Poisson-Voronoi Spanning Trees

with Applications to the Optimization of Communication Networks

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Abstract

We define a family of random trees in the plane. Their nodes of level k, $k = 0, \ldots, m$ are the points of a homogeneous Poisson point process Π_k, whereas their arcs connect nodes of level k and $k+1$, according to the least distance principle: if V denotes the Voronoi cell w.r.t. Π_{k+1} with nucleus x, where x is a point of Π_{k+1}, then there is an arc connecting x to all the points of Π_k which belong to V. This creates a family of stationary random trees rooted in the points of Π_m. These random trees are useful to model the spatial organization of several types of hierarchical communication networks. In relation with these communication networks, it is natural to associate various cost functions with such random trees. Using point process techniques, like the exchange formula between two Palm measures, and integral geometry techniques, we show how to compute the average cost in function of the intensity parameters of the Poisson processes. The formulas which are derived for the average value of the cost function are then exploited for parametric optimization purposes. Several applications to classical and mobile cellular communication networks are studied in detail.

OR/MS Subject Classification: Probability: stochastic model applications; Networks/graphs: stochastic; Communications; Transportation: network models.

Keywords: SPANNING TREE, STOCHASTIC MODELING, CELLULAR NETWORK, VORONOI TESSELLATION, POINT PROCESS, POISSON PROCESS

1 Introduction

Consider the following distribution problem: a family of *concentration* points is given, characterized by their random coordinates in the plane. Another family of *subscribers* is also given, with a similar characterization via the set of their random coordinates. A typical example, which is the one considered throughout this paper, is that when both point processes are homogeneous Poisson processes, although more general cases can also be considered along the same lines.

Assume one has to connect each subscriber to exactly one concentrator. A typical example is that where the concentrators are all linked by a connected network, so that when each subscriber is connected to one concentrator, it is actually interconnected to all subscribers as well.

For connecting each subscriber to one concentrator, one can either use a direct link, or an indirect link via some *distribution* point. A distribution point is a location where several links originating from subscribers can be grouped into one further link to some concentrator with some economy of scale (see Figure 2).

The first problem that we consider in this paper is that of the optimal use of such distribution points. We do not address the problem of a minimal spanning tree structure, which is a quite different combinatorial optimization question. We restrict ourselves to a parametric optimization which consists in introducing an independent Poisson process of distribution points and in determining the intensity of this process which minimizes a cost function which reflects the economy of scale alluded to above. The general assumption is that connections are always to the closest point (i.e. each subscriber is connected to the closest distribution point, and each distribution point to the closest concentrator); the Voronoi cells associated with the Poisson point processes play therefore a central role in the construction of these trees.

The paper is structured as follows: Section 2 gives the basic model and a precise definition of the cost function, together with the main characteristics of the optimal parametric model. Section 3 focuses on an extension of this to a hierarchical model with several layers which was proposed in Baccelli et al. (1996) to describe the structure of certain communication networks, and which includes the basic model of §2 as a special case. Finally, Sections 4 and 5 give some extensions of the optimization method proposed in §1-2 to related problems, also arising in communication network modeling, and also based on Poisson-Voronoi tesselations and trees: a non purely hierachical model is considered in §4, a localization problem in mobile communications in §5.1, and a class of infinite spanning graphs in §5.2.

Although the models are described throughout the paper are originating from communication problems, it is worthwhile stressing that they are generic in that they could in principle be applied to other planar or spatial distribution problems, where randomness is intrinsic.

2 Three level connection systems

2.1 Stochastic Assumptions

Consider 3 independent homogeneous Poisson processes Π_i $(i = 0, 1, 2)$ in the plane $\mathbb{R}^2$ with intensities λ_i, representing *subscribers* (process Π_0), *distribution points* (process Π_1) and *concentrators* (process Π_2), respectively.

Random trees are built from any realizations of these three point processes as follows: each point of the process Π_0 is connected to a point of process Π_1 according to the least distance principle. Similarly, each points of Π_1 is connected to the closest point of the process Π_2 (this closest point is almost surely uniquely defined under the above Poisson and independence assumptions).

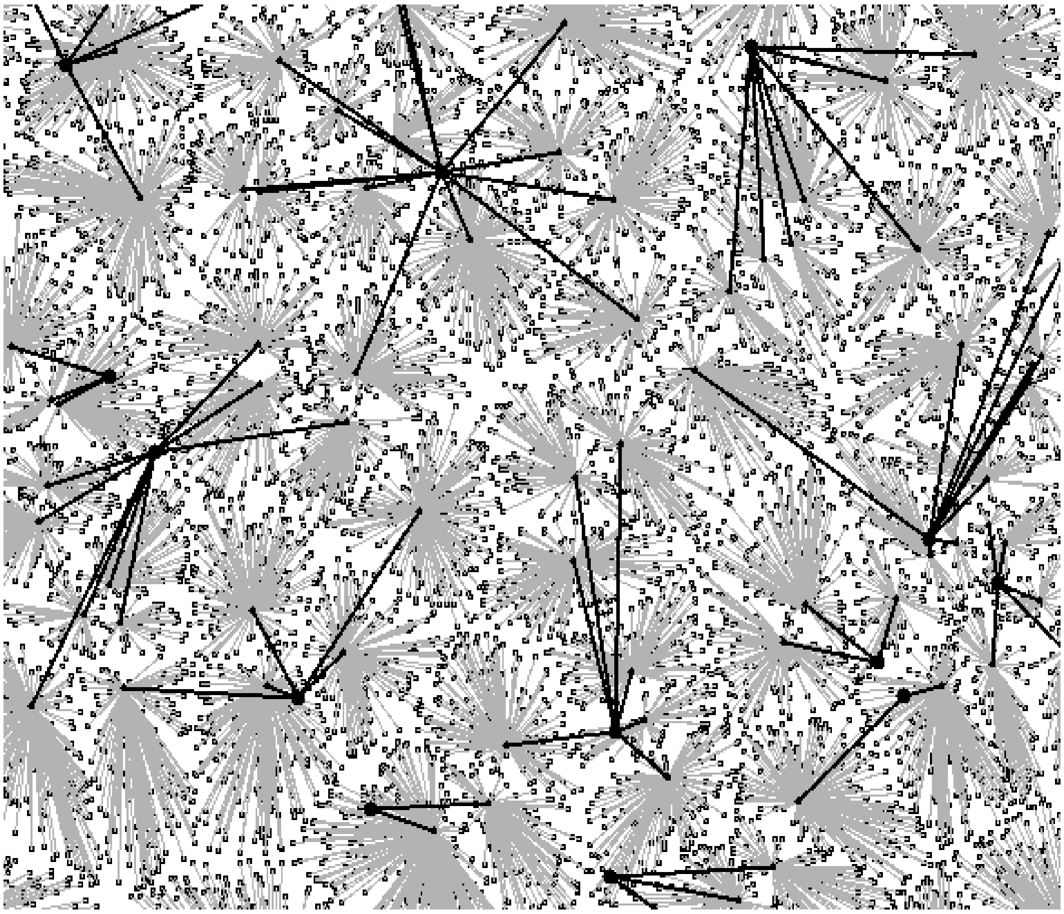

Figure 1: A family of Poisson-Voronoi trees of height two.

A typical realization of such Poisson-Voronoi trees is shown in Figure 1.

2.2 Cost function via an example

There is a cost function associated with such random trees. This cost function is perhaps best described through a physical example. In this example, the connections represent links. Each link has an associated cost which consists of two parts:

1. the *capacity cost* which represents the cost of the communication medium (e.g. optical fiber, copper etc.) and

2. the *infrastructure cost*, which represents the cost of the civil engineering (trench, poles etc.) which supports the medium.

In our example, the capacity cost for connecting a point of Π_i to the closest point of Π_{i+1}, is given by a function of the form $A_{i,i+1}r^{\alpha_{i,i+1}}$, where r is the distance between the points and $A_{i,i+1},\ \alpha_{i,i+1}$ are some non-negative parameters. This example of non–linear dependence in the variable r (more general cases will be considered in the next section) stems from the fact that the type of medium to be used depends on the distance r.

The infrastructure cost is also assumed to be given by a polynomial function of the form $B_{i,i+1}r^{\beta_{i,i+1}}$, where $B_{i,i+1},\ \beta_{i,i+1} > 0$. Note that the cables from a distribution point to a concentrator share the same civil engineering, and it is reasonable to assume that the cost of civil engineering does not depend on the number of cables which are grouped there. The various elements of this cost structure are illustrated in Figure 2.

