STOCHASTIC GEOMETRY AND ARCHITECTURE OF CELLULAR COMMUNICATION NETWORKS

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Problem

- Strategic planning of telecommunication networks
  - characterization of future demand in communications
  - evaluation of network evolution scenarios
  - choice between potential architectures

Necessity of Global Models capturing the variability of spatial characteristics

State of the Art

- Regular lattices (honeycomb, square, triangular) as a model of station zones (cells)
- No structural variability of the characteristics of the cells

Probabilistic models of network structure – general idea

– Positions of stations (concentrators, antennas) and of subscribers (fixed, mobile) in plane are considered as realizations of stochastic point processes

– Other characteristics (routing of calls, traffic etc.) are functionals of these processes

Interest of probabilistic models

– Reduction of the number of structuring parameters

– Analysis and Optimisation of the network as a function of its parameters

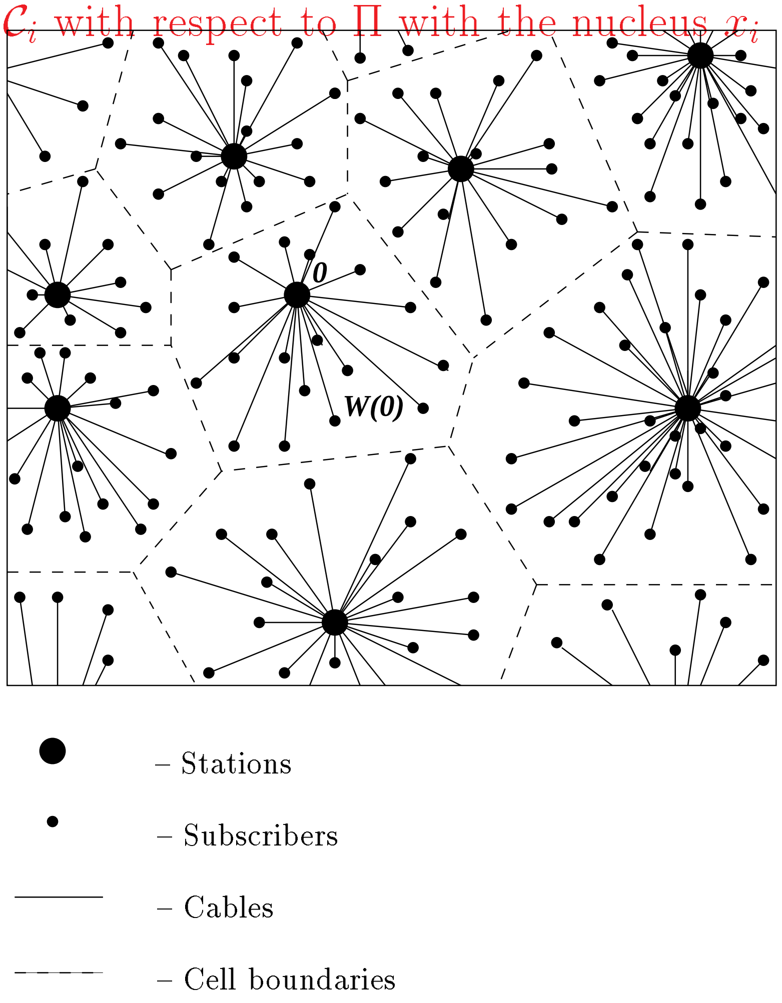
Basic Cellular Network Models

Voronoi Cells

Π is a point process in $\mathbf{R}^2$ describing positions of stations.

The cell of a station $x_i \in \Pi$ consists of the points of the plane, which are closer to x_i than to any other $x_j \in \Pi$

