STOCHASTIC DISCRETE EVENT SYSTEMS

F. Baccelli

INRIA Sophia-Antipolis, France

CDC December 1995

1. Part 1 Discrete Event Systems: Petri Nets

    1.1. Manufacturing

    – Kanban, Assembly, Jobshop, Flowshop

2. Part 2 Algebraic Framework

    2.1. The $(\max,+)$ algebra

    2.2. Stochastic Event Graphs as $(\max,+)$-Linear Systems

    2.3. Free Choice nets as $(\min,+,\times)$-Systems

3. Part 3 Timed and Stochastic Event Graphs

    3.1. Stability

    3.2. Simulation

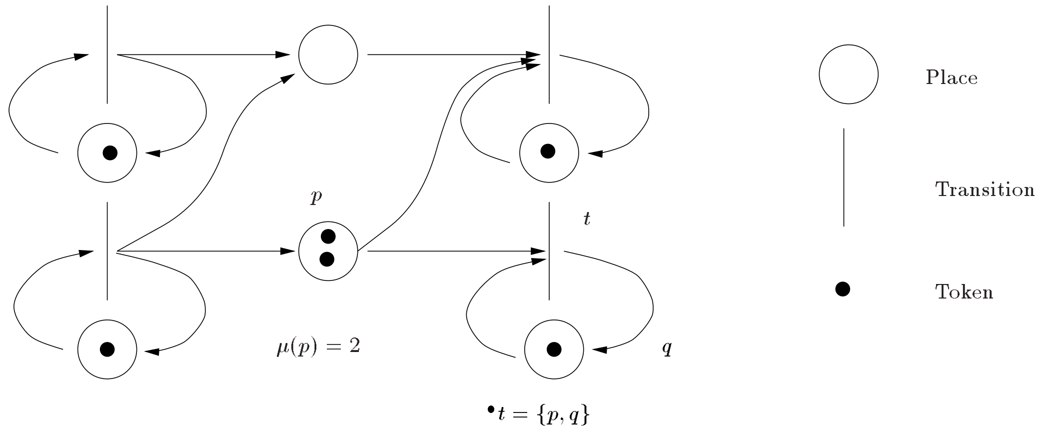
    3.3. Computational aspects

4. Part 4 Timed and Stochastic Free Choice Nets

    4.1. Stability

PETRI NETS AS A GRAPHICAL TOOL FOR DISCRETE EVENT SYSTEMS

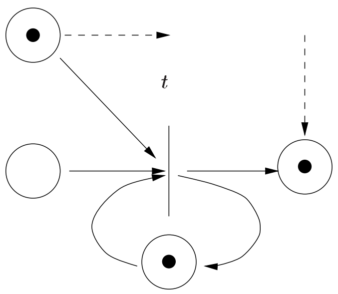
1. Bipartite Directed Graph with two types of nodes
   - $\mathcal{P}$ set of places reservoirs
   - $\mathcal{T}$ set of transitions synchronizers
2. Arcs
   - $p^\bullet$ $({}^\bullet p)$ set of transitions which follow (precede) p
   - $t^\bullet$ $({}^\bullet t)$ set of places which follow (precede) t
3. Initial Marking $\mu : \mathcal{P} \to I\!N$



LOCAL DYNAMICS OF SYNCHRONIZATION

1. Enabling of a transition $\forall p \in \ {}^{\bullet}t, \quad \mu(p) \geq 1$

   Transition t is not enabled



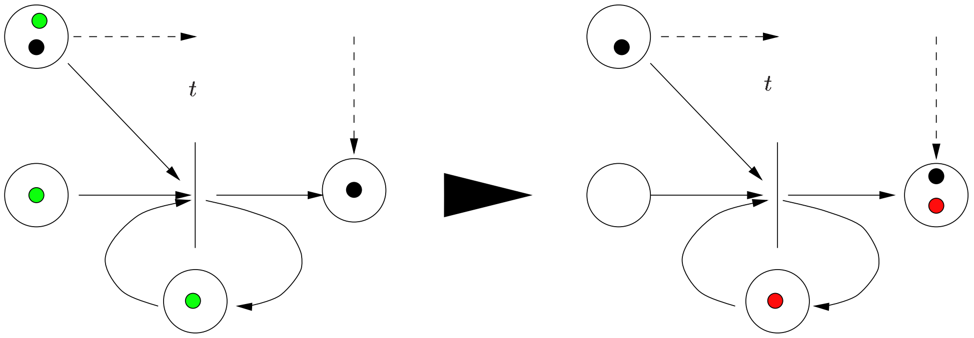
2. Firing of an enabled transition

   Tokens consumed by the firing of t

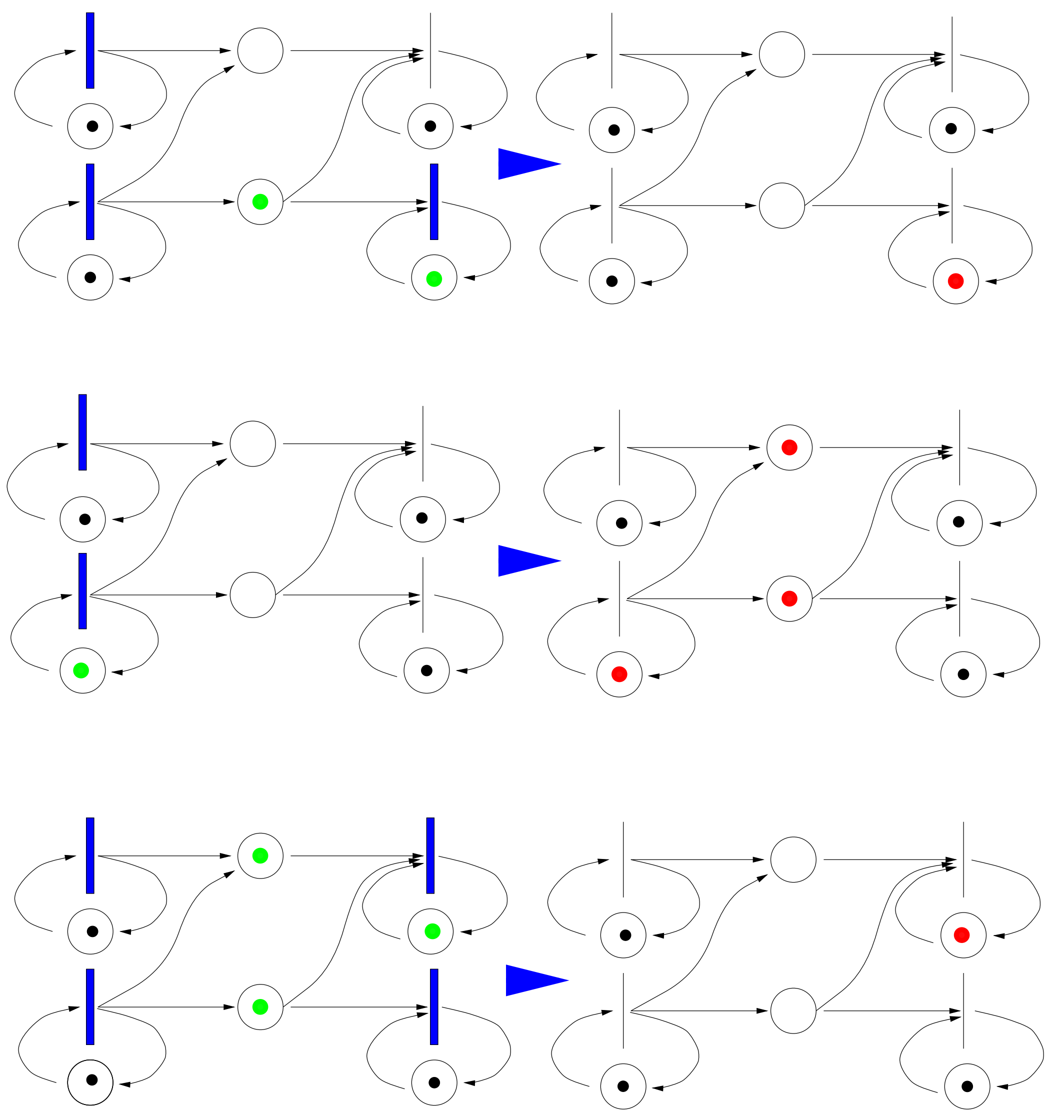
   $\forall p \in^{\bullet} t, \quad \mu(p) := \mu(p) - 1$

   Tokens produced by the firing of t

   $\forall p \in \ t^{\bullet}, \quad \mu(p) := \mu(p) + 1$



EXAMPLE OF GLOBAL DYNAMICS



BASIC CONSTRUCTS

– And Convergence, And Divergence synchronization

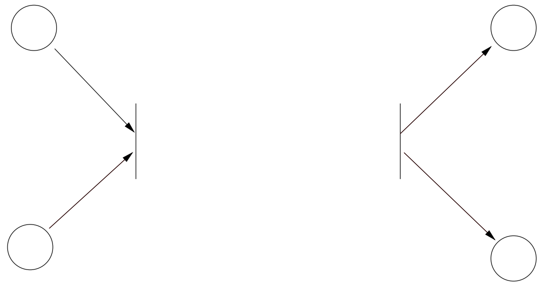


Figure 1: And Convergence (in synchro.) – And Divergence (out synchro.)