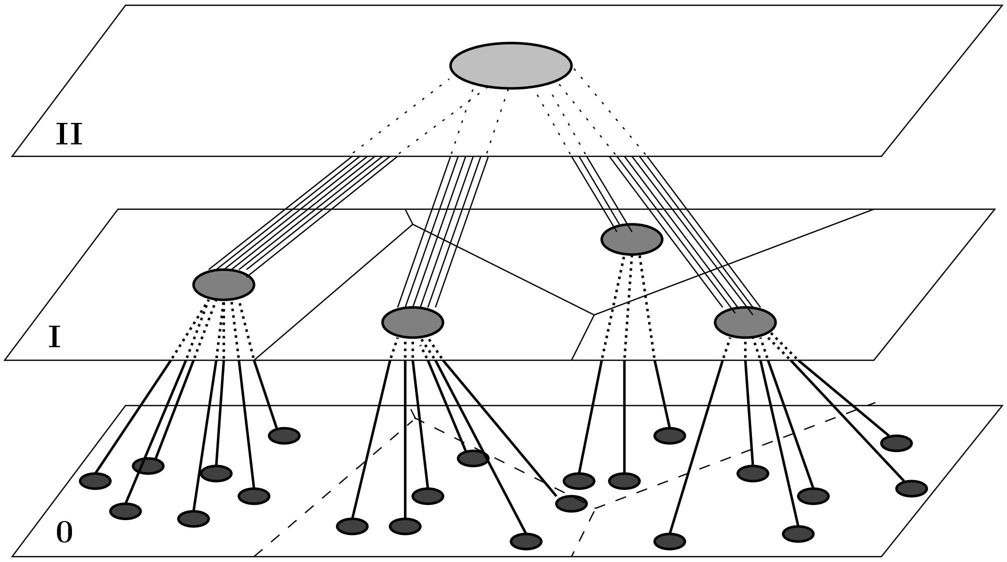

Figure 2: Links in a three-level model

Our first question is the following: given the intensities $\lambda_0,\ \lambda_2$ of the subscriber and concentrator point processes and a cost C_1 associated with the introduction of a distribution point, what is the intensity λ_1 of the distribution point process which minimizes the average total cost?

2.3 Mathematical formulation

More precisely, the cost function is the following:

$$G \stackrel{\text{def}}{=} \mathbf{E}_2^0 \Bigg[\sum_{y_i \in \Pi_1 \cap V_0(\Pi_2)} \Big[C_1 + B_{1,2}|y_i|^{\beta_{1,2}} + A_{1,2}|y_i|^{\alpha_{1,2}} \mathcal{N}_{y_i} + \sum_{x_j \in \Pi_0 \cap V_{y_i}(\Pi_1)} \big(B_{0,1}|x_j - y_i|^{\beta_{0,1}} + A_{0,1}|x_j - y_i|^{\alpha_{0,1}} \big) \Big] \Bigg]. \quad (1)$$

Here and below,

- $\mathbf{E}_k^0$ denotes the expectation with respect to the Palm distribution $\mathbf{P}_k^0$ with respect to the point process Π_k;
- $V_x(\Pi_k)$ is the Voronoi cell constructed with respect to Π_k, with nucleus $x \in \Pi_k$;
- $\mathcal{N}_x$ is the number of Π_0-points linked to the point x.

From now on we make an extensive use of the following Neveu's exchange formula, which describes the relationship between the two Palm distributions $\mathbf{P}_1^0$ and $\mathbf{P}_2^0$:

$$\lambda_1 \, \mathbf{E}_1^0 \, f(N) = \lambda_2 \, \mathbf{E}_2^0 \int_{V_0(\Pi_2)} f(\theta_x N) \, N_1(dx) \tag{2}$$

for any measurable $f : \Omega \mapsto \mathbb{R}_+$ (see the remark following Proposition 1 in Neveu (1976), p.201). Now the expression for the cost function simplifies to

$$\frac{\lambda_1}{\lambda_2} C_1 + \frac{\lambda_1}{\lambda_2} \mathbf{E}_1^0 \left[B_{1,2} |z_0|^{\beta_{1,2}} + A_{1,2} |z_0|^{\alpha_{1,2}} \mathcal{N}_0 + \sum_{x_j \in \Pi_0 \cap V_0(\Pi_1)} (B_{0,1} |x_j|^{\beta_{0,1}} + A_{0,1} |x_j|^{\alpha_{0,1}}) \right], \tag{3}$$

where z_0 here denotes the point of the process Π_2 which is the closest to 0.

On the other hand, for all $a > 0$, if one denotes $S_{|x|}(u)$ the open ball centered in u and with radius $|x|$,

$$\begin{aligned}
\mathbf{E}_1^0 \, |z_0|^a &= \mathbf{E}_1^0 \int |x|^a \; \mathbb{I}\{\Pi_2(S_{|x|}(0)) = 0\} \, \Pi_2(dx) \\
&= \mathbf{E} \int |x|^a \; \mathbb{I}\{\Pi_2(S_{|x|}(0)) = 0\} \, \Pi_2(dx) \\
&= \lambda_2 \int |x|^a \; \mathbf{P}_2^0[\Pi_0(S_{|x|}(-x)) = 0] \, dx \\
&= 2\pi\lambda_2 \int_0^\infty r^{a+1} \exp\{-\lambda_2 \pi r^2\} \, dr \; = \frac{\Gamma\left(\frac{a}{2}+1\right)}{(\pi\lambda_2)^{\frac{a}{2}}} ,
\end{aligned} \tag{4}$$

where the second equality follows from the independence of Π_1 and Π_2, and the third from Campbell's refined formula ($\Gamma(\cdot)$ here is the Euler Gamma-function).

Denote by T_i, $i = 1, 4$, the i-th term of the sum in the expectation in (3). In view of (4), we have

$$\mathbf{E}_1^0 \, T_1 = \frac{B_{1,2}}{(\pi\lambda_2)^{\frac{\beta_{1,2}}{2}}} \, \Gamma\left(\frac{\beta_{1,2}}{2} + 1\right).$$

To compute the expectation of the second term, note that when the origin is occupied by a point of the process Π_1 the variables $|z_0|$ and $\mathcal{N}_0$ are independent since they are functions of two independent processes Π_2 and $\Pi_0 \cup \Pi_1$, respectively. It is easily seen that $\mathbf{E}_1^0 \, \mathcal{N}_0 = \lambda_0 / \lambda_1$ (more complete results on geometry of random points in a typical Voronoi cell can be found in Foss and Zuyev (1996)) and we obtain

$$\mathbf{E}_1^0 \, T_2 = A_{1,2} \, \mathbf{E}_1^0 \, |z_0|^{\alpha_{1,2}} \; \mathbf{E}_1^0 \, \mathcal{N}(0) = \frac{A_{1,2} \lambda_0}{\lambda_1 (\pi\lambda_2)^{\frac{\alpha_{1,2}}{2}}} \, \Gamma\left(\frac{\alpha_{1,2}}{2} + 1\right).$$

Similarly, using once more the above exchange formula (2), but this time between $\mathbf{E}_1^0$ and $\mathbf{E}_0^0$, we get

$$\lambda_1\,\mathbf{E}_1^0[T_3+T_4] = \frac{B_{0,1}\lambda_0}{(\pi\lambda_1)^{\frac{\beta_{0,1}}{2}}}\,\Gamma\left(\frac{\beta_{0,1}}{2}+1\right) + \frac{A_{0,1}\lambda_0}{(\pi\lambda_1)^{\frac{\alpha_{0,1}}{2}}}\,\Gamma\left(\frac{\alpha_{0,1}}{2}+1\right).$$

Combining the above expressions, we get the following explicit expression for the cost function:

$$G = H_1\lambda_1 + H_2 + H_3\lambda_1^{-\frac{\beta_{0,1}}{2}} + H_4\lambda_1^{-\frac{\alpha_{0,1}}{2}}, \tag{5}$$

where

$$\begin{aligned}
H_1 &= \frac{B_{1,2}}{\pi^{\frac{\beta_{1,2}}{2}}\lambda_2^{\frac{\beta_{1,2}}{2}+1}}\Gamma\left(\frac{\beta_{1,2}}{2}+1\right)+\frac{C_1}{\lambda_2};\\
H_2 &= \frac{A_{1,2}\lambda_0}{\pi^{\frac{\alpha_{1,2}}{2}}\lambda_2^{\frac{\alpha_{1,2}}{2}+1}}\Gamma\left(\frac{\alpha_{1,2}}{2}+1\right);\\
H_3 &= \frac{B_{0,1}\lambda_0}{\pi^{\frac{\beta_{0,1}}{2}}\lambda_2}\Gamma\left(\frac{\beta_{0,1}}{2}+1\right);\\
H_4 &= \frac{A_{0,1}\lambda_0}{\pi^{\frac{\alpha_{0,1}}{2}}\lambda_2}\Gamma\left(\frac{\alpha_{0,1}}{2}+1\right).
\end{aligned}$$

We see that the cost function has a unique minimal value, attained in the point λ_1^* which solves the equation

$$H_3\beta_{0,1}\lambda_1^{-(\frac{\beta_{0,1}}{2}+1)} + H_4\alpha_{0,1}\lambda_1^{-(\frac{\alpha_{0,1}}{2}+1)} = 2H_1\,.$$

In particular case when $\beta_{0,1}=\alpha_{0,1}\stackrel{\text{def}}{=}\alpha$ we obtain

$$\lambda_1^* = \left[\frac{\alpha(H_3+H_4)}{2H_1}\right]^{\frac{2}{2+\alpha}}$$

and the minimal value of the cost function is equal to

$$G(\lambda_1^*) = \frac{\alpha+2}{\alpha}H_1^{\frac{\alpha}{2+\alpha}}\left[\frac{\alpha}{2}(H_3+H_4)\right]^{\frac{2}{2+\alpha}} + H_2.$$

For $\alpha=1$ and $\alpha_{1,2}=\beta_{1,2}=1$, we obtain

$$\lambda_1^* \;=\; \lambda_2^{\frac{1}{3}}\left[\frac{\lambda_0(A_{0,1}+B_{0,1})}{2B_{1,2}+4C_1\lambda_2^{1/2}}\right]^{\frac{2}{3}}$$

and

$$\mathbf{E}_2^0\,G \;=\; \frac{1}{\lambda_2}\left[3\Big(B_{1,2}\lambda_2^{1/3}+C_1\Big)^{1/3}\Big(\frac{\lambda_0}{4}\big(A_{0,1}+B_{0,1}\big)\Big)^{2/3} + \frac{A_{12}}{2}\lambda_0\lambda_2^{1/3}\right].$$

The upper curve in Figure 4 shows a typical example of the cost function of a three level system. A more detailed discussion can be found in Section 4.3.