Voronoi cell $\mathcal{C}_i$ with respect to Π with the nucleus x_i

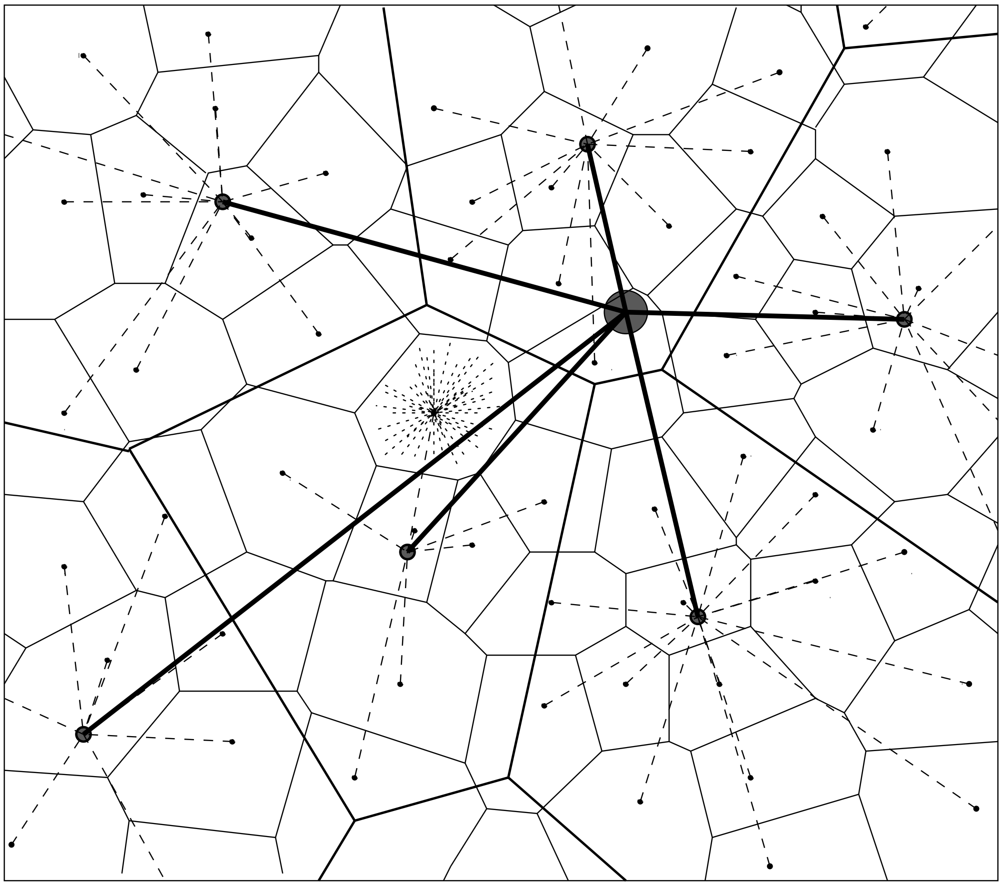


Hierarchical Model

Π_i, $i = 0, 1, \ldots, N$ point processes describing positions of stations of i-th level (0-level = subscribers)

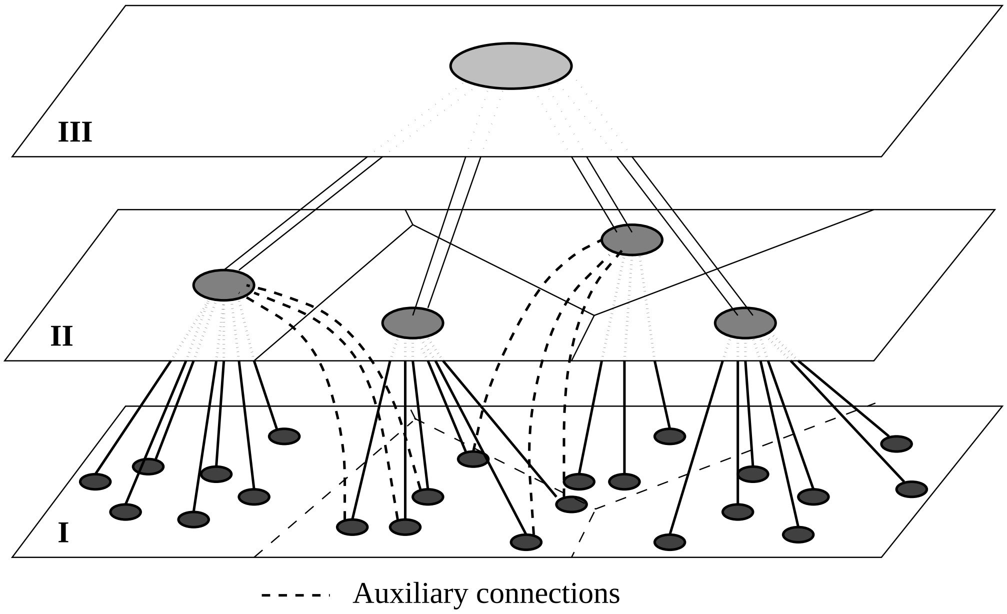
Poisson Model: Π_i are independent homogeneous Poisson processes of intensity λ_i ($\lambda_i >> \lambda_{i+i}$).