– Or Convergence, Or Divergence reservoirs

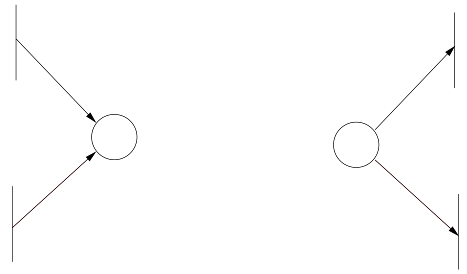


Figure 2: Or Convergence (superposition) – Or Divergence (choice)

CLASSES OF NETS

1. Event Graphs Each place has

   – at most one input transition

   – at most one output transition

   Pure Synchronization

2. Free Choice Nets
   In-Synchronization and Choice are separated

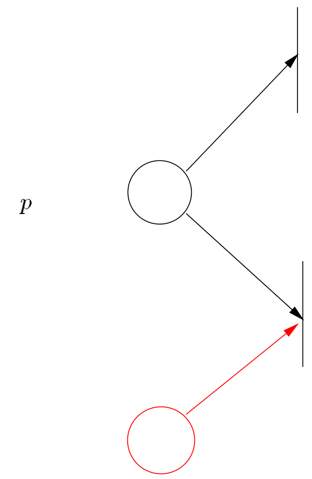


Figure 3: $Card(p^\bullet) > 1$ implies $Card(^\bullet t) = 1$ for all $t \in p^\bullet$