3 Hierarchical model

3.1 The model

Consider the following extension of the models introduced in §2: there are $m \geq 3$ independent homogeneous Poisson processes Π_k, $k = 0, 1 \ldots, m$, in $\mathbb{R}^2$ with respective intensities λ_k.

We will use a terminology which continues that of our typical example so as to make things more easy to name and manipulate: the point process Π_0 represents the network *subscribers*, whereas the point processes Π_1, Π_2, Π_3, etc. represent *distribution points* or stations of various levels in the hierarchy. Of course in practice, we should have $\lambda_0 > \lambda_1 > \ldots > \lambda_m$, although this is not mandatory in our mathematical treatment.

Associated with the realizations of these point processes, which can be seen as denumerable collections of nodes located in the plane, one then defines arcs between certain pairs of nodes, which are built following the least distance hierarchical principle: for all $k = 1, \ldots, m$, each point of Π_{k-1} is connected to the point of Π_k which is the closest. Equivalently, with each station x_j of level k, if one denotes $V_{x_j}(\Pi_k)$ the Voronoi cell with nucleus x_j constructed with respect to Π_k, each point of Π_{k-1} lying inside $V_{x_j}(\Pi_k)$ has an arc connecting it to x_j. These arcs represent connections or links in the communication network.

This set of arcs defines a random graph on the nodes, which can be decomposed into a denumerable family of connected subgraphs which are all random trees of level m at most; there is one such tree rooted in each point of Π_m (see Fig. 2), and this tree has subscribers or points of Π_k, $k = 1, \ldots, m$, as leaves. For instance, if the Voronoi cell $V_x(\Pi_m)$ is empty of points of Π_{m-1}, then x is both a root and a leaf of the tree rooted in x. The random trees in this family are strongly dependent, but identically distributed. In that, this family is a collection of spanning trees allowing one to reach a certain subset of the set of subscribers from each root via some distribution points of levels $m-1, \ldots, 1$. So if the roots are interconnected, each subscriber can be connected via a finite set of arcs to any other subscriber of the plane.

3.2 Description of the cost function via the communication example

Roughly speaking, the cost of each such tree consists of the cost of all the connections and stations of lower levels linked to its root. In our communication example:

- each point of Π_k has a cost associated with the installation and maintenance of the corresponding station of level k, which we denote C_k.
- each arc from a node x_j of Π_k to a node y_l of Π_{k+1} has two components in its cost:
    1. The *capacity cost*, namely the cost of the communication medium allowing the interconnection between the subtree rooted in x_j and y_l; this communication

medium should allow the network to cope with the set of all subscribers being leaves of this subtree. The capacity cost of this medium is assumed to be given by a function of the form

$$H_{k,k+1}(n)A_{k,k+1}(r),$$

when the number $\mathcal{N}_{x_j}$ of subscribers of the subtree rooted in x_j is n, and when the arch connecting x_j to y_l is of length r. Here H and A are non-decreasing real-valued mapping. A typical example is that with $H_{k,k+1}(n) \equiv n$, in which case this capacity cost is purely additive in $\mathcal{N}_{x_j}$. However, the capacity of a connection is often designed to take advantage of the so-called statistical multiplexing between subscribers, and the function H is often sub-additive. The function $A(r)$ will be referred to as the *unit capacity cost* (the capacity cost of the additive case when the variable $\mathcal{N}_{x_j}$ is equal to 1). The particular case $A(r) = Const \cdot r$ and its polynomial generalizations were already commented in §1.

2. The *infrastructure cost*, which is given by a function $B_{k,k+1}(r)$ which does not depend on $\mathcal{N}_{x_j}$ by assumption, although it may of course depend on k.

3.3 Definition of the cost function

The cost function is defined recursively through the costs of the subtrees rooted in the points of Π_k for all k.

Let again $\mathbf{P}_k^0$ denote the Palm probability of the point process Π_k, and $\mathbf{E}_k^0$ expectation w.r.t. $\mathbf{P}_k^0$. On the Palm probability space of Π_k, there is a station of level k at the origin, and we define the following random variables (where Π denotes a realization of the family of all point processes $\Pi_0, \ldots, \Pi_m$, and $\{\theta_x\}$ the group of translations of all these point processes):

- $\mathcal{C}(\Pi)$ is the cost of the station in the origin (i.e. C_k);
- $G(\Pi)$ is the cost of the subtree rooted in the origin;
- $\mathcal{N}(\Pi)$ $(= \mathcal{N}_0)$ is the number of subscribers in this subtree;
- $\mathcal{A}(\Pi)$ the unit capacity cost of the arc from 0 to the nearest point of level $k+1$;
- $\mathcal{B}(\Pi)$ the infrastructure cost of this arc.
- $\mathcal{H}(\Pi, \cdot)$ is the mapping giving the dependence of the capacity cost in $\mathcal{N}(\Pi)$ (i.e. $H_{k,k+1}(\cdot)$).

For $k = 0$, we obviously have $G(\Pi) = C_0 = \mathcal{C}(\Pi)$ and $\mathcal{N}(\Pi) = 1$. For $k = 1$,

$$\begin{aligned} G(\Pi) &= C_1 + \sum_{x_j \in \Pi_0 \cap V_0(\Pi_1)} C_0 + H_{0,1}(1)A_{0,1}(|x_j|) + B_{0,1}(|x_j|) \\ &= \mathcal{C}(\Pi) + \int\limits_{V_0(\Pi_1)} \left[G(\theta_x \Pi) + \mathcal{H}(\theta_x \Pi, \mathcal{N}(\theta_x \Pi))\mathcal{A}(\theta_x \Pi) + \mathcal{B}(\theta_x \Pi)\right] \Pi_0(dx)\,. \end{aligned}$$

In general, on the Palm space of Π_k,

$$G(\Pi) = \mathcal{C}(\Pi) + \int\limits_{V_0(\Pi_k)} \left[G(\theta_x\Pi) + \mathcal{H}(\theta_x\Pi, \mathcal{N}(\theta_x\Pi))\mathcal{A}(\theta_x\Pi) + \mathcal{B}(\theta_x\Pi)\right] \Pi_{k-1}(dx)\,.$$

Since under the Palm distribution $\mathbf{P}_k^0$ ($k \geq 1$) there is a.s. a station of level k in the origin, we can apply the exchange formula (see (2)) with $N_1 = \Pi_{k-1}$ and $N_2 = \Pi_k$ to the Palm expectation of the last expression to obtain

$$\mathbf{E}_k^0\, G(\Pi) = C_k + \frac{\lambda_{k-1}}{\lambda_k}\, \mathbf{E}_{k-1}^0 \left[G(\Pi) + \mathcal{H}(\Pi, \mathcal{N}(\Pi))\mathcal{A}(\Pi) + \mathcal{B}(\Pi)\right]. \tag{6}$$

To compute $\mathbf{E}_{k-1}^0\, \mathcal{B}(\Pi)$, we use Campbell's refined formula as we did it (4) to obtain

$$\begin{aligned}
\mathbf{E}_{k-1}^0\, \mathcal{A}(\Pi) &= \mathbf{E}_k^0 \int B_{k-1,k}(|x|)\, \mathbb{1}\{\Pi_k(B_{|x|}(0)) = 0\}\, \Pi_k(dx) \\
&= \lambda_k \int B_{k-1,k}(|x|)\, \mathbf{P}_k^0[\Pi_k(B_{|x|}(-x)) = 0]\, dx \\
&= 2\pi\lambda_k \int_0^\infty r B_{k-1,k}(r) \exp\{-\lambda_k \pi r^2\}\, dr\,.
\end{aligned}$$

Next observe that the random variable $\mathcal{N}(\Pi)$ in (6) depends only on $\Pi_0 \ldots, \Pi_{k-1}$, whereas $\mathcal{A}(\Pi)$, like $\mathcal{B}(\Pi)$, depends only on Π_k. Therefore

$$\mathbf{E}_{k-1}^0[H_{k-1,k}(\mathcal{N}(\Pi))\mathcal{A}(\Pi)] = \mathbf{E}_{k-1}^0\, [H_{k-1,k}(\mathcal{N}(\Pi))]\, \mathbf{E}_k^0\, \mathcal{A}(\Pi).$$