Hierarchical connections the i-th level stations in a cell of a $(i+1)$-st level station are directly connected to the latter.



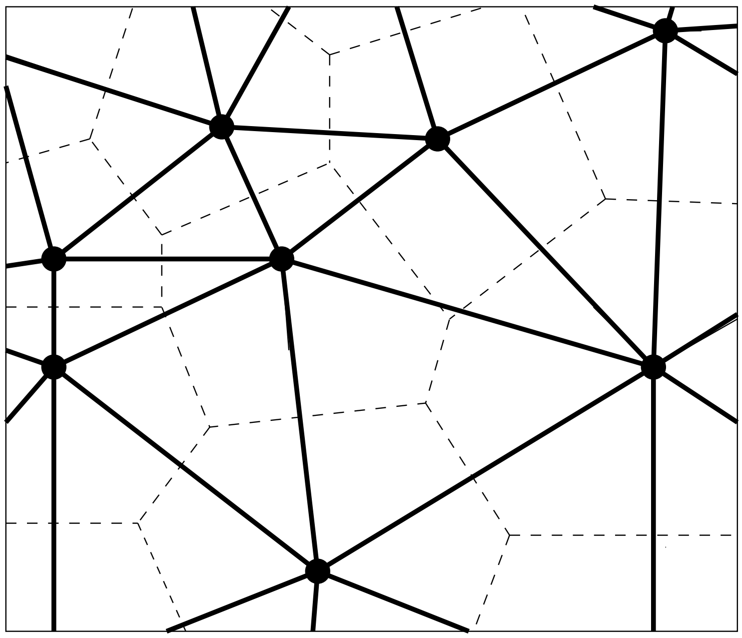
Hierarchical Model (*continued*)

Auxiliary connections to the 2-nd closest station of the next level:



Hierarchical Model (*continued*)

Auxiliary connections at highest level



– stations

-- connections between stations via Delaunay triangulation

-- borders of the Voronoi cells

Analytical Results – Poisson Case

Cell Characteristics

Many analytical results on Voronoi tessellations: Gilbert 62, Møller 89 94, Z 94

Additive functionals

$$\Sigma_{x_j \in \Pi_i} f(x_j) 1\{x_j \in \mathcal{C}\}$$

for a given function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$, where $\mathcal{C}$ is a "typical" $(i+1)$-level cell
Examples:

- $\mathcal{N}$ the number of stations of i-level in $\mathcal{C}$ $(f(x) = 1)$;
- $\mathcal{L}_\beta$ cost of all direct connections in $\mathcal{C}$ – if the cost for connecting a level i-station at x to its level $i+1$-station at y is

$$f(x,y) = |x-y|^\beta, \ \beta \geq 1$$

The first and the second moments

There are explicit integral expressions for many additive functionals. In particular

Theorem 1 , *Guilbert 62, Foss & Z 93*

$$(i)\quad \mathbf{E}\mathcal{N} = \frac{\lambda_i}{\lambda_{i+1}}$$

$$\mathbf{E}\mathcal{L}_1 = \frac{\lambda_i}{2\lambda_{i+1}^{3/2}}$$

$$(ii)\quad \mathbf{var}\mathcal{N} = \frac{\lambda_i}{\lambda_{i+1}} + 0.280\frac{\lambda_i^2}{\lambda_{i+1}^2}$$

$$\mathbf{var}\,\mathcal{L}_1 = \frac{\lambda_i}{\pi\lambda_{i+1}^2} + 0.147\frac{\lambda_i^2}{\lambda_{i+1}^3}$$

$$\mathbf{cov}\,(\mathcal{N}, \mathcal{L}_1) = \frac{\lambda_i}{2\lambda_{i+1}^{3/2}} + 0.197\frac{\lambda_i^2}{\lambda_{i+1}^{5/2}}$$

Tail Decays

Theorem 2 , *Foss & Z 93 There exist positive constants* A_1, A_2, C_1, C_2 *depending on* λ_{i+1} *and* λ_i *only, such that*

$$A_1 x^{-1/(2+\beta)} \exp(-C_1 x^{2/(2+\beta)}) < \mathbf{P}\{\mathcal{L}_\beta > x\} < A_2 \exp(-C_2 x^{2/(2+\beta)})$$

for all $x > 0$.