An Event Graph is a Free Choice net

MANUFACTURING EXAMPLE: KANBAN

Event Graph

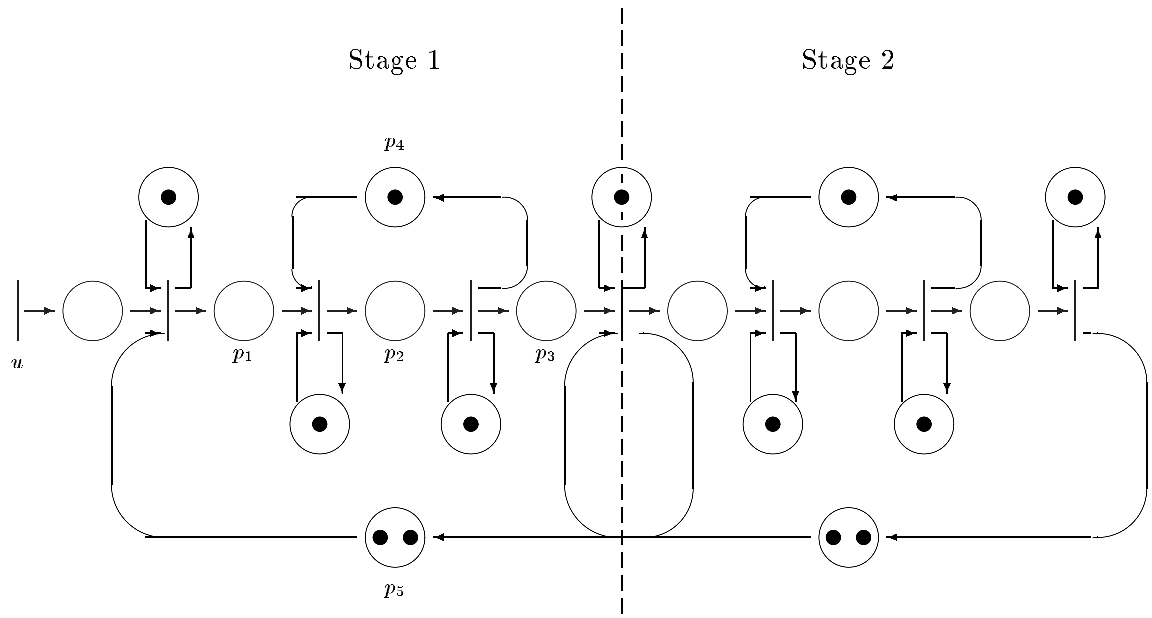


Figure 4: A Two Stage Kanban

– u: orders

– p_1: input buffer of machine 1

– p_2: working buffer of machine 1

– p_3: output buffer of machine 1

– $\mu(p_5)$: number of kanbans of machine 1

MANUFACTURING EXAMPLE: JOBSHOP

Scheduling

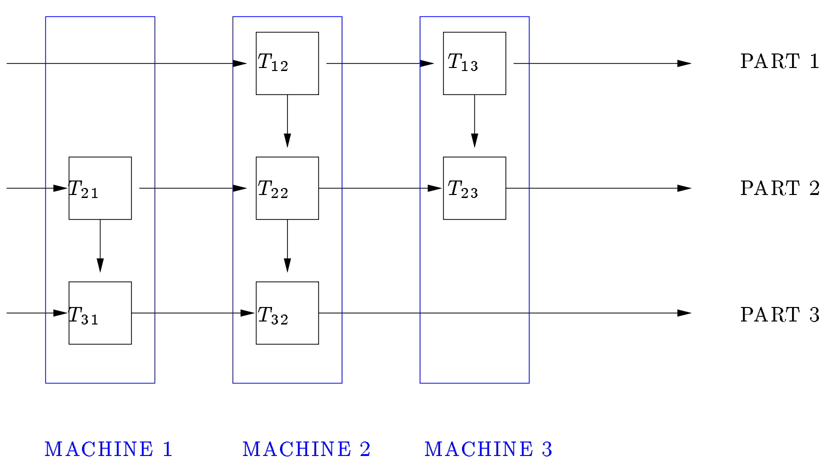


Figure 5: Process Plan

MANUFACTURING EXAMPLE: JOBSHOP

Event Graph

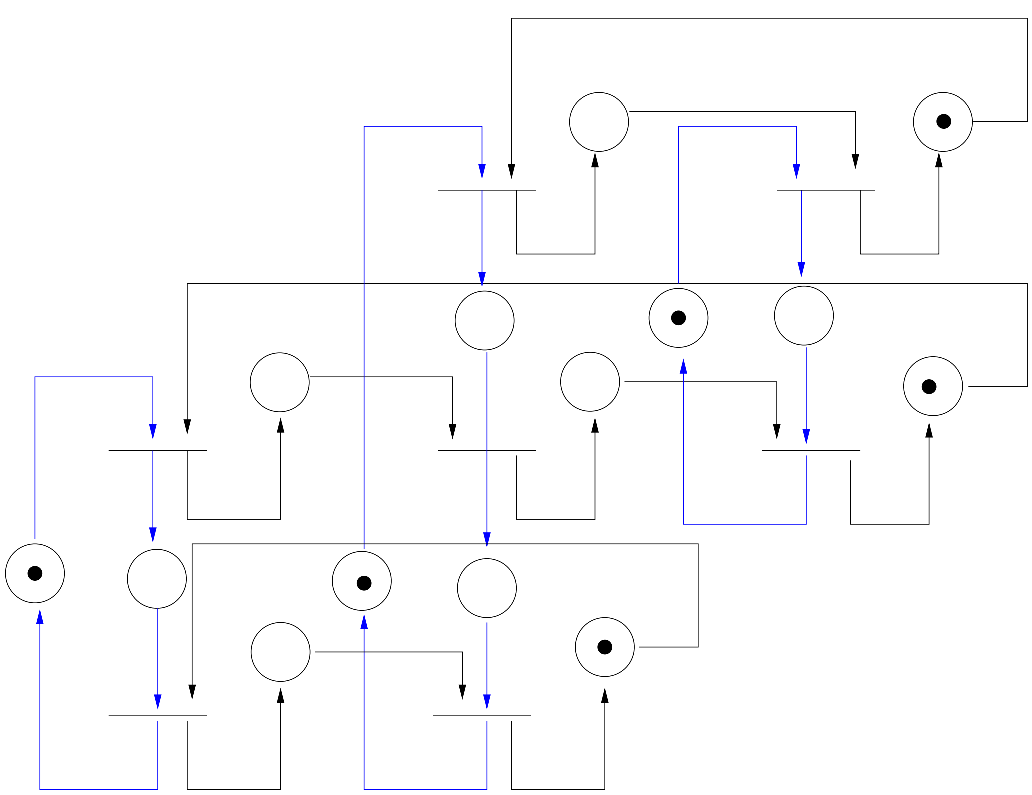


Figure 6: Associated event graph: 1 palet per part

ASSEMBLY/DISASSEMBLY

Event Graph

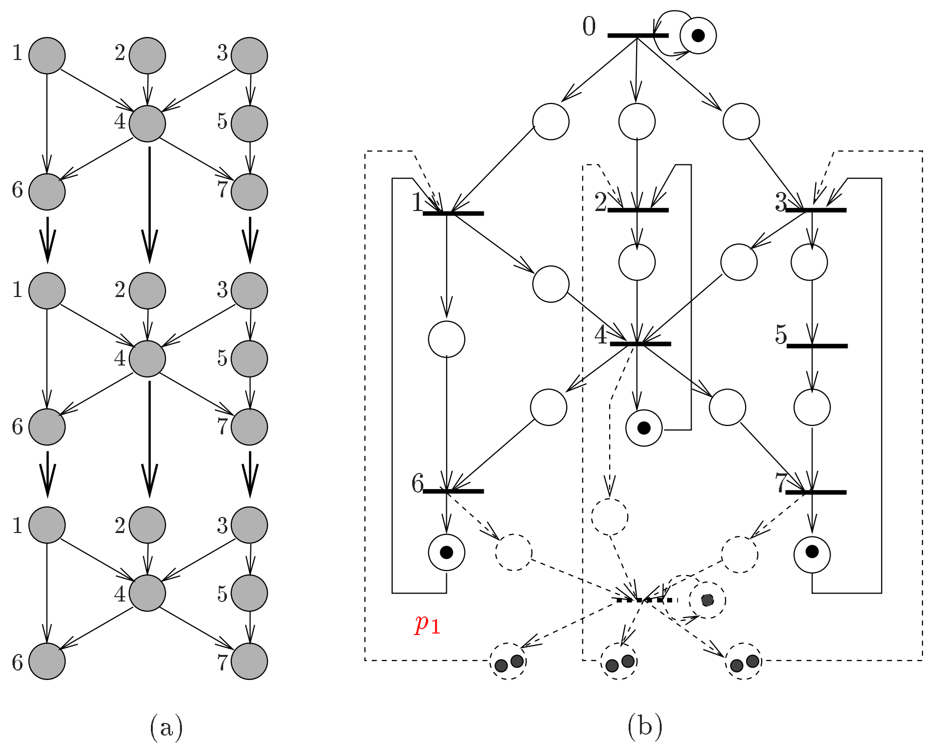


Figure 7: (a) Chart of Assembly Operations– (b) Associated Petri Net

– Assembly Task 4 and Task 2: to be performed by Machine 2

– Tasks $1, 6$: to be performed by Machine 1

– Transition 0: Raw Materials

– Place p_1: Machine 1

– Dashed: Kanban Type Control

THE (max, +) ALGEBRA

Maslov 1973, Gondran-Minoux 77, Cuninghame Green 79, Cohen-Quadrat-Viot 83, B+Cohen-Olsder-Quadrat 1992

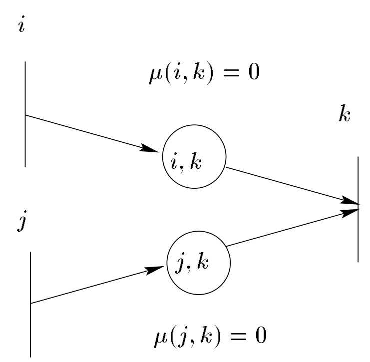
1. Scalar Semi-Field $(I\!R_{max}, \oplus, \otimes)$
   - $I\!R_{max} = I\!R \cup \{\varepsilon = -\infty\}$, $\oplus = \max$, $\otimes = +$
   - $(I\!R_{max}, \oplus)$ commutative idempotent monoid
   $$a \oplus b = b \oplus a$$
   $$a \oplus \varepsilon = \varepsilon \oplus a = a$$
   $$a \oplus a = a$$
   - $(I\!R_{max}, \otimes)$ commutative group neutral element $e = 0$
   - $\otimes$ distributes over $\oplus$
   $$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$
2. Matrix Semi-Ring $(I\!R_{max}^{K,K}, \oplus, \otimes)$
   - Matrix addition Neutral $\mathcal{E}$, $\mathcal{E}_{ij} = \varepsilon$
   $$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$$
   - Matrix multiplication Neutral E, $E_{ij} = \varepsilon$, $i \neq j$, $E_{ii} = e$
   $$(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj}$$

TIMED EVENT GRAPHS – (max, +)-LINEAR SYSTEMS

– $M \in I\!N$: Maximal initial marking

– $\sigma_n^k \in I\!R$: Duration of n-th firing of transition k

– $X_n^k \in I\!R$: Date of n-th enabling of transition k



Evolution equation in FIFO case

$$X_n^k = \max_{l \in {}^{\bullet\bullet}k} X^l_{n-\mu(l,k)} + \sigma^l_{n-\mu(l,k)} = \bigoplus_p \bigoplus_{l \in S_{k,p}} X^l_{n-p} \otimes \sigma^l_{n-p}$$

Evolution equation in $I\!R^K_{max}$

$$X_n = A(n,0) \otimes X_n \oplus A(n,1) \otimes X_{n-1} \oplus \ldots \oplus A(n,M) \otimes X_{n-M}$$

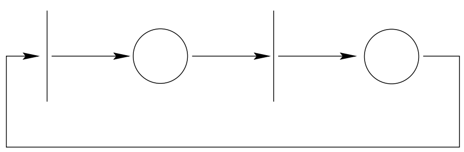
– X_n $I\!R^{K\times 1}_{max}$ vector X_n^k

– $A(n,p)$ $I\!R^{K\times K}_{max}$ matrix

$$A(n,p)_{kl} = \begin{cases} \sigma^l_{n-p} & \text{if } l \in {}^{\bullet\bullet}k,\ \mu(l,k)=p \\ \varepsilon & \text{otherwise} \end{cases}$$

TRANSFORMATION 1: CONSTRUCTIVE RECURRENCES

1. Deadlock After a finite number of firings, no transition can fire



2. Liveness $\Rightarrow$ no deadlock An event graph is *live* iff the matrix $A(n,0)$ is strictly lower triangular up to a change of basis
   For a live net, the series
   $$A(n,0)^\star = E \oplus A(n,0) \oplus A(n,0)^2 \oplus \ldots$$
   converges to a finite value

3. Transformation
   $$X_n = A(n,0) \otimes X_n \oplus A(n,1) \otimes X_{n-1} .. A(n,M) \otimes X_{n-M}$$
   $$\overline{A}(n,p) = A(n,0)^\star \otimes A(n,p)$$
   $$X_n = \overline{A}(n,1) \otimes X_{n-1} \oplus \ldots \oplus \overline{A}(n,M) \otimes X_{n-M}$$

TRANSFORMATION 2: RECURRENCE OF ORDER 1

1. State extension

$$\begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_{n-M+1} \end{pmatrix} = \begin{pmatrix} \overline{A}_{n,1} & \overline{A}_{n,2} & \dots & \dots & \overline{A}_{n,M} \\ E & \mathcal{E} & \dots & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & E & \ddots & \vdots & \mathcal{E} \\ \vdots & \ddots & E & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & \dots & \mathcal{E} & E & \mathcal{E} \end{pmatrix} \begin{pmatrix} X_{n-1} \\ X_{n-2} \\ \vdots \\ X_{n-M} \end{pmatrix}$$

2. Standard form

$$\widetilde{X}_{n+1} = \widetilde{A}_n \otimes \widetilde{X}_n$$

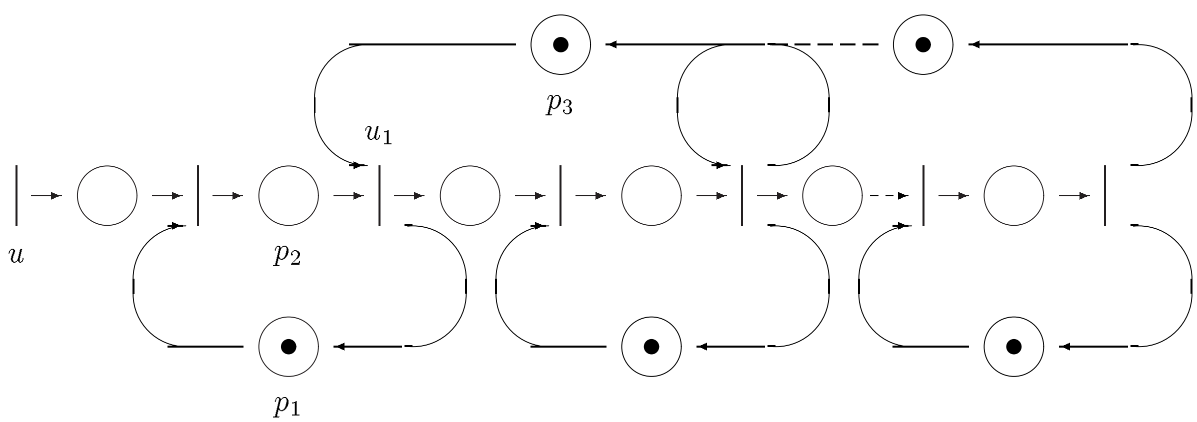
Deterministic case $\sigma_n^k = \sigma^k, \forall n$