Finally, in the particular case when $H_{k-1,k} = Id$, it follows from (2) that

$$\begin{aligned}
\mathbf{E}_k^0\, \mathcal{N}(\Pi) &= \mathbf{E}_k^0 \int\limits_{V_0(\Pi_k)} \mathcal{N}(\theta_x\Pi)\, \Pi_{k-1}(dx) \\
&= \frac{\lambda_{k-1}}{\lambda_k}\, \mathbf{E}_{k-1}^0\, \mathcal{N}(\Pi) = \ldots \\
&= \frac{\lambda_{k-1}}{\lambda_k} \frac{\lambda_{k-2}}{\lambda_{k-1}} \cdots \frac{\lambda_0}{\lambda_1}\, \mathbf{E}_0^0\, \mathcal{N}(\Pi) = \frac{\lambda_0}{\lambda_k}\,,
\end{aligned}$$

and therefore

$$\mathbf{E}_k^0\, G(\Pi) = C_k + \frac{\lambda_{k-1}}{\lambda_k}\, \mathbf{E}_{k-1}^0\, G(\Pi)$$

$$+2\pi \int_0^\infty \left[\lambda_0 A_{k-1,k}(r) + \lambda_{k-1} B_{k-1,k}(r)\right] r \exp\{-\lambda_k \pi r^2\}\, dr\,.$$

Thus we have proved the following

Theorem 1. *In the case $H_{k,k+1} = Id$ for all k, for any $k \geq 1$, we have*

$$\lambda_k\, \mathbf{E}_k^0\, G = \lambda_{k-1}\, \mathbf{E}_{k-1}^0\, G + W_k = \sum_{n=0}^{k} W_n \tag{7}$$

where $W_0 = \lambda_0 C_0$,

$$W_n = \lambda_n C_n + \lambda_0 \mathcal{R}_{\lambda_n}\left(A_{n-1,n}(\bullet)\right) + \lambda_{n-1} \mathcal{R}_{\lambda_n}\left(B_{n-1,n}(\bullet)\right) \tag{8}$$

for $n \geq 1$ and the integral operator

$$\mathcal{R}_\lambda\big(f(\bullet)\big) \stackrel{\text{def}}{=} 2\pi\lambda \int_0^\infty r f(r) e^{-\lambda\pi r^2}\, dr\,. \tag{9}$$

Remark 1. Operator $\mathcal{R}_\lambda\big(f(\bullet)\big)$ can be viewed as the expectation of random variable $f(\rho_\lambda)$ where ρ_λ follows Rayleigh distribution with the parameter $1/(2\pi\lambda)$.

Equation (7) above receives the following natural interpretation: the increment of the cost density between subtrees of level $k-1$ and k is equal to the sum of the cost density of the inter-connections between these two levels and the cost density of the k-level stations.

3.4 Example of optimization

Consider a four level system with the unit capacity cost and infrastructure cost depending linearly from the distance: $A_{i,j}(r) = a_{i,j} r$ and $B_{i,j}(r) = b_{i,j} r$, respectively. It is readily seen that the cost function $\mathbf{E}_3^0 G$ is smooth in $\mathbb{R}_+^2$, tends to infinity if λ_1 or λ_2 tends to 0 or to infinity and the stationary point is given by the following system of equations:

$$\lambda_3 \frac{\partial\, \mathbf{E}_3^0 G}{\partial \lambda_1} = C_1 + \frac{b_{1,2}}{2\sqrt{\lambda_2}} - \frac{\lambda_0 b_{0,1}}{4\lambda_1^{\frac{3}{2}}} = 0\,;$$
$$\lambda_3 \frac{\partial\, \mathbf{E}_3^0 G}{\partial \lambda_2} = C_2 + \frac{b_{2,3}}{2\sqrt{\lambda_3}} - \frac{\lambda_0 a_{1,2} + \lambda_1 b_{1,2}}{4\lambda_2^{\frac{3}{2}}} = 0\,.$$

Expressing λ_1 from the second equation and substituting it in the first one we obtain

$$\lambda_1 = \Phi\lambda_2^{\frac{3}{2}} - \frac{\lambda_0 a_{1,2}}{b_{1,2}}\,,$$
$$C_1 + \frac{b_{1,2}}{2\sqrt{\lambda_2}} - \frac{\lambda_0 b_{0,1}}{4\Big(\Phi\lambda_2^{\frac{3}{2}} - \frac{\lambda_0 a_{1,2}}{b_{1,2}}\Big)^{\frac{3}{2}}} = 0\,,$$

where

$$\Phi = \frac{4}{b_{1,2}}\Big(C_2 + \frac{b_{2,3}}{2\sqrt{\lambda_3}}\Big)\,.$$

Denoting $\nu = \sqrt{\lambda_2}$ after some algebra, we get

$$\Big(\Phi\nu^3 - \frac{\lambda_0 a_{1,2}}{b_{1,2}}\Big)^3 = \frac{\lambda_0^2 b_{0,1}^2 \nu^2}{4(C_1\nu + b_{1,2})^2}\,. \tag{10}$$

We see that in the LHS of (10) stands a continuous function which increases from a negative value to infinity, whereas the function in the RHS is positive and decreasing for $\nu > 0$. Therefore there exists a unique positive solution to this equation. Noting that the value in the LHS is exactly λ_1^3 we conclude that there is a unique positive pair (λ_1, λ_2) providing the minimal value of the cost function.

4 Case of direct connections

4.1 Model

In the hierarchical model considered in the previous sections, a $k-1$-level station is always attached to a k-level station, even if there are higher level stations closer to it. In this section we let such a station be linked to the station of levels $k\dots,m$ (see Fig. 3) which is the closest. We consider here only a 3-level system and the case with $H(n)=n$; higher level models can in principal be treated by the same method.

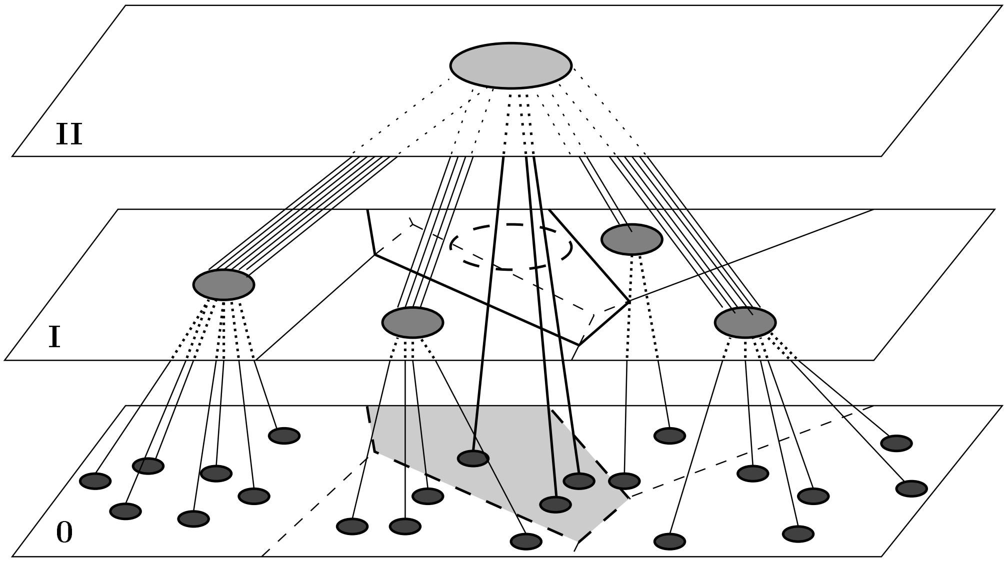

Figure 3: 3-level model with direct connections

4.2 Cost function

We introduce marks associated with the points of Π_0, namely functionals of configurations having a 0-level station located at the origin:

> $\mathcal{A}'(\Pi)$ is the unit capacity cost of the link from the origin to the nearest 1-st level station if there is no closer 2-nd level station, and 0 otherwise;
>
> $\mathcal{A}''(\Pi)$ is the unit capacity cost of the link from the origin to the nearest 2-nd level station if there is no closer 1-st level, and 0 otherwise;

Infrastructure cost variables $\mathcal{B}'$ and $\mathcal{B}''$ are defined similarly. Let $\mathcal{A}(\Pi)=\mathcal{A}'(\Pi)+\mathcal{A}''(\Pi)$ and $\mathcal{B}(\Pi)=\mathcal{B}'(\Pi)+\mathcal{B}''(\Pi)$.