Corollary 1 *for all positive* x

$$A_1 x^{-1/3} \exp(-C_1 x^{2/3}) < \mathbf{P}\{\mathcal{L}_1 > x\} < A_2 \exp(-C_2 x^{2/3})$$

For all positive n

$$(1 + 4\lambda_{i+1}/\lambda_i)^{-n} \leq \mathbf{P}\{\mathcal{N} \geq n\} < C^6_{n+6}(1 + 0.346\lambda_{i+1}/\lambda_i)^{-n}$$

Traffic Matrices

Assumption Communications are established according to the hierarchical principle

Basic Construct

- $h(s^{(i)}, s^{(j)})$ height of a communication between $s^{(i)}$ and $s^{(j)}$
- $H_{ij}(r,h)$ the probability, that station $s^{(i)}$ has a communication of height h with some $s^{(j)}$ at the distance r

Theorem 3 , *B & Z 94*

$$H_{ij}(r,h) = \begin{cases} 0, & \textit{if } h < i \vee j \textit{ or } h = i = j < N; \\ \exp(-\lambda_{i\vee j}\pi r^2), & \textit{if } h = i \vee j < N,\ i \neq j; \\ 1, & \textit{if } h = i \vee j = N; \end{cases}$$

$$H_{ij}(r,h) = \begin{cases} (1-\exp(-\lambda_{i\vee j}\pi r^2))\, Q(r\sqrt{\lambda_h})\, \Pi_{m=i\vee j+1}^{h-1}(1-Q(r\sqrt{\lambda_m})), \\ \qquad \textit{if } h = i \vee j + 1, \ldots, N-1,\ i \neq j; \\ (1-\exp(-\lambda_{i\vee j}\pi r^2))\, \Pi_{m=i\vee j+1}^{N-1}(1-Q(r\sqrt{\lambda_m})), \\ \qquad \textit{if } h = N,\ i \neq j; \end{cases}$$

$$H_{ii}(r,h) = \begin{cases} Q(r\sqrt{\lambda_h})\, \Pi_{m=i+1}^{h-1}(1-Q(r\sqrt{\lambda_m})), & \textit{if } h = i+1, \ldots, N-1; \\ \Pi_{m=i+1}^{N-1}(1-Q(r\sqrt{\lambda_m})), & \textit{if } h = N. \end{cases}$$

$Q(\rho)$ probability that two fixed points at the distance ρ belong to the same Voronoi cell

Traffic Matrices (*continued*)

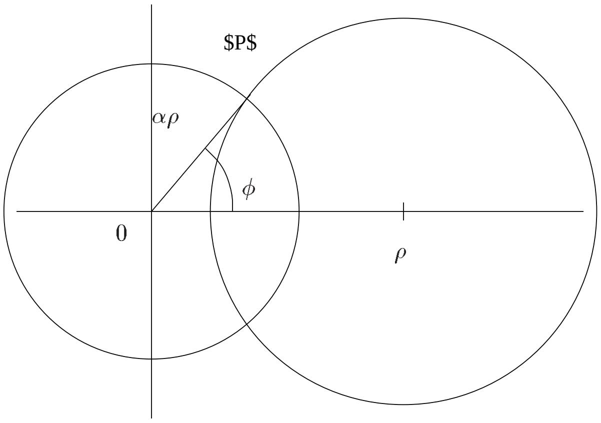
Lemma 1

$$Q(\rho) = 2\rho^2 \int_0^{\pi} d\phi \int_0^{\infty} \alpha \exp(-\rho^2 A(\alpha, \phi))\, d\alpha$$

where $A(\alpha, \phi))$

$$= \alpha \sin\phi + \alpha^2(\pi - \phi) + (\alpha^2 + 1 - 2\alpha\cos\phi)\left(\pi - \arccos\frac{1 - \alpha\cos\phi}{\sqrt{\alpha^2 + 1 - 2\alpha\cos\phi}}\right)$$

Two points 0 and $(0, \rho)$ belong to the same Voronoi cell if there exists a concentrator P that lies in the intersection of the boundaries of two empty discs centered in 0 and $(0, \rho)$ respectively. The



probability of having these discs empty given the position of P is

$$\exp(-A(\alpha, \phi))$$

$A(\alpha, \phi)$ is the area of this union of discs

Traffic Matrices (*continued*)