A live timed event graph admits a representation as a $(\max, +)$-linear recurrence of order 1

$$\widetilde{X}_{n+1} = \widetilde{A} \otimes \widetilde{X}_n$$

Cohen-Dubois-Quadrat-Viot 83

EXAMPLE: MANUFACTURING BLOCKING (NON-AUTONOMOUS)



$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$A(n,0) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon \end{pmatrix}, A(n,1) = \begin{pmatrix} 1 & 0 & \varepsilon & \varepsilon \\ \varepsilon & 2 & 0 & \varepsilon \\ \varepsilon & \varepsilon & 3 & 0 \\ \varepsilon & \varepsilon & \varepsilon & 4 \end{pmatrix}, B(n,0) = \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$$

Assume $\sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \sigma_4 \quad \rightarrow \quad X_{n+1} = AX_n \oplus BU_{n+1}$

$$A = A(1,0)^{\star} \otimes A(1,1) = \begin{pmatrix} 1 & 0 & \varepsilon & \varepsilon \\ 12 & 2 & 0 & \varepsilon \\ 123 & 23 & 3 & 0 \\ 1234 & 234 & 34 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 12 \\ 123 \\ 1234 \end{pmatrix}$$

STOCHASTIC EVENT GRAPHS

σ_n^k random variables on a probability space

$$(\Omega, \mathcal{F}, P)$$

- G: the sequences $\{\sigma_n^k\}$ are jointly stationary and ergodic
- GI: the sequences $\{\sigma_n^k\}$ are independent and i.i.d.
- M: the sequences $\{\sigma_n^k\}$ are independent, i.i.d. and exponentially distributed $\longrightarrow$ Markov chains on the marking – AJMONE MARSAN-BALBO-CONTE 84

G-Stochastic Event Graph

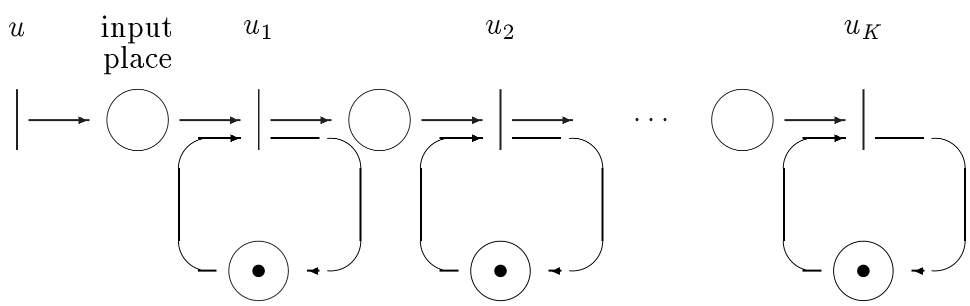
A live FIFO stochastic event graph admits a representation as a $(\max, +)$-linear recurrence of order 1 'in a random medium', i.e.

$$\widetilde{X}_{n+1} = \widetilde{A}_n \otimes \widetilde{X}_n$$

where the sequence of matrices $\{\widetilde{A}_n\}$ is stationary and ergodic

B 92

NON-AUTONOMOUS CASE: TANDEM QUEUES



$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$A(n,0) = \begin{pmatrix} \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma_n^2 & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \cdots & \sigma_n^{K-2} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_n^{K-1} & \varepsilon \end{pmatrix}, \quad B(n,0) = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$$

$$A(n,1) = \begin{pmatrix} \sigma_{n-1}^1 & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \sigma_{n-1}^2 & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma_{n-1}^3 & \cdots & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \sigma_{n-1}^{K-1} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_{n-1}^{K} \end{pmatrix}$$

QUEUES IN TANDEM

Transformation 1

$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$X_n = A(n,0)^\star A(n,1) \otimes X_{n-1} \oplus A(n,0)^\star \otimes B(n,0)U_n$$

Standard Representation for non-autonomous systems

$$\widetilde{X}_{n+1} = \widetilde{A}_n \otimes \widetilde{X}_n \oplus \widetilde{B}_{n+1} U_{n+1}$$

Here

$$(\widetilde{A}_n)_{ij} = \begin{cases} \varepsilon & \text{if} \quad i < j \\ \sum_{k=j}^{i-1} \sigma_{n+1}^k + \sigma_n^j & \text{if} \quad i \geq j \end{cases}$$

$$\widetilde{B}_n = \begin{pmatrix} 0 \\ \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \sigma_n^1 + \sigma_n^2 + \sigma_n^3 \\ \vdots \\ \sum_{k=1}^{K-1} \sigma_n^k \end{pmatrix}.$$

PART III – Timed and Stochastic Event Graphs

– Stability

  * Spectral Theory of Matrices
  * Input-Output Map
  * First Order Ergodic Theorems
  * Second Order Ergodic Theorems

– Parallel Simulation in place of Table of Events

  * Time parallel simulation
  * Space parallel simulation

– Analytic Methods Poisson input

  * Mean Values
  * Laplace Transforms

SPECTRAL THEORY OF MATRICES

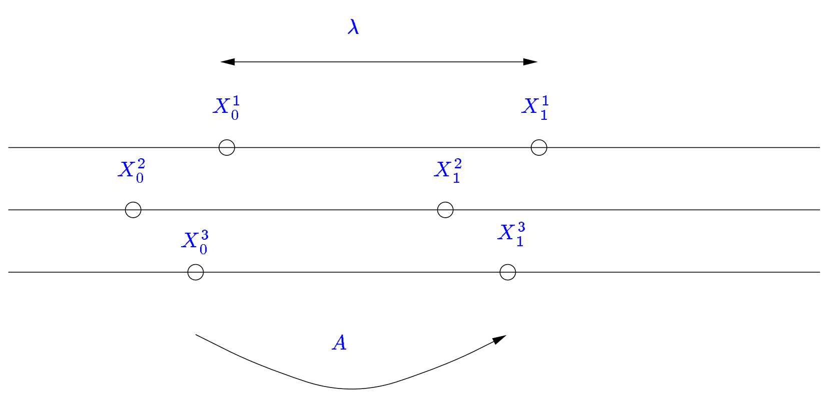
Eigenvalue problem – irreducible case

A deterministic matrix with an irreducible graph

Problem: find $X \in I\!R^K_{max}$ and $\lambda \in I\!R_{\max}$ s.t.

$$A \otimes X = \lambda \otimes X$$

Eigenpair $\Leftrightarrow$ periodic regime of period 1



- Circuit $\xi = (t_1, \ldots, t_j, t_{j+1} = t_j)$ with $t_1 \rightarrow t_2, \ldots$
- Cycle mean of ξ $p(\xi) = \frac{A_{t_1 t_2} + \ldots + A_{t_j t_1}}{j}$

A has a unique eigenvalue $\gamma = \max_\xi p(\xi)$

Romanovski 67, Ramchandani 74

EIGENVECTORS

- Critical circuit: circuit of cycle mean γ
- Critical graph: set of critical circuits
- A SCS n: # of mscs in the critical graph is n
- Cyclicity of a strongly connected graph: gcd of the lengths of its circuits
- A CYC k: lcm of the cyclicities of the mscs of critical graph

For a normalized matrix A (subtract γ to all entries)

$$A^+ = A \oplus A^2 \oplus A^3 \oplus \ldots \oplus A^K$$

1. For all i in critical graph, $A^+_{.i}$ is an eigenvector
2. For all i, j in same mscs, $A^+_{.i} = \beta \otimes A^+_{.j}$
3. All eigenvectors: $I\!R^K_{\max}$-linear combinations of these

For all irreducible matrix A with eigenvalue γ, $\exists\ d,\ N \in I\!N$ s.t.