Now under $\mathbf{P}_1^0$,

$$\begin{aligned} G(\Pi) &= C_1 + \sum_{x_j\in\Pi_0\cap V_0(\Pi_1\cup\Pi_2)} C_0 + A_{0,1}(|x_j|) + B_{0,1}(|x_j|) \\ &= C_1 + \int_{V_0(\Pi_1\cup\Pi_2)} \left[G(\theta_x\Pi) + \mathcal{A}'(\theta_x\Pi) + \mathcal{B}'(\theta_x\Pi)\right]\Pi_0(dx)\,. \end{aligned} \tag{11}$$

First of all note that due to the well known property of Poisson processes, $\Pi_1 \cup \Pi_2$ is also a homogeneous Poisson process with intensity $\lambda_1 + \lambda_2$. Therefore by Campbell's theorem

$$\mathbf{E}_1^0 \sum_{x_j \in \Pi_0 \cap V_0(\Pi_1 \cup \Pi_2)} C_0 = \lambda_0 C_0 |V_0(\Pi_1 \cup \Pi_2)| = \frac{\lambda_0 C_0}{\lambda_1 + \lambda_2}\,.$$

In addition, we have $V_0(\Pi_1 \cup \Pi_2) \subseteq V_0(\Pi_1)$. But for $x \in V_0(\Pi_1) \setminus V_0(\Pi_1 \cup \Pi_2)$, there exists $z_j \in \Pi_2$ such that $|x - z_j| < |x|$, so that $\mathcal{A}'(\theta_x \Pi) = \mathcal{B}'(\theta_x \Pi) = 0$. Therefore we can replace the integration domain in (11) by $V_0(\Pi_1)$ and then apply (2) to write

$$\mathbf{E}_1^0\, G = C_1 + \frac{\lambda_0 C_0}{\lambda_1 + \lambda_2} + \frac{\lambda_0}{\lambda_1}\, \mathbf{E}_0^0 [\mathcal{A}'(\Pi) + \mathcal{B}'(\Pi)]\,.$$

By the same arguments as in the proof of (7), we obtain

$$\begin{aligned}\mathbf{E}_0^0\, \mathcal{B}'(\Pi) &= \mathbf{E} \int B_{0,1}(|x|)\ \mathbb{I}\{\Pi_1(S_{|x|}(0)) = 0\}\ \mathbb{I}\{\Pi_2(S_{|x|}(0)) = 0\}\, \Pi_1(dx) \\ &= 2\pi\lambda_1 \int_0^\infty r B_{0,1}(r) \exp\{-\pi r^2 (\lambda_1 + \lambda_2)\}\, dr = \frac{\lambda_1}{\lambda_1 + \lambda_2} \mathcal{R}_{\lambda_1 + \lambda_2}\big(B_{0,1}(\bullet)\big) \qquad (12)\end{aligned}$$

and a similar formula for $\mathbf{E}_0^0\, \mathcal{A}'(\Pi)$. Combining all the above results, we obtain

$$\mathbf{E}_1^0\, G = C_1 + \frac{\lambda_0 C_0}{\lambda_1 + \lambda_2} + \frac{\lambda_0}{\lambda_1 + \lambda_2} \mathcal{R}_{\lambda_1 + \lambda_2}\big(A_{0,1}(\bullet) + B_{0,1}(\bullet)\big)\,. \qquad (13)$$

Consider now the Palm space of Π_2. On this space,

$$G(\Pi) = C_2 + \int_{V_0(\Pi_2)} \big[G(\theta_x \Pi) + \mathcal{B}(\theta_x \Pi) + \mathcal{N}(\theta_x \Pi) \mathcal{A}(\theta_x \Pi)\big]\, \Pi_1(dx) \qquad (14)$$

$$+ \int_{V_0(\Pi_1 \cup \Pi_2)} \big[G(\theta_x \Pi) + \mathcal{B}''(\theta_x \Pi) + \mathcal{A}''(\theta_x \Pi)\big]\, \Pi_0(dx) \qquad (15)$$

and as in (11), the last integration over $V_0(\Pi_1 \cup \Pi_2)$ can be replaced by an integration over $V_0(\Pi_2)$. Therefore using (2), we get

$$\mathbf{E}_2^0\, G = C_2 + \frac{\lambda_1}{\lambda_2}\, \mathbf{E}_1^0 [G + \mathcal{B} + \mathcal{N}\mathcal{A}] + \frac{\lambda_0}{\lambda_2}\, \mathbf{E}_0^0 [G + \mathcal{B}'' + \mathcal{A}'']\,. \qquad (16)$$

We have

$$\begin{aligned}\mathbf{E}_0^0\, G &= C_0\,; \\ \mathbf{E}_0^0\, \mathcal{A}''(\Pi) &= \mathbf{E} \int A_{0,2}(|z|)\ \mathbb{I}\{\Pi_1(S_{|z|}(0)) = 0\}\ \mathbb{I}\{\Pi_2(S_{|z|}(0)) = 0\}\, \Pi_2(dz) \\ &= 2\pi\lambda_2 \int_0^\infty r A_{0,2}(r) \exp\{-\pi r^2 (\lambda_1 + \lambda_2)\}\, dr\,; \\ \mathbf{E}_0^0\, \mathcal{B}''(\Pi) &= 2\pi\lambda_2 \int_0^\infty r B_{0,2}(r) \exp\{-\pi r^2 (\lambda_1 + \lambda_2)\}\, dr\,.\end{aligned}$$

The expression for $\mathbf{E}_1^0\mathcal{B}$ is given by (7) with $k=2$, but some difficulties arise when computing $\mathbf{E}_1^0\mathcal{N}\mathcal{A}$ since these variables are no longer independent. However, we can write

$$\begin{aligned}
\mathbf{E}_1^0\mathcal{N}\mathcal{A} &= \mathbf{E}_1^0 \int\limits_{V_0(\Pi_1\cup\Pi_2)} \Pi_0(dx) \int A_{1,2}(|z|)\,\mathbb{I}\{\Pi_2\big(S_{|z|}(0)\big)=0\}\,\Pi_2(dz) \\
&= \mathbf{E}_1^0 \iint A_{1,2}(|z|)\;\mathbb{I}\{\Pi_1\big(S_{|x|}(x)\big)=0\} \\
&\qquad\qquad \mathbb{I}\{\Pi_2\big(S_{|x|}(x)\cup S_{|z|}(0)\big)=0\}\,\Pi_0(dx)\Pi_2(dz) \\
&= \lambda_0\,\mathbf{E} \iint A_{1,2}(|z|)e^{-\lambda_1\pi|x|^2}\;\mathbb{I}\{\Pi_2\big(S_{|x|}(x)\cup S_{|z|}(0)\big)=0\}\,dx\Pi(dz) \\
&= \lambda_0\lambda_2 \iint A_{1,2}(|z|)e^{-\lambda_1\pi|x|^2}\;\mathbf{P}_2^0[\theta_{-z}\Pi_2\big(S_{|x|}(x)\cup S_{|z|}(0)\big)=0]\,dxdz\,.
\end{aligned}$$

Notice that the probability

$$\mathbf{P}_2^0[\theta_{-z}\Pi_2\big(S_{|x|}(x)\cup S_{|z|}(0)\big)=0]$$

above is 0 unless $z\notin S_{|x|}(x)$. Switching to polar coordinates, the last expression reads

$$\begin{aligned}
&2\pi\lambda_0\lambda_2 \iint rA_{1,2}(r)e^{-\lambda_1\pi|x|^2}\,\mathbf{P}_2^0[\theta_{(r,0)}\Pi_2\big(S_{|x|}(x)\cup S_r(0)\big)=0]\,dxdr \\
&= 2\pi\lambda_0\lambda_2 \iint\limits_{\Delta} rA_{1,2}(r)\exp\{-\lambda_1\pi|x|^2-\lambda_2|S_{|x|}(x)\cup S_r(0)|\}\,dxdr\,,
\end{aligned}$$

where the integration is over

$$\Delta=\{(x,r)\in\mathbb{R}^2\times\mathbb{R}_+ \text{ such that } (r,0)\notin S_{|x|}(x)\}.$$

This integral can be simplified if we represent x in polar coordinates of the form $(r\rho,\phi)$ and adopt ρ as new integration variable. Then

$$\mathbf{E}_1^0\mathcal{N}\mathcal{A} = 2\pi\lambda_0\lambda_2 \int_0^\infty r^3A_{1,2}(r)dr \iint\limits_{\delta} \rho\exp\big\{-r^2\big(\lambda_1\pi\rho^2+\lambda_2U(\rho)\big)\big\}\,d\rho d\phi\,,$$

where $U(\rho)$ is the area of the union of the discs $S_\rho(\rho\cos\phi,\rho\sin\phi)$ and $S_1(0)$, and where the double integral is over the set δ of couples (ρ,ϕ) such that $S_\rho(\rho\cos\phi,\rho\sin\phi)$ does not contain point $(1,0)$, i. e. over

$$\phi\in\Big[\mathbb{I}\{\rho>1/2\}\arccos\frac{1}{2\rho},2\pi-\mathbb{I}\{\rho>1/2\}\arccos\frac{1}{2\rho}\Big]\,.$$

Thus we finally get

$$\begin{aligned}
\mathbf{E}_1^0\mathcal{N}\mathcal{A} = 4\pi\lambda_0\lambda_2 \int_0^\infty r^3A_{1,2}(r)dr \int_0^\infty \rho\Big(\pi-\mathbb{I}\{\rho>1/2\}\arccos\frac{1}{2\rho}\Big) \\
\times\exp\{-r^2\big(\lambda_1\pi\rho^2+\lambda_2U(\rho)\big)\}\,d\rho\,,
\end{aligned} \tag{17}$$

where

$$U(\rho)=\begin{cases}\pi & \text{if } \rho\in[0,1/2] \\ \pi+\frac{1}{2}\sqrt{4\rho^2-1}+(2\rho^2-1)\arccos\frac{1}{2\rho} & \text{if } \rho>1/2\end{cases} \tag{18}$$

as elementary geometrical considerations show.