Assumptions A level i-station $s^{(i)}$ asks for a communication with a level j station $s^{(j)}$ at distance r with temporal intensity $f_{ij}(r)$, independently of all the rest

Traffic Matrices

- Mean number of communication requests of $s^{(i)}$ to stations of level j with height $h \geq i \vee j$ per time unit:

  $$M_{iJ}(h) = 2\pi\lambda_j \int_0^{+\infty} r f_{ij}(r) H_{ij}(r,h)\, dr.$$

- Mean number of level h-communications in progress initiated by $s^{(i)}$ If the process of requests is Poisson (M/G/∞)

  $$2\pi \sum_{j=0}^{N} \lambda_j \int_0^{+\infty} r f_{ij}(r)\, \overline{T}_{ij}(r,h)\, H_{ij}(r,h)\, dr.$$

  $\overline{T}_{ij}(r,h)$: mean duration of a communication

Mobile Communications

Mobile Models

Basic Data for Traffic on Roads A random line process with phase space: $\mathbf{R} \times [0,\pi) \times \mathcal{S} \times \mathcal{T}$

- $D = (p,\alpha) \in \mathbf{R} \times [0,\pi)$ distance to the line (road) and its inclination;
- $\sigma \in \mathcal{S}$ type of road (highway, one-way road etc.);
- $\tau \in \mathcal{T}$ traffic characteristics (density, distribution of positions, velocity etc.)

Main assumption Poisson process with the intensity measure

$$\lambda_r \, dp \, \mathcal{O}(d\alpha) \, Q(\alpha, d\sigma) \, T(\alpha,\sigma,d\tau),$$

λ_r intensity of roads;
$\mathcal{O}(d\alpha)$ distribution of the orientation (if uniform – isotropic road system, if discrete with two atoms $\{0\}$ and $\{\pi/2\}$ – Manhattan type).
$Q(\alpha, d\sigma)$ type distribution, depends in general on the orientation.
$T(\alpha,\sigma,d\tau)$ traffic distribution, depends on everything.

Poisson traffic model

– The initial positions of mobiles on a line $(p_i, \alpha_i, \sigma_i)$: homogeneous Poisson point process of intensity $\lambda_t(\alpha_i, \sigma_i)$.

– Distribution of velocities: $\mathcal{V}(\alpha_i, \sigma_i)(dv)$, chosen independently for each vehicle.

Barlett's theorem: *distribution of positions time stationary*: Poisson

Communications from Mobiles Each mobile can be in two alternative states: *call* and *think* modes. The time spent in the think and the call modes have exponential distributions with parameters ν_0, ν_1, respectively, and the modes of different mobiles are independent. $\theta = \nu_0/(\nu_0 + \nu_1)$ is an average proportion of time spent in the call mode in the stationary regime.

Additive functionals

Take $\mathcal{T} = \mathbf{R}_+$. Functionals of the type $\Sigma = \Sigma_i \tau_i$ where τ_i is some non-negative characteristic associated with i-th road.

Theorem 4 , *B & Z 94 Let $\phi_D(t)$ is the Laplace transform of the distribution of $\tau(D) = \tau(p, \alpha, \sigma)$. Then the Laplace transform of Σ is*

$$\Phi(t) = \exp\left\{\int_E (\phi_D(t) - 1)\, M(dD)\right\}$$

where $M(dD) = \lambda_r\, dp\, \mathcal{O}(d\alpha)\, Q(\alpha, d\sigma)$. In particular,

$$\mathbf{E}\Sigma = \int_E \mathbf{E}\tau_D\, M(dD) \ et$$
$$\mathbf{var}\Sigma = \int_E \mathbf{E}\tau_D^2\, M(dD).$$

Proof: (Poisson summation formula)

Applications

Number of Communicating Vehicles in a Cell

- $\mathcal{M}(Z)$ number of communicating mobiles in fixed domain Z
- $l(p,\alpha)$ length of the intersection of Z with the line (p,α)
- $hit(Z) = \{(p,\alpha) : \ l(p,\alpha) \neq 0\}$

In the isotropic Poisson traffic model

$$\mathbf{E}\mathcal{M}(Z) = \theta\lambda_r|Z| \int_{\mathcal{S}} \lambda_t(\sigma)Q(d\sigma)$$

$$\mathbf{var}\mathcal{M}(Z) = \mathbf{E}\mathcal{M}(Z) + \frac{\lambda_r\theta^2}{\pi} \int_{\mathcal{S}} \lambda_t^2(\sigma)Q(d\sigma) \int_{hit(Z)} l^2(p,\alpha)\, d\alpha dp.$$