$$\forall n \geq N \quad A^{n+d} = \gamma^d \otimes A^n$$

The smallest such d is the cyclicity of A.
Cohen Dubois Quadrat Viot 85
Possible eventually periodic regimes of period d=cyclicity

FIRST ORDER ERGODIC THEOREMS

Cycle Time – Throughput

$\{A_n\}$ stationary and ergodic sequence of $I\!R_{max}^{K\times K}$ matrices
Lyapunov exponent – strictly positive case

If $\forall i,j,\ \ P((A_0)_{ij} = \varepsilon) = 0$ and $(A_0)_{ij}$ integrable, there exists a constant $\gamma \in I\!R$ / $\forall X_0, \forall i$

$$\lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \gamma, \quad P - a.s.$$

J.E. Cohen 88, from Kingman's Subadditive Ergodic Theorem (73)

$$|A \otimes B|_\oplus \leq |A|_\oplus \otimes |B|_\oplus$$

$$|A_1 \otimes \ldots A_n \otimes \ldots A_p|_\oplus \leq |A_1 \otimes \ldots A_n|_\oplus \otimes |A_{p+1} \otimes \ldots A_n|_\oplus$$

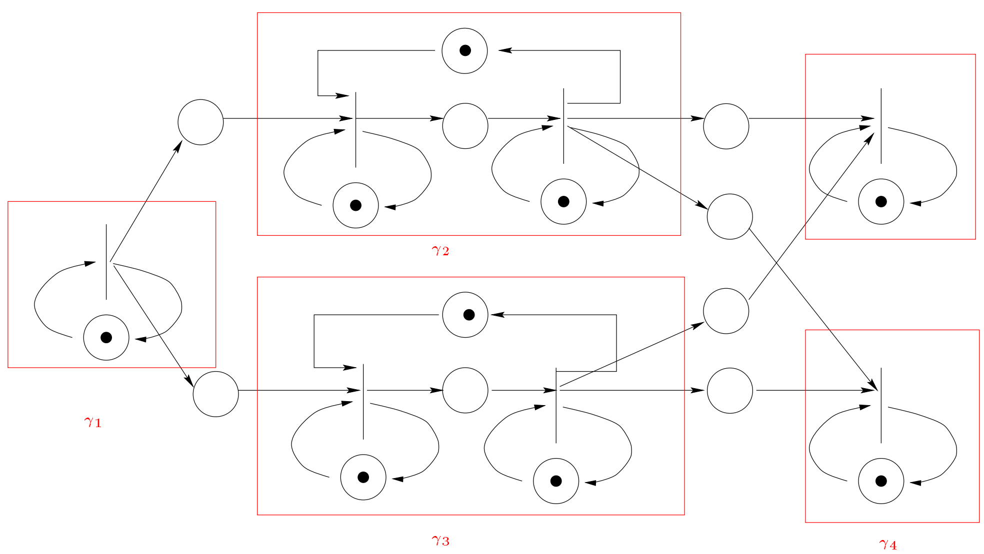
Estimates of γ

1. By convex ordering:B + Z. Liu, 90
2. By large deviations:B + T. Konstantopoulos, 91

STRUCTURE OF LYAPUNOV EXPONENTS

Graph of A_0 directed graph $1 \leq j \to k \leq K$ iff $(A_0)_{kj} > \varepsilon$ a.s

MSCS Equivalence classes of $i\mathcal{R}j$ iff $i \to^\star j$ and $j \to^\star i$



$\forall i \in mscs\ \tilde{u}, \lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \oplus_{\tilde{v} \in {}^\bullet\tilde{u}} \gamma_{\tilde{v}}$

- ${}^\bullet\tilde{u}$: set of all ancestors of $\tilde{u}$ in reduced graph
- $\gamma_{\tilde{v}}$: Lyapunov exponent of the restriction of A_n to the coordinates of $\tilde{v}$

B 92

$$\lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4, \ i \in mscs\ 4$$

SECOND ORDER ERGODIC THEOREMS
Non-autonomous case

Reference equation – 1 dim input

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1} \tag{1}$$

Second Order Variables

$$W_n^i = X_n^i - T_n$$

Second Order Equation by subtracting T_{n+1} on both sides of (1)

$$W_{n+1} = A_n \otimes C(\tau_n) \otimes W_n \oplus B_{n+1}$$

$C(x) = x \otimes E$, $\tau_n = T_{n+1} - T_n$

Assumptions:

– $\{A_n, B_n, \tau_n\}$ is stationary and ergodic

– γ maximal Lyapunov exponent of $\{A_n\}$

– all entries non ε are integrable

If $\rho = \lambda\gamma < 1$, $\lambda^{-1} = E(\tau_0)$, $\{W_n\}$ converges in total variation to a unique stationary sequence. The stationary law of this sequence is that of the random variable

$$W = B_0 \oplus \bigoplus_{n\geq 1} C(T_{-n}) \otimes D_n$$

where

$$D_n = A_{-1} \otimes \ldots \otimes A_{-n} \otimes B_{-n}$$

If $\rho > 1$, $|W_n|_\oplus$ tends to ∞ a.s. B 92

EXTENSIONS

Higher dimension input

Division $u \in I\!R^m_{max}$, $v \in I\!R^n_{max}$

$$\frac{u}{v} \in I\!R^{n\times m}_{max}, \quad \left(\frac{u}{v}\right)_{ij} = u_i - v_j$$

T dim J $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$

Second Order Variables

$$W_n = \frac{X_n}{T_n} \in I\!R^{K\times J}_{max}$$

Second Order Equation $U^+_n = \frac{T_{n+1}}{T_n}$, $U^-_n = \frac{T_n}{T_{n+1}}$

$$W_{n+1} = A_n \otimes W_n \otimes U^-_n \oplus B_{n+1} \otimes U^+_n \otimes U^-_n$$

For extending the ergodic theorem, $\{U^+_n\}$ should be stationary ergodic with $EU^+_{ii} = \lambda^{-1}$

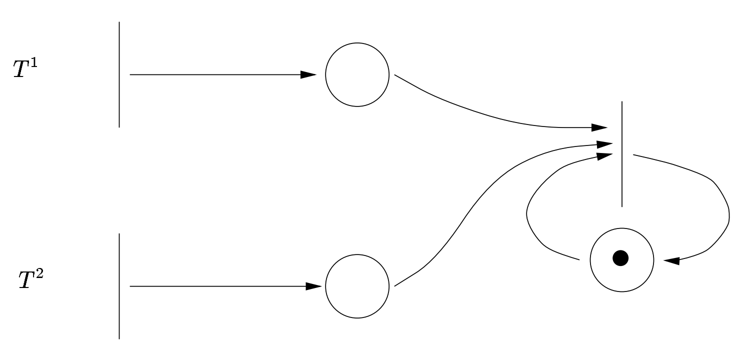


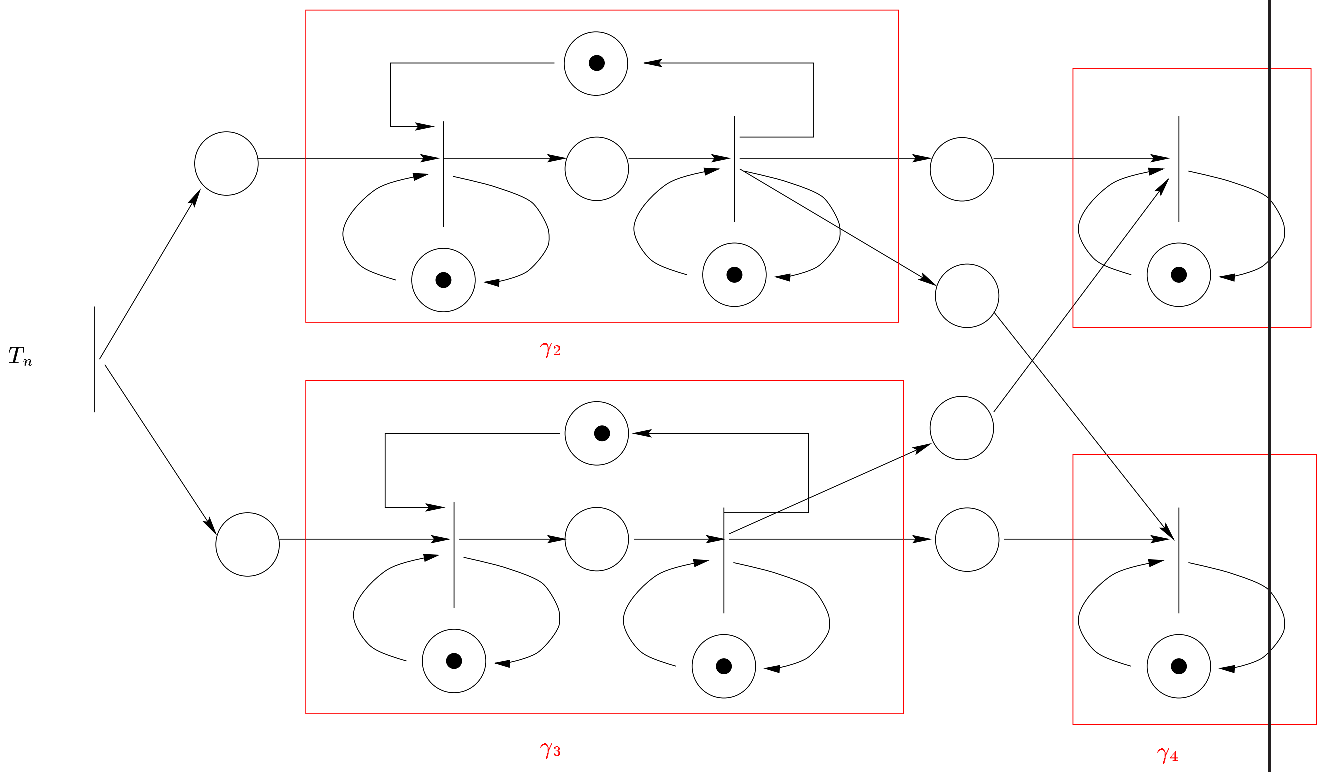
Figure 8: Solution to Harrison's stability problem in assembly lines

