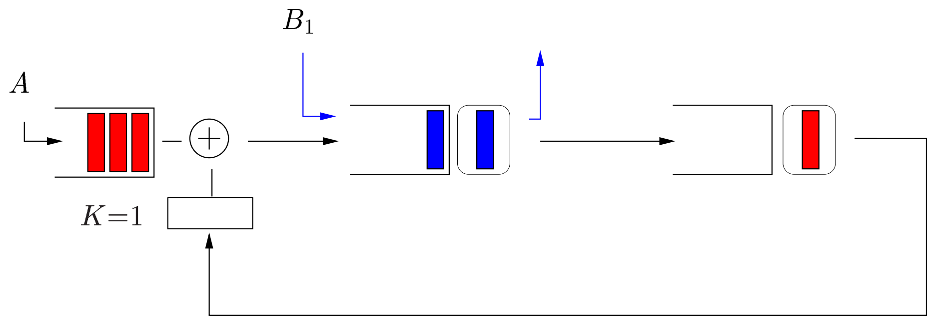
EXAMPLE

The simplest non–trivial case:

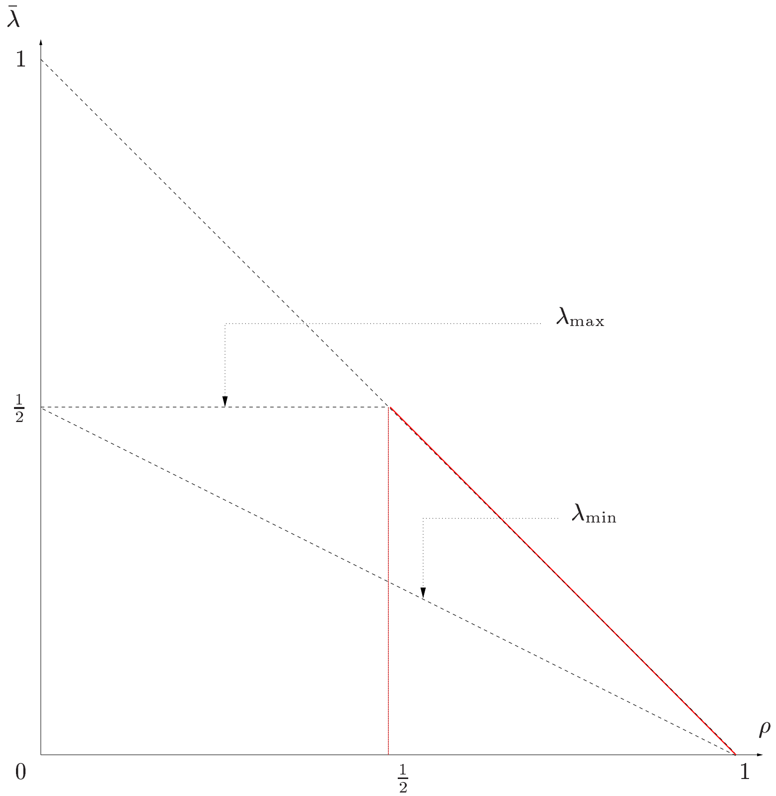
$$N = 2, \quad K = 1.$$

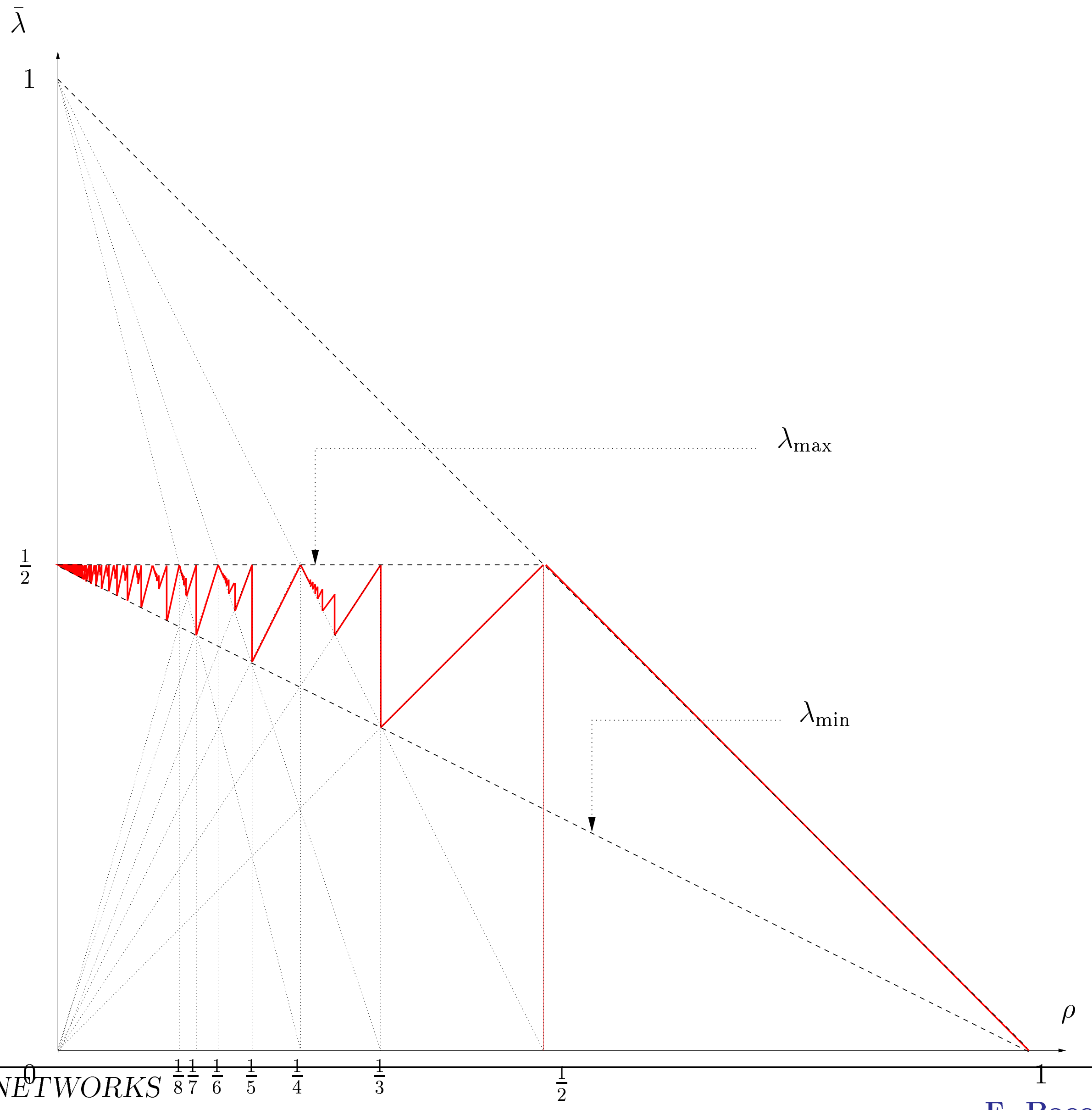
$$\mu_1 = \mu_2 = 1 \quad \text{and} \quad \mu = \frac{1}{2}$$

B_1 periodic, $B_2 = \emptyset$.



STABILITY REGION




$\bar{\lambda}$
1
$\frac{1}{2}$
$\lambda_{\max}$
$\lambda_{\min}$
ρ
0
$\frac{1}{8}$ $\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ 1

Questions at: Francois.Baccelli@ens.fr

Papers and links available at:

http://www.inria.fr/mistral/personnel/Francois.Baccelli/me.html

Software of A. Jean-Marie on (max,plus) and related systems:

http://www.inria.fr/mistral/soft/ers.html