Tail behavior

For dimensioning the network the following result is important:

Theorem 5 *Let Z be a convex bounded domain, $|\partial Z|$ and $diam(Z)$ being its perimeter and its diameter. Assume that*

$$\lambda_t^* \stackrel{def}{=} \sup_{\mathcal{S}} \lambda_t(\sigma) < \infty$$

Then in the isotropic road model

$$\mathbf{P}\{\mathcal{M}(Z) \geq n\} < U(\lambda_r |\partial Z| \pi^{-1}, \theta \lambda_t^* \, diam(Z), n),$$

$$\text{where } U(a,b,n) = \sqrt{\frac{L(n)}{2\pi n}} \left(\frac{eb}{L(n)}\right)^n (1 + o(1)\,)$$

$L(x) = L_{a,b}(x)$ is the inverse of the function $x(L) = eaL(L/eb)^L$ in the interval $[b, \infty)$.

So the decay rate is superexponential

Integration into the Poisson Random Cell Model

If Z is a typical Voronoi cell $\mathcal{C}$ then

$$\mathbf{E}\mathcal{M}(\mathcal{C}_0) = \frac{\theta\lambda_r\overline{\lambda_t}}{\lambda_c}$$

$$\mathbf{var}\mathcal{M}(\mathcal{C}_0) = \frac{\theta\lambda_r\overline{\lambda_t}}{\lambda_c} + 0.672\frac{\theta\lambda_r\overline{\lambda_t^2}}{\lambda_c^{3/2}} + 0.280\frac{\theta\lambda_r\overline{\lambda_t}}{\lambda_c^2},$$

where

$$\overline{\lambda_t^k} = \int_{\mathcal{S}} \lambda_t^k(\sigma) Q(d\sigma).$$

Number of Crossings of a Cell Border – Handover

$\mathcal{I}(Z)$ number of crossings of the border ∂Z of zone Z by mobiles in communication

$$\overline{M_1} = \int_S \lambda_t(\sigma)\mathbf{E}_\sigma|V|Q(d\sigma);$$
$$\overline{M_2} = \int_S (\lambda_t(\sigma)\mathbf{E}_\sigma|V| + (\lambda_t(\sigma)\mathbf{E}_\sigma|V|)^2)\, Q(d\sigma).$$