EXTENSIONS

A_n non-irreducible

– If $\oplus_{\tilde{v}} \gamma_{\tilde{v}} < \lambda^{-1}$, W_n converges in total variation to a ! finite limit

– Otherwise, for $\tilde{u}_0$ s.t. $\oplus_{\tilde{v} \in \bullet \tilde{u}_0} \gamma_{\tilde{v}} > \lambda^{-1}$, $(W_n)_{i.}$ diverges a.s. for all i in mscs $\tilde{u}_0$



SPECIAL DETERMINIST CASE

Legendre Transform

Formal analogue of the z-transform: Legrendre Transform:

$$u^*(z) = \bigoplus_{n\geq 1} u(n) \otimes z^n = \sup_{n\geq 1} u(n) + nz$$

Example of application Input-Output Map

$$X_{n+1} = AX_n \oplus BU_{n+1}, \quad Y_n = CX_n, \quad n \geq 0$$

with $X_0 = \varepsilon$; then

$$X_n = \bigoplus_{p=0}^{n-1} A^p BU(n-p), \quad Y_n = CX_n$$

With $H(p) = CA^pB$, $p \geq 0$, ε otherwise:

$$Y(n) = \bigoplus_{p=0}^{n-1} H(p) \otimes U(n-p)$$

Sup-convolution representation of the input-output map

$$Y^*(z) = \bigoplus_{n\geq 1} Y(n) z^n$$

$$H^*(z) = \bigoplus_{n\geq 1} H(n) z^n$$

$$Y^*(z) = z \otimes H^*(z) \otimes U^*(z)$$

SECOND ORDER ERGODIC THEOREMS

Stochastic irreducible autonomous case

$\{A_n\}$ stationary and ergodic.
Projective Space
$u\mathcal{R}v$ in $I\!R_{max}^K$ iff for $\lambda \in I\!R$

$$u^i = v^i + \lambda$$

$\pi(u)$ equivalence class of u

Suppose there is an integer N such that among the 'patterns' of $A_{N-1} \otimes \cdots \otimes A_0$, there exists an irreducible matrix C SCS 1 CYC 1. Then $\pi(X_n)$ converges to a unique stationary distribution independently of the initial condition. If C is in the interior of the support, there is total variation convergence to the stationary distribution. Otherwise there is only weak convergence.
Mairesse 94

PARALLEL SIMULATION

Time parallel algorithm – Assembly Model

Estimation of throughput via the ergodic theorem

$$\gamma^i = \lim_n \frac{X_n^i}{n} \quad P \ a.s.$$

Non-autonomous equation

$$X_{n+1} = A_n \otimes X_n \oplus H \otimes T_{n+1}$$

Autonomous equivalent form: $Y_{n+1} = B_n \otimes Y_n$

$$Y_{n+1} = \begin{pmatrix} T_{n+2} \\ X_{n+1} \end{pmatrix} = \begin{pmatrix} \tau_{n+1} & \varepsilon \\ H & A_n \end{pmatrix} \otimes \begin{pmatrix} T_{n+1} \\ X_n \end{pmatrix}$$

The matrices $\{B_n\}$ are i.i.d. if

– input process is renewal

– task graphs are i.i.d.

Possibility of sampling the B_n matrices independently

ALGORITHM AND COMPLEXITY

Simulation problem Compute

$$Y_{n+1} = B_n \otimes B_{n-1} \otimes \ldots \otimes B_0 \otimes Y_0$$

for n very large

Parallel algorithm – due to

1. associativity of $\otimes$ in $\mathbb{R}_{max}^{K,K}$
2. possibility of sampling the matrices B_n independently

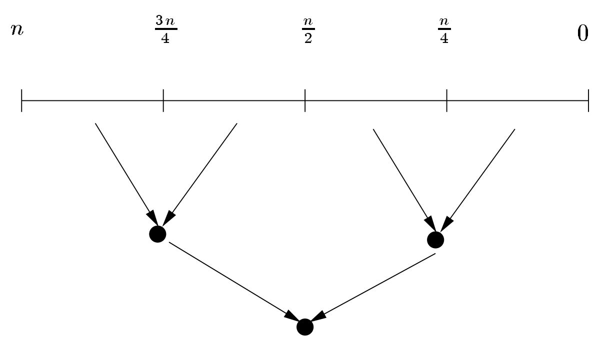


Figure 9: Parallel prefix algorithm – 4 processors

Time complexity of the computation of X_n with P processors

$$\mathcal{O}\left(\frac{nK^3}{P} + K^3 \log P\right)$$

B+Canales, 93 Extension of Greenberg, Mitrani, Lubatchevsky, 91 for $G/G/1$

Extensions to space parallel algorithms

PARALLEL SIMULATION

Space parallel algorithm

$$\begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_{n-M+1} \end{pmatrix} = \begin{pmatrix} \overline{A}_{n,1} & \overline{A}_{n,2} & \dots & \dots & \overline{A}_{n,M} \\ E & \mathcal{E} & \dots & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & E & \ddots & \vdots & \mathcal{E} \\ \vdots & \ddots & E & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & \dots & \mathcal{E} & E & \mathcal{E} \end{pmatrix} \otimes \begin{pmatrix} X_{n-1} \\ X_{n-2} \\ \vdots \\ X_{n-M} \end{pmatrix}$$

with $\overline{A}_{n,l} = (A_{n,0})^* \otimes A_{n,l}$, $(A_{n,0})^* = (E \oplus A_{n,0})^L$

$$\begin{array}{ll}
\texttt{for(k=0; k<n; k++)} & \\
\texttt{\{} & \\
\quad \texttt{Generation of } (\mathtt{A}_{\mathtt{k},1}, \dots, \mathtt{A}_{\mathtt{k},\mathtt{M}}) & O(1) \\
\quad \mathtt{X}_\mathtt{k} = (\mathtt{A}_{\mathtt{k},1}, \dots, \mathtt{A}_{\mathtt{k},\mathtt{M}}) \otimes \widetilde{\mathtt{X}}_{\mathtt{k}-1} & O(\log MT) \\
\quad \texttt{Generation of } \mathtt{A}_{\mathtt{k},0} & O(1) \\
\quad \texttt{for(i=0; i<L; i++)} & O(L \log T) \\
\qquad \mathtt{X}_\mathtt{k} = \mathtt{X}_\mathtt{k} \otimes (\mathtt{E} \oplus \mathtt{A}_{\mathtt{k},0}) & \\
\quad \tilde{\mathtt{X}}_\mathtt{k} = \mathtt{f}(\mathtt{X}_\mathtt{k}, \tilde{\mathtt{X}}_{\mathtt{k}-1}) & O(\log MT) \\
\texttt{\}} &
\end{array}$$

Time complexity with more than MT^2 processors

$$\mathcal{O}\left(n(L \log T + \log MT)\right).$$

L: Longest empty path B+Canales, 93

TAYLOR EXPANSIONS FOR RESPONSE TIMES

Non-autonomous event graph with 1-dim input – $W_n^i = X_n^i - T_n$

Assumptions:

- $\{T_n\}$ Poisson process of intensity λ
- $\{\sigma_n^k,\ k = 1, \ldots, K\}$ i.i.d. sequence
- For all k, $E[(\sigma_n^k)^{m+3}] < \infty$

$$W = B_0 \oplus \bigoplus_{n \geq 1} C(T_{-n}) \otimes D_n, \quad D_n = A_{-1} \otimes \ldots \otimes A_{-n} \otimes B_{-n}$$

If $\lambda\gamma < 1$,

$$EW^i = \sum_{k=0}^{m} \lambda^k E p_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) \ + \ \mathcal{O}(\lambda^{m+1})$$

$$p_k(x_0, \ldots, x_{k-1}) = \sum_{i=(i_0,\ldots,i_{k-1}) \in S_k} (-1)^{q_k(i)} \frac{x_0^{i_0}}{i_0!} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!}$$

$$S_k = \{ i \in \{I\!N\}^k \ : \ i_0 + \cdots + i_{k-1} = k; \ \text{ if } \ i_s = l > 1,\ i_{[s-1]_{mod\ k}} = \ldots = i_{[s-l+1]_{mod\ k}} = 0 \}$$

$$q_k(i_0, i_1, \ldots, i_{k-1}) = 1 + \Sigma_{s=0}^{k-1} \mathbf{1}(i_s > 0)$$

B + V. Schmidt, 1995

POLYNOMIALS

$$
\begin{aligned}
p_1(x_0) &= x_0 \\
p_2(x_0, x_1) &= \frac{1}{2}\left[x_0^2 + x_1^2 - 2x_0x_1\right] \\
p_3(x_0, x_1, x_2) &= \frac{1}{6}\left[x_0^3 + x_1^3 + x_2^3\right. \\
&\quad -3(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) \\
&\quad \left.+6x_0x_1x_2\right] \\
p_4(x_0, \ldots, x_3) &= \frac{1}{24}\left[x_0^4 + x_1^4 + x_2^4 + x_3^4\right. \\
&\quad -4(x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0) \\
&\quad -6(x_0^2x_2^2 + x_1^2x_3^2) \\
&\quad +12(x_0^2x_1x_2 + x_1^2x_2x_3 + x_2^2x_3x_0 + x_3^2x_0x_1) \\
&\quad \left.-24x_0x_1x_2x_3\right]
\end{aligned}
$$

Properties

1. invariance by circular permutation: $k \geq 1$

$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_1, x_2 \ldots, x_{k-1}, x_0)$$

2. 1-invariance for all $t \in I\!R, k \geq 2$,

$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_0{+}t, x_1{+}t, \ldots, x_{k-1}{+}t)$$

3. Integral recurrence

$$p_{k+1}(x_0,\ldots,x_k) = \sum_{n=0}^{k-1}\int_{x_n-x_0}^{x_{n+1}-x_0} \Big[p_k(\underbrace{x_0,\ldots,x_0}_{n},x_{n+1}-u,\ldots,x_k-u) - p_k(\underbrace{x_0,\ldots,x_0}_{n+1},x_{n+1}-u,\ldots,x_{k-1}-u)\Big]\, du.$$

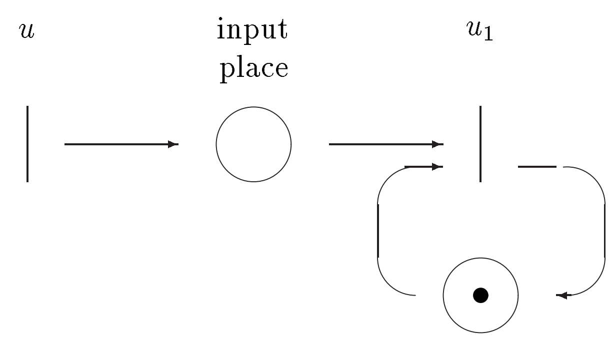
4. For all $k \geq 2$,

$$p_k(0,\ldots,0) = 0$$
$$p_k(1,\ldots,k) = \frac{1}{2}.$$

5. For all $t \in I\!R$,

$$p_k(tx_0,tx_1,\ldots,tx_{k-1}) = t^k p_k(x_0,x_1\ldots,x_{k-1}).$$

EXAMPLE: SINGLE SERVER QUEUE



Lindley's equation $W_{n+1} = (\sigma_n \otimes (-\tau_n) \otimes W_n) \oplus e$
$D_0 = 0$ and $D_k = \Sigma_{n=1}^{k} \sigma_{-n}$

$$p_2(D_0, D_1) = \frac{\sigma_{-1}^2}{2}, \; p_3(D_0, D_1, D_2) = \frac{\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2}{6}$$

$$\begin{aligned} p_4(D_0, \ldots, D_3) = \; & \frac{1}{24}(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 \\ & +4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3)) \\ & -6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) \\ & +12\left(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2\right) \end{aligned}$$

Assume $\{\sigma_n\}$ i.i.d.

$$Ep_2(D_0, D_1) = \frac{E[\sigma_n^2]}{2}, \quad Ep_3(D_0, D_1, D_2) = \frac{E[\sigma_n^2]E[\sigma_n]}{2}$$

$$Ep_4(D_0, D_1, D_2, D_3) = \frac{E[\sigma_n^2]E[\sigma_n]^2}{2}$$

POLLACZEK-KHINCHINE FORMULA

Assume $\lambda E[\sigma_n] < 1$, $E[\sigma_n^6] < \infty$: Corollaries

1. Pollaczek-Khinchine Formula

$$EW = \lambda \frac{E[\sigma_n^2]}{2} + \lambda^2 \frac{E[\sigma_n^2] E[\sigma_n]}{2} + \lambda^3 \frac{E[\sigma_n^2] E[\sigma_n]^2}{2} + \mathcal{O}(\lambda^4).$$

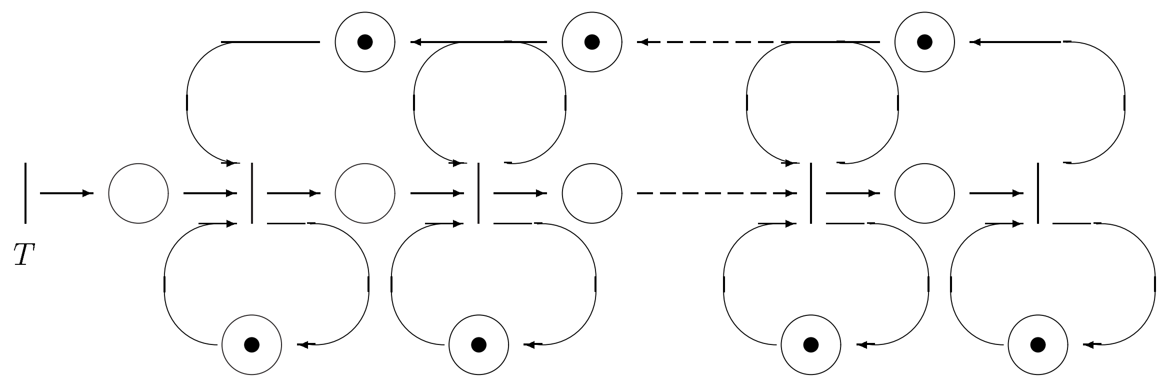
$$EW = \frac{\lambda E[\sigma_n^2]}{2(1 - \lambda E[\sigma_n])} = \frac{E[\sigma_n^2]}{2} \sum_{k=1}^{\infty} \lambda^k E[\sigma_n]^{k-1}$$

2. Conditional expectation formula

   $\mathcal{F}_\sigma$: σ–algebra generated by $\{\sigma_n\}$

$$\begin{aligned} E[W|\mathcal{F}_\sigma] &= \frac{\lambda}{2}\sigma_{-1}^2 + \frac{\lambda^2}{6}\left(\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2\right) \\ &+ \frac{\lambda^3}{24}(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 + 4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3) \\ &\quad -6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) + 12(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2)) + \dots \end{aligned}$$

EXAMPLE M/D-TYPE: BLOCKING



$$D_0 = \begin{pmatrix} 0 \\ 1 \\ 12 \\ 123 \\ 1234 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 12 \\ 123 \\ 1234 \\ 123\gamma \\ 1234\gamma \end{pmatrix}, \quad D_2 = \begin{pmatrix} 12^23 \\ 123^24 \\ 1234\gamma \\ 123\gamma^2 \\ 1234\gamma^2 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} 12^23^24 \\ 123^24\gamma \\ 1234\gamma^2 \\ 123\gamma^3 \\ 1234\gamma^3 \end{pmatrix}, \quad D_n = \begin{pmatrix} 12^23^24\gamma^{n-3} \\ 123^24\gamma^{n-2} \\ 1234\gamma^{n-1} \\ 123\gamma^n \\ 1234\gamma^n \end{pmatrix}, \quad n \geq 4, \quad \gamma = 45$$

$$EW^1 = \frac{\lambda}{2}(\sigma^1+\sigma^2)^2 + \frac{\lambda^2}{6}\left[(\sigma^1+\sigma^2)^3 + (\sigma^1+2\sigma^2+\sigma^3)^3 - 3(\sigma^1+\sigma^2)^2(\sigma^1+2\sigma^2+\sigma^3)\right] + \mathcal{O}(\lambda^3)$$