We combine all the above results in the following theorem.

Theorem 2. *For 3-level hierarchical model with direct connections we have*

$$\mathbf{E}_1^0 G = C_1 + \frac{\lambda_0}{\lambda_1+\lambda_2} C_0 + \frac{\lambda_0}{\lambda_1+\lambda_2} \mathcal{R}_{\lambda_1+\lambda_2}\big(A_{0,1}(\bullet) + B_{0,1}(\bullet)\big)\,; \tag{19}$$

$$\begin{aligned} \mathbf{E}_2^0 G = C_2 &+ \frac{\lambda_0}{\lambda_2}\Big[C_0 + \frac{\lambda_1}{\lambda_1+\lambda_2}\mathcal{R}_{\lambda_1+\lambda_2}\big(A_{0,2}(\bullet)+B_{0,2}(\bullet)\big)\Big] \\ &+ \frac{\lambda_1}{\lambda_2}\Big[\,\mathbf{E}_1^0 G + \mathcal{R}_{\lambda_2}\big(B_{1,2}(\bullet)\big) + \mathbf{E}_1^0 \mathcal{NA}\Big]\,, \end{aligned} \tag{20}$$

where $\mathbf{E}_1^0\mathcal{NA}$ *is given by (17) and* $\mathcal{R}_\lambda\big(f(\bullet)\big)$ *is defined by (9).*

In the case when $A_{1,2}(r) = a_{1,2} r^{\alpha_{1,2}}$ the integration over r in (17) can be explicitly made leading to

$$\begin{aligned} \mathbf{E}_1^0 \mathcal{NC} &= 4\pi\lambda_0\lambda_2 a_{1,2}\Gamma\Big(\frac{\alpha_{1,2}+3}{2}\Big) \\ &\qquad \int_0^\infty \rho\Big(\pi - \mathbb{I}\{\rho > 1/2\}\arccos\frac{1}{2\rho}\Big)\Big(\lambda_1\pi\rho^2 + \lambda_2 A(\rho)\Big)^{-\frac{\alpha_{1,2}+3}{2}} d\rho \\ &= \frac{2\lambda_0\lambda_2 a_{1,2}\Gamma\Big(\frac{\alpha_{1,2}+1}{2}\Big)}{\lambda_1\pi^{\frac{\alpha_{1,2}-1}{2}}}\Big[\lambda_2^{-\frac{\alpha_{1,2}+1}{2}} - (\lambda_1/4+\lambda_2)^{-\frac{\alpha_{1,2}+1}{2}}\Big] \\ &+ 2^{\alpha_{1,2}+3}\pi\lambda_0\lambda_2 a_{1,2}\Gamma\Big(\frac{\alpha_{1,2}+3}{2}\Big) \\ &\qquad \int_0^{\pi/2} \frac{(\pi-\gamma)\sin\gamma\cos^{\alpha_{1,2}}\gamma\, d\gamma}{[\pi(\lambda_1+2\lambda_2) + \lambda_2\sin 2\gamma + 2\lambda_2(\pi-\gamma)\cos 2\gamma]^{\frac{\alpha_{1,2}+3}{2}}}\,, \end{aligned} \tag{21}$$

where $\gamma = \arccos\frac{1}{2\rho}$. Using

$$\mathcal{R}_\lambda(ar^\alpha) = a(\pi\lambda)^{-\alpha/2}\Gamma(\alpha/2+1)\,,$$

an explicit expression for the cost is obtained after substitution of (21) into (20).

In particular, if all $\alpha_{i,j} = \beta_{i,j} = 1$ we get

$$\begin{aligned} \mathbf{E}_2^0 G = C_2 &+ \frac{\lambda_1}{\lambda_2}C_1 + \frac{\lambda_0(2\lambda_1+\lambda_2)}{\lambda_2(\lambda_1+\lambda_2)}C_0 + \frac{b_{1,2}\lambda_1}{2\lambda_2^{3/2}} + \frac{2\lambda_0\lambda_1 a_{1,2}}{\lambda_2(\lambda_1+4\lambda_2)} \\ &+ 8\pi\lambda_0\lambda_1 a_{1,2}\int_0^{\pi/2}\frac{(\pi-\gamma)\sin 2\gamma\, d\gamma}{[\pi(\lambda_1+2\lambda_2)+\lambda_2\sin 2\gamma + 2\lambda_2(\pi-\gamma)\cos 2\gamma]^2} \\ &+ \frac{\lambda_0(a_{0,2}+b_{0,2})}{2(\lambda_1+\lambda_2)^{3/2}} + \frac{\lambda_0\lambda_1(a_{0,1}+b_{0,1})}{2\lambda_2(\lambda_1+\lambda_2)^{3/2}}\,. \end{aligned} \tag{22}$$

4.3 Numerical examples and comparison

Unfortunately, there are no closed form expressions of the optimal value of parameters for this model. Figure 4 shows a numerical example for the costs of the strictly hierarchical model and the model with direct connections, for the following choice of the parameters.

- intensities: $\lambda_0 = 10.91,\ \lambda_2 = 0.018\,;$
- station costs: $C_0 = 4,\ C_1 = 18\,;$

- capacity and infrastructure cost functions: $A_{i,j}(r) = 4r,\ B_{i,j}(r) = 9r$ for all $i, j \in \{0, 1, 2\}$.

(the second level station cost C_2 is not pertinent for the optimal λ_1 and can be taken equal to 0). As we see, for these values, allowing direct connections always improves the cost of

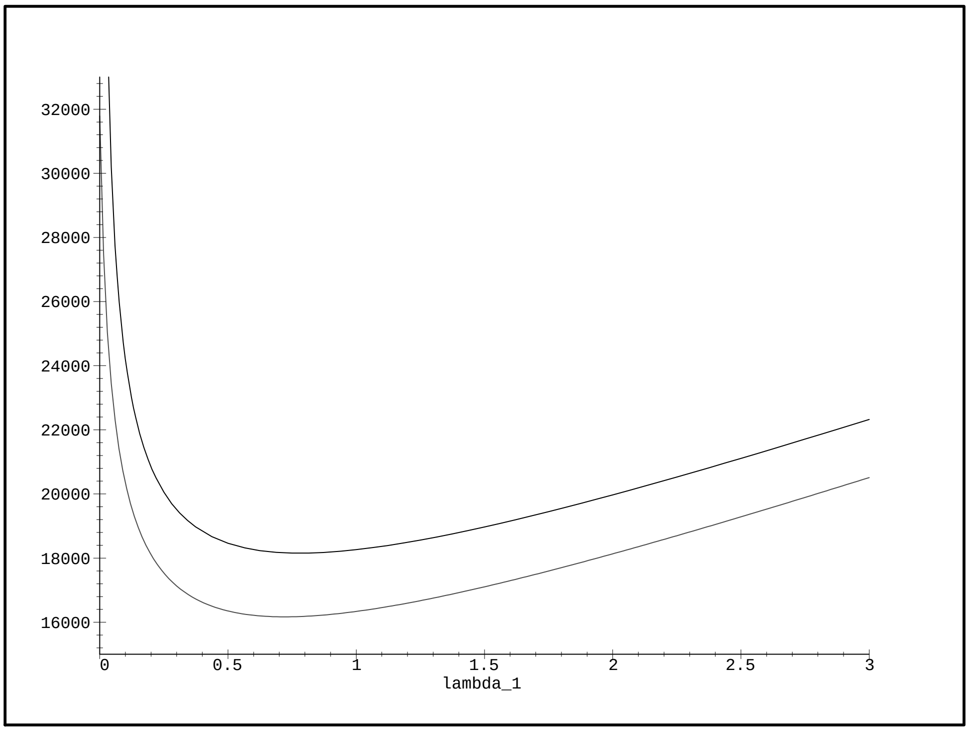

Figure 4: Cost function in the hierarchical model (the upper curve) and the model with direct connections ($a_{i,j} = 4,\ b_{i,j} = 9$).

the system. The optimal value of the intensity of Π_1 in the strictly hierarchical model is $\lambda_1 \approx 0.779$, with cost $\mathbf{E}_2^0 G \approx 18154$. For the model with direct connections, the optimal value is $\lambda_1 \approx 0.721$, and the cost is $\mathbf{E}_2^0 G \approx 16168$.

Note the difference in behavior of the cost functions for small λ_1. For λ_1 small, there are mostly direct connections between 0 and 2-nd level stations, if they are allowed. The topology of the system becomes star-shaped as in two level systems. The cost of the model with direct connections is $\mathbf{E}_2^0 G \approx 31789$ for $\lambda_1 = 0$. In contrast, in the strictly hierarchical model, the 0-level stations are connected to 1-st level stations even if they are far away, which makes the cost of the system explode as λ_1 vanishes.