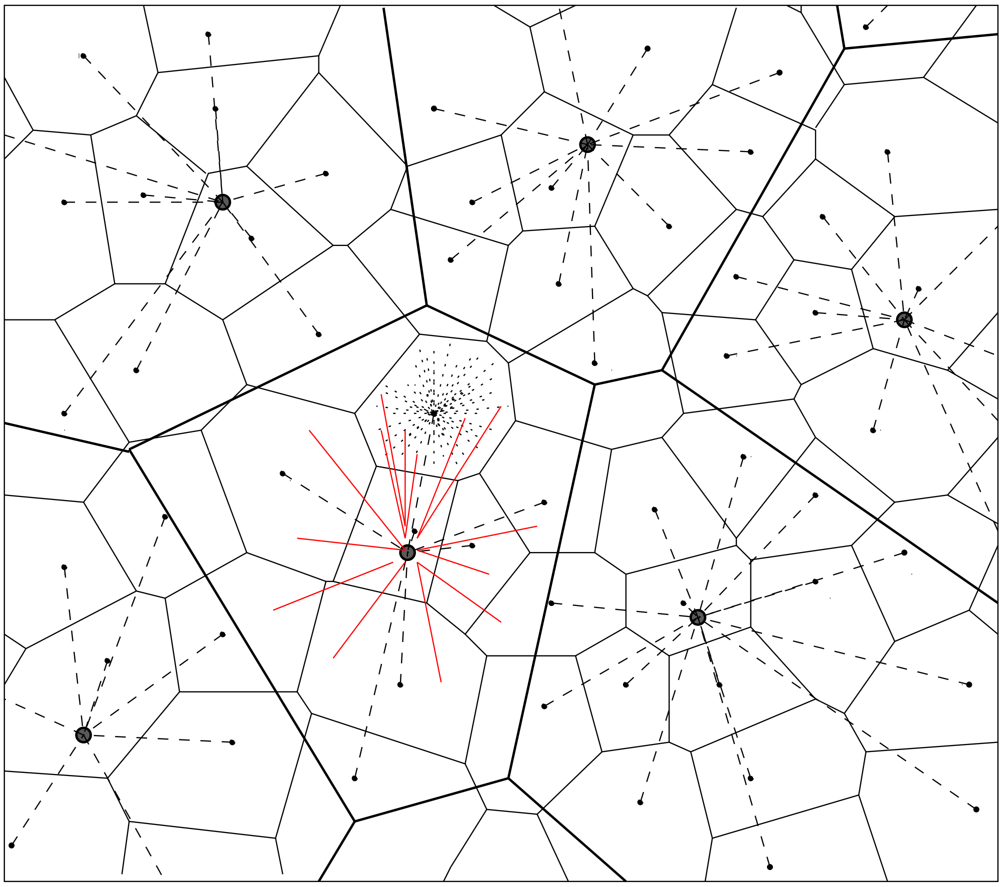
Theorem 6 , *B & Z, 94*

$$\mathbf{E}\mathcal{I}(Z) = \frac{2}{\pi}|\partial Z|\theta\lambda_r\overline{M_1};$$
$$\mathbf{E}\mathcal{I}(\mathcal{C}_0) = \frac{8\theta\lambda_r}{\pi\sqrt{\lambda_c}}\overline{M_1}$$

$$\frac{2}{\pi}\theta\lambda_r|\partial Z|\overline{M_2} \leq \mathbf{var}\mathcal{I}(Z) \leq \frac{4}{\pi}\theta\lambda_r|\partial Z|\overline{M_2};$$
$$\frac{\theta\lambda_r}{\pi}\left[\frac{0.945\overline{M_1}}{\lambda_c} + \frac{8\overline{M_2}}{\sqrt{\lambda_c}}\right] \leq \mathbf{var}\mathcal{I}(\mathcal{C}_0) \leq \frac{\theta\lambda_r}{\pi}\left[\frac{0.945\overline{M_1}}{\lambda_c} + \frac{16\overline{M_2}}{\sqrt{\lambda_c}}\right]$$

Example of Architecture Optimization

Distribution Points



Example of Architecture Optimization (*continued*)

Optimal intensity of distribution points

Cost Structure – Example

1. an additive cost (e.g. cable)
2. a cost with economy of scale (e.g. civil engineering)

Assumption:

$$G = \mathbf{E}_2 \Big\{ D_1 \mathcal{N}_1(0) + \sum_{y_i \in \Pi_1 \cap V_2(0)} \Big[B_{12} |y_i|^{\varepsilon_{12}} + C_{12} |y_i|^{\beta_{12}} \mathcal{N}_0(y_i)$$

$$+ \sum_{x_j \in \Pi_0 \cap V_1(y_i)} \left(B_{01} |x_j - y_i|^{\varepsilon_{01}} + C_{01} |x_j - y_i|^{\beta_{01}} \right) \Big] \Big\}.$$

Let

$$A_1 = \frac{B_{12}}{\pi^{\varepsilon_{12}/2} \lambda_2^{\varepsilon_{12}/2+2}} \Gamma\left(\frac{\varepsilon_{12}}{2} + 1\right) + \frac{D_1}{\lambda_2}; \quad A_3 = \frac{B_{01}\lambda_0}{\pi^{\varepsilon_{01}/2}\, \lambda_2} \Gamma\left(\frac{\varepsilon_{01}}{2} + 1\right);$$

$$A_2 = \frac{C_{12}\lambda_0}{\pi^{\beta_{12}/2}\, \lambda_2^{\beta_{12}/2+2}} \Gamma\left(\frac{\beta_{12}}{2} + 1\right); \quad A_4 = \frac{C_{01}\lambda_0}{\pi^{\beta_{01}/2}\, \lambda_2} \Gamma\left(\frac{\beta_{01}}{2} + 1\right)$$

Theorem 7 , *B & Z, 95 The cost function G has a unique minimal value in λ_1, attained in the point λ_1^* which solves the equation*

$$A_3 \varepsilon_{01} \lambda_1^{-(\varepsilon_{01}/2+1)} + A_4 \beta_{01} \lambda_1^{-(\beta_{01}/2+1)} = 2A_1$$

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