In another example we take the same values of the parameters except for $B_{0,1}(r) = B_{0,2}(r) = 0.1r$. As in the previous example, direct connections improve the cost. But although for this model there is a local minimum of the cost function in point $\lambda_1 \approx 0.233$, the global minimum of the cost $\mathbf{E}_2^0 G \approx 11686$ is attained in $\lambda_1 = 0$, i.e when there are no distribution points at all (see Figure 5).

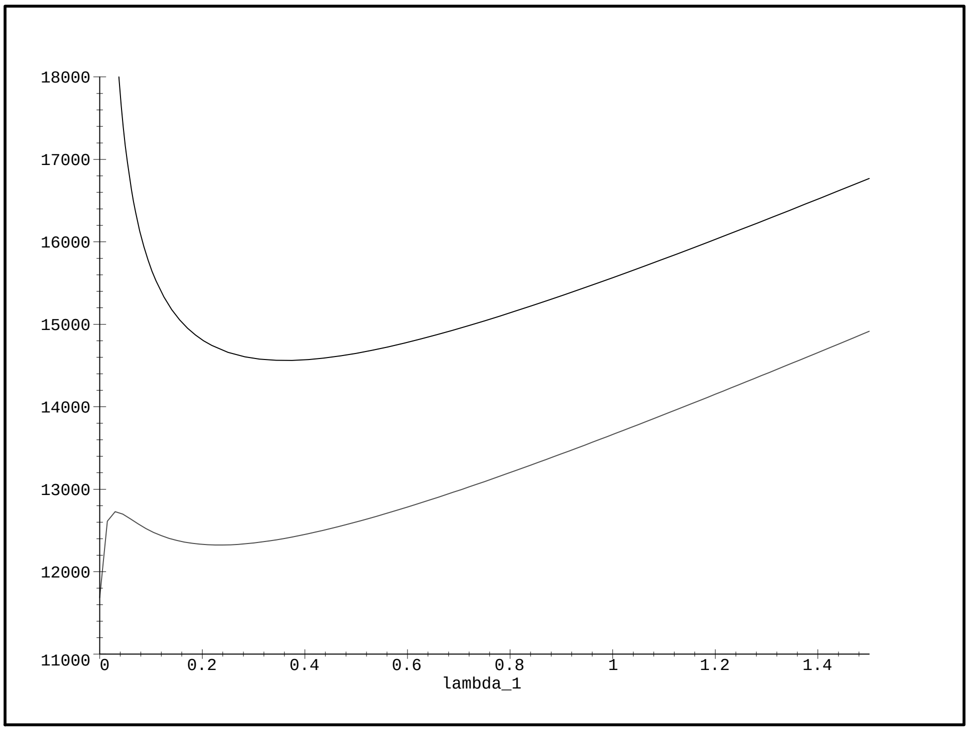

Figure 5: Cost function in the hierarchical model (the upper curve) and the model with direct connections with a small infrastructure cost on the lowest level ($a_{i,j} = 4,\ b_{0,1} = b_{0,2} = 0.1,\ b_{1,2} = 9$).

In the last example $A_{i,j}(r) = 0.4r$, for all $i, j \in \{0, 1, 2\}$, $B_{0,1}(r) = 0.1r$, $B_{0,2} = 0.4r$ and $B_{1,2} = 9r$. We see that the optimal architecture here is strictly hierarchical, except for very small values of λ_1. The optimal intensity is $\lambda_1 \approx 0.089$ and the minimal cost gives the hierarchical model: $\mathbf{E}_2^0 G \approx 4091$ (see Fig. 6). The corresponding optimal topology is visualized on Figure 1 and was drawn with the help of the program ARC (see Zuyev, Desnogues and Rakotoarisoa (1996)).

5 Related spanning problems

The methods considered in the previous sections can easily be adapted to other applications. Two examples of such applications are considered below.

5.1 Localization in mobile communication systems

Consider two independent Poisson point processes Π_0 and Π_1 representing respectively the positions of mobile subscribers at time 0 and the positions of the *base stations* of the network.

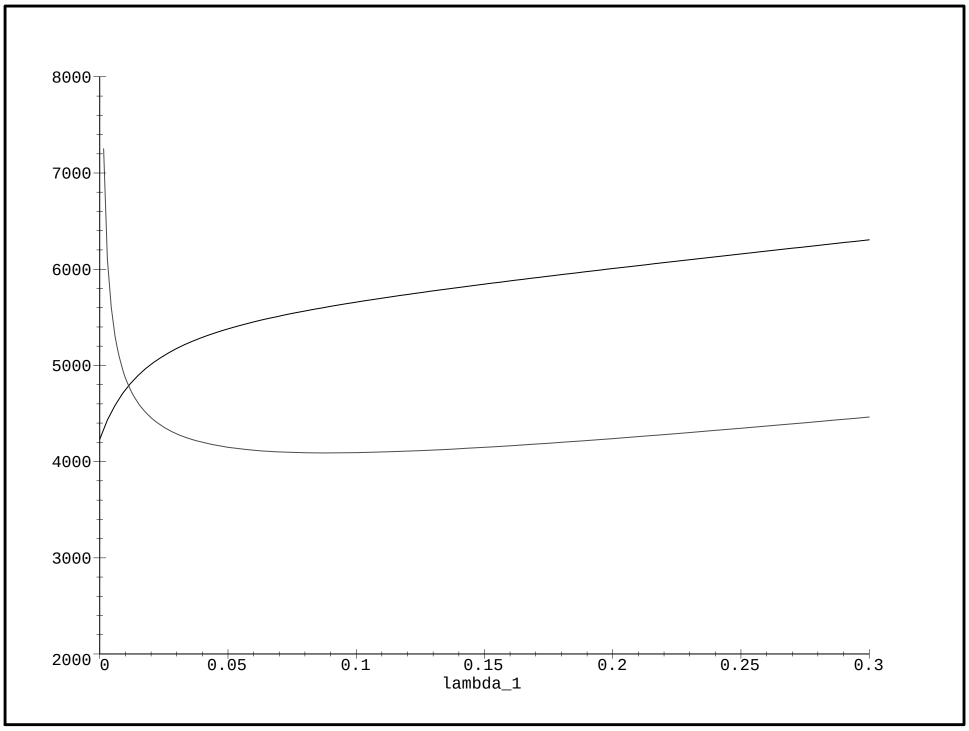

Figure 6: Example when the strictly hierarchical model provides the optimal cost ($a_{i,j} = 0.4,\ b_{0,1} = 0.1,\ b_{0,2} = 0.4,\ b_{1,2} = 9$).

The base station located in point x of Π_1 is an antenna allowing for hertzian communication with any mobile located inside the Π_1-Voronoi cell with nucleus x. In case attenuation increases with distance in a translation-invariant and isotropic way, and if all stations have the same emission power, this cell is also the part of the plane where the reception level of the antenna located in x is best.

However the mobility of the points of Π_0 (which we will discuss below) creates a new problem which is that of *mobile localization*. The system has to know where each mobile is located at any given time. This is particularly true for mobiles which are not in communication. Should any of those be called, the system has to be able to find the cell where the mobile is located at the time of the call.

This is done by updating the position of each mobile in some database, which is organized as follows: there is a third level of stations which we will refer to as *location* stations. This level will be represented as an independent Poisson process Π_2. The database is updated each time a mobile crosses a Π_2–cell boundary. So, one always knows the Π_2–cell in which the called mobile is located. In case of a call for a mobile, it is then enough to broadcast a search message to all base stations in this Π_2 cell to find the mobile and to establish the communication.

There are two costs associated with this scheme:

1. an *update cost* which comes from the update messages sent by all mobiles to the database; we will represent this as a linear function of the number of mobiles crossing Π_2–cell boundaries per unit of time; let X denote the cost of one update message.

2. a *search cost* which comes from the search messages sent to all called mobiles; let $Y(n)$ denote the cost to broadcast the search messages for a mobile located in a Π_2–cell with n base stations.

Here, the problem consists in finding the intensity of the location stations, which minimizes the localization cost.

We now give a precise definition of the cost function.

Assume that the reference probability space carries three independent point processes Π_0, Π_1 and Π_2, together with some marks associating a random motion in $\mathbb{R}^2$, with each point of Π_0. Below, Π denotes a realization of these point processes together with their marks.

We assume that these marks are such that the *temporal* point process which gives the epochs when the mobile initially located in 0 crosses Π_2-cell boundaries, is stationary.

A simple example is that when the mark is chosen independently for each mobile and represents the trajectory of the mobile together with its velocity v, which it keeps constant. It is naturally to assume that the trajectory of a mobile is continuous and piecewise 1-differentiable. From Møller (1989), Formula (7.22), the intensity of intersections of Π_2-cell boundaries with a straight line is equal to $4\lambda_2^{1/2}/\pi$. Therefore approximating each differentiable part of the trajectory by a segment, we see that the mean number of crossings during time Δt equals $4v\Delta t\sqrt{\lambda_2}/\pi + o(\Delta t)$. Thus the *temporal* intensity of the cell boundary crossings by a mobile is given by

$$\mathcal{X} = \frac{4v\lambda_2{}^{\frac{1}{2}}}{\pi}$$

(see also Mecke (1981) on intersections of a stationary fiber process with curves).

On the Palm space of Π_0, let

- $\mathcal{X}(\Pi)$ be the temporal intensity of the crossings of Π_2–cell boundaries by the mobile located in 0.

- $\mathcal{Y}(\Pi)$ be the search cost of the mobile located in 0.

If we denote μ the intensity of calls directed to mobile 0 (we assume that this is a constant), then the localization cost per unit of time and space is naturally defined as

$$\mathbf{E}\,G = \mathbf{E}\int_{[0,1]^2} \left(\mathcal{X}(\theta_x\Pi) + \mu\mathcal{Y}(\theta_x\Pi)\right)\Pi_0(dx), \tag{23}$$

that is, using Campbell's refined formula and the independence assumptions

$$
\begin{aligned}
\mathbf{E}\, G &= \lambda_0\, \mathbf{E}_0^0[\mathcal{X}(\Pi)] + \lambda_0 \mu\, \mathbf{E}_0^0[\mathcal{Y}(\Pi)] \\
&= \lambda_0 X \frac{4\overline{v}{\lambda_2}^{\frac{1}{2}}}{\pi} + \lambda_0 \mu\, \mathbf{E}[Y(\Pi_1(\tilde{V_0}(\Pi_2))] \,, \qquad (24)
\end{aligned}
$$

where $\overline{v}$ is the mean velocity of the mobiles and $\tilde{V_0}(\Pi_2)$ is the Voronoi cell constructed with respect to Π_2 *containing* the origin in its interior. In case $Y(n) = Y \cdot n$ we have

$$
\begin{aligned}
\mathbf{E}\, \Pi_1(V_0(\Pi_2)) &= \mathbf{E} \sum_{y_i \in \Pi_1} \sum_{z_j \in \Pi_2} \mathbb{I}\{y_i \in V_{z_j}(\Pi_2)\}\ \mathbb{I}\{0 \in V_{z_j}(\Pi_2)\} \\
&= \lambda_1\, \mathbf{E} \sum_{z_j \in \Pi_2} \int \mathbb{I}\{y \in V_0(\theta_{z_j}\Pi_2)\}\ \mathbb{I}\{-z_j \in V_0(\theta_{z_j}\Pi_2)\}\, dy \\
&= \lambda_1 \lambda_2\, \mathbf{E}_2^0 \iint \mathbb{I}\{y \in V_0(\Pi_2)\}\ \mathbb{I}\{z \in V_0(\Pi_2)\}\, dydz \\
&= \lambda_1 \lambda_2\, \mathbf{E}_2^0\, |V_0(\Pi_2)|^2 \,,
\end{aligned}
$$

since $V_0(\Pi_2)$ and $-V_0(\Pi_2)$ have the same distribution. Using the fact that $\mathbf{E}_2^0\, |V_0(\Pi_2)|^2 = 35/(8\pi^2\lambda_2^2)$ (see e.g. Okabe, Boots and Sugihara (1992), p. 324) we obtain the following explicit expression for the cost function:

$$
\mathbf{E}\, G = H_1 {\lambda_2}^{\frac{1}{2}} + H_2 \frac{\lambda_1}{\lambda_2} \qquad (25)
$$

with

$$
H_1 = \frac{4\overline{v}\lambda_0 X}{\pi}, \quad H_2 = \frac{35}{8\pi^2}\lambda_0\lambda_1\mu Y
$$

and the minimum is reached for

$$
\lambda_2^* = \left(\frac{2H_2}{H_1}\right)^{\frac{2}{3}} = \left(\frac{35\lambda_1\mu Y}{32\pi\overline{v}X}\right)^{\frac{2}{3}} .
$$

Remark 2. Since the location stations are used to broadcast search messages to the antennas and eventually to concentrate communications from or to mobiles connected to these antennas, they can also be seen as a higher level of stations allowing to reach the base stations according to the least distance connection principle. So, one can also naturally consider a cost associated with these connections which is similar to what was considered in the previous sections, namely a capacity cost and an infrastructure cost.

5.2 Infinite spanning graphs

In all preceding cases, we assumed that the points of highest level (e.g. the roots of the trees) were linked via some connected network, which was then ignored. The aim of this subsection is to take this additional structure into account within our parametric setting, and to show that optimization problems similar to those considered above can be also addressed.

Consider the following problem which is an infinite volume analog of the classical minimal spanning graph problem in a bounded region (cf. Steele (1988)). Let Π_λ be a homogeneous Poisson process with intensity λ in $\mathbb{R}^2$. Can we find a connected planar graph with vertices in all the points of the Poisson process with minimal "connection cost density" that is such that the cost of the edges of the graph per unit area is minimal. We have put quotes because such a quantity is only well defined for finite graphs, and it may not exist for the infinite volume case.

Suppose that for all realizations of a homogeneous Poisson process Π_ν with intensity ν, we are given a rule to construct a connected graph $\mathcal{G}(\Pi_\nu)$ with vertex set Π_ν. Let $S_N,\ N \in \mathbb{N}$ be a family of compact sets, where S_1 is a 1-connected figure with unit area containing the origin in its interior, and S_N is homothetic to S_1 with coefficient $N^{1/2}$. Let $L_N(\Pi_\nu)$ be the total length of (the parts of) the graph's edges lying in S_N and assume that the following almost sure limit

$$C_{\mathcal{G}} \stackrel{\text{def}}{=} \lim_{N\to\infty} L_N(\Pi_\nu)/(N\nu)^{1/2}$$

exists and does not depend on S_1. Examples of such graphs include the Delaunay triangulation, for which $C_{\mathcal{G}} = 32/(3\pi) \approx 3.395$ (cf. Okabe, Boots and Sugihara (1992), p.324). Another example is the optimal spanning tree, for which $C_{\mathcal{G}} \approx 0.656$ (see Roberts (1968); the existence of the limit was proved in Steele (1988). As already mentioned above, there is a basic difference between this and the preceding case as minimal spanning trees are only well defined in the finite volume case.

Of course the optimal spanning tree does much better than the the Delaunay triangulation. Nevertheless, such optimal spanning trees are rarely used in public communication networks due to their lack of robustness to the addidition or deletion of points and to reliability questions. In fact the Delaunay triangulation or a subset of it seems to be a good first approximation of what is observed in existing communications networks.

Consider now the following variant of the basic problem of §2-3: we split the initial process Π_λ into independent subprocesses $\Pi_1, \ldots, \Pi_n$ with intensities $\lambda_1, \ldots, \lambda_n$, $\sum_{i=1}^n \lambda_i = \lambda$, say by a Bernoulli thinning. Then the spanning graph consists of the set of Poisson-Voronoi hierarchical spanning trees rooted in the points of the process Π_n (see Section 3) and the graph $\mathcal{G}(\Pi_n)$ at the highest level. We assume that the cost of a tree edge is its length and the cost of an edge of $\mathcal{G}$ is its length times a weight σ.

By Theorem 1 the edge-cost density of this spanning graph is

$$F = \sum_{k=2}^{n} \frac{\lambda_{k-1}}{2\sqrt{\lambda_k}} + \sigma C_{\mathcal{G}} \sqrt{\lambda_n}$$

and the problem now consists in finding the parameters $n, \lambda_1, \ldots, \lambda_n$ which minimize the value of F.

Numerical evidence shows that the optimal value n of the number of levels is closely related to the value of $\sigma C_{\mathcal{G}}$. The larger $\sigma C_{\mathcal{G}}$, the larger the optimal value of n and the larger the optimal proportion of points connected through the trees. On the opposite, if $\sigma C_{\mathcal{G}} < 1$

then $n = 1$ is optimal and the graph $\mathcal{G}(\Pi_\lambda)$ itself is the optimal topology in the described class.

If $\mathcal{G}(\Pi)$ is the Delaunay triangulation on the point set $\operatorname{supp}\Pi$, and if $\sigma = 1$, then the optimal values of the parameters are the following: $n = 3$, $\lambda_1 \approx 0.7121\lambda$, $\lambda_2 \approx 0.2466\lambda$ and $\lambda_3 \approx 0.0413\lambda$. In this case the minimal value of the cost is approximately $2.014\sqrt{\lambda}$.

5.3 Simulation approach

Quite often analytical expressions for the average cost functions of stochastic models of the considered type are difficult to compute, even in the case considered here where all point processes are Poisson processes. In many such cases, the stochastic gradient method introduced in Baccelli, Klein and Zuyev (1995) can then be applied for estimating the optimum. It consists in finding efficient ways of calculating or of estimating the gradients $\frac{\partial}{\partial \lambda_i} \mathbf{E}\, G$ and then using steepest descent type algorithms to find the optimum. For details and examples of applications of this technique in telecommunication studies see Zuyev, Desnogues and Rakotoarisoa (1996) and the references therein.

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