EXAMPLE M/D-TYPE: ASSEMBLY
Poisson input

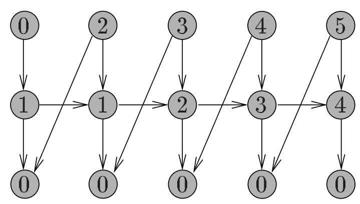


Figure 10: Repetitive tasks to be performed by 5 machines

State reduction

X_n^i time when first subtask of task n starts on machine i

$$X_{n+1} = A_n \otimes X_n \oplus H \otimes T_{n+1}$$

A_n depends on n-th task graph only

$$A_n = A \; = \begin{pmatrix} 1 & 2 & \varepsilon & \varepsilon & \varepsilon \\ 2 & 3 & 3 & \varepsilon & \varepsilon \\ 4 & 5 & 5 & 4 & \varepsilon \\ 7 & 8 & 8 & 7 & 5 \\ 11 & 12 & 12 & 11 & 9 \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Evolution equation for waiting times

$$W_{n+1} = A \otimes C(\tau_n) \otimes W_n \oplus H$$

$$W_n^i = X_n^i - T_n,$$

EXPANSION FOR ASSEMBLY MODEL

$$D_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \\ 12 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 5 \\ 8 \\ 12 \\ 17 \\ 21 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} 10 \\ 15 \\ 21 \\ 26 \\ 30 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 18 \\ 24 \\ 30 \\ 35 \\ 39 \end{pmatrix}, \quad D_5 = \begin{pmatrix} 26 \\ 33 \\ 39 \\ 44 \\ 48 \end{pmatrix}$$

$D^k_{n+1} = D^k_n + 9$ for all k and for all $n \geq 5$

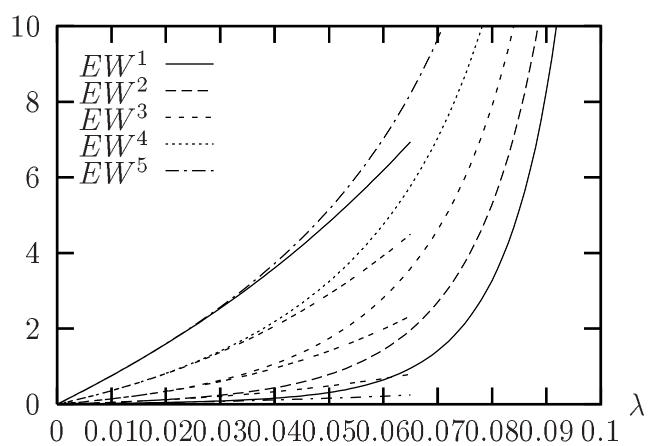
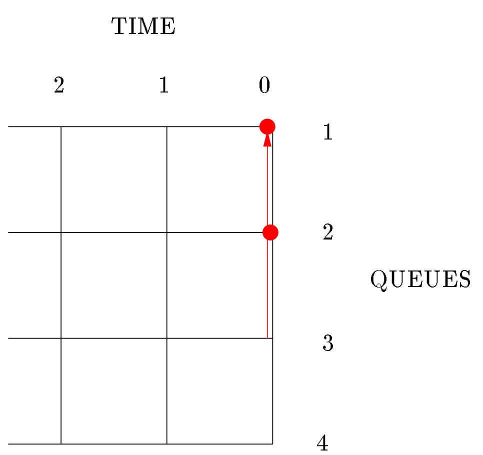


Figure 11: Expected waiting times for a task

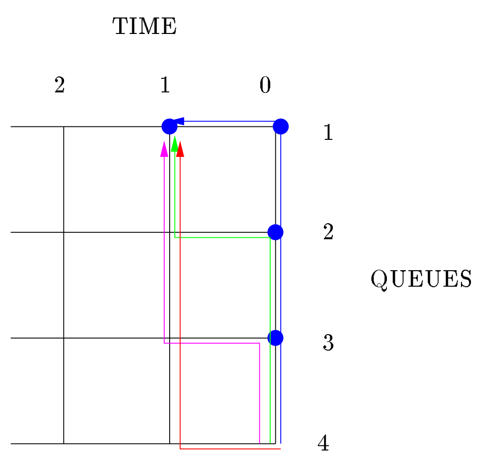
EXAMPLE M/GI-TYPE: QUEUES IN TANDEM

General Case

$$D_n^i = \max_{1 \le l_n \le \ldots \le l_1 \le i} \left\{ \sum_{k=l_1}^{i-1} \sigma_0^k + \sum_{k=l_2}^{l_1} \sigma_{-1}^k + \ldots + \sum_{k=1}^{l_n} \sigma_{-n}^k \right\}$$



D_0^3



D_1^4

$$D_0 = \begin{pmatrix} e \\ 1_0 \\ 1_0 2_0 \\ 1_0 2_0 3_0 \end{pmatrix}, D_1 = \begin{pmatrix} 1_1 \\ 1_0 1_1 \oplus 2_1 1_1 \\ 2_0 1_0 1_1 \oplus 2_0 2_1 1_1 \oplus 3_1 2_1 1_1 \\ 3_0 2_0 1_0 1_1 \oplus 3_0 2_0 2_1 1_1 \oplus 3_0 3_1 2_1 1_1 \oplus 4_1 3_1 2_1 1_1 \end{pmatrix}$$

EXAMPLE M/GI-TYPE: QUEUES IN TANDEM

$$D_1^i - D_0^i = \max_{1 \le l \le i} \left\{ \sigma_{-1}^l + \sum_{k=1}^{l-1} (\sigma_{-1}^k - \sigma_0^k) \right\}$$

$$D_2^i - D_0^i = \max_{1 \le l_2 \le l_1 \le i} \left\{ \sigma_{-2}^{l_2} + \sum_{k=1}^{l_2-1} (\sigma_{-2}^k - \sigma_0^k) + \sigma_{-1}^{l_1} + \sum_{k=l_2}^{l_1-1} (\sigma_{-1}^k - \sigma_0^k) \right\}$$

$$Ep_1(D_0^i) = ED_0^i = \sum_{k=1}^{i-1} E\sigma_0^k$$

$$Ep_2(D_0^i, D_1^i) = \frac{1}{2} E(D_1^i - D_0^i)^2$$

$$\begin{aligned} Ep_3(D_0^i, D_1^i, D_2^i) &= \frac{1}{6} \{ E(D_1^i - D_0^i)^3 + E(D_2^i - D_0^i)^3 \\ &\quad -3E(D_1^i - D_0^i)^2 (D_2^i - D_0^i) \} \end{aligned}$$

Case with identical random service times on all nodes

$$\begin{aligned} D_0^3 &= 2\sigma(0) \\ D_1^3 &= \sigma(1) + 2\max\{\sigma(0), \sigma(1)\} \\ D_2^3 &= \sigma(1) + \sigma(2) + 2\max\{\sigma(0), \sigma(1), \sigma(2)\}, \end{aligned}$$

$$E[W^3] = 2E[\sigma(0)] + \lambda \frac{E[\sigma(0)] + 2E[(\sigma(1) - \sigma(0))^+]}{2}$$

$$+\lambda^2 \frac{2E[\sigma(0)] + 2E[\max\{(\sigma(1) - \sigma(0))^+, \sigma(2)\sigma(0)\}]}{6} + \mathcal{O}(\lambda^3)$$

EXPANSIONS FOR LAPLACE TRANSFORMS

$$E[e^{-sW^i}] = \sum_{k=0}^{m} \lambda^k \, E[\, q_{k+1}(s; D_0^i, D_1^i, \ldots, D_k^i) \,] + \mathcal{O}(\lambda^{m+1}),$$

where $q_1(s; x_0) = e^{-sx_0}$,

$$q_{k+1}(s; x_0, x_1, \ldots, x_k)$$

$$= \frac{1}{s^k} \sum_{n=0}^{k} \binom{k}{n} (-1)^n e^{-sx_n} - \sum_{j=0}^{k-1} \frac{1}{s^j} \sum_{n=0}^{j} \binom{j}{n} (-1)^n e^{-sx_n} \{p_{k-j}(x_{n+1}, \ldots, x_{k-j+n}) - p_{k-j}(x_n, \ldots, x_{k-j+n-1})\}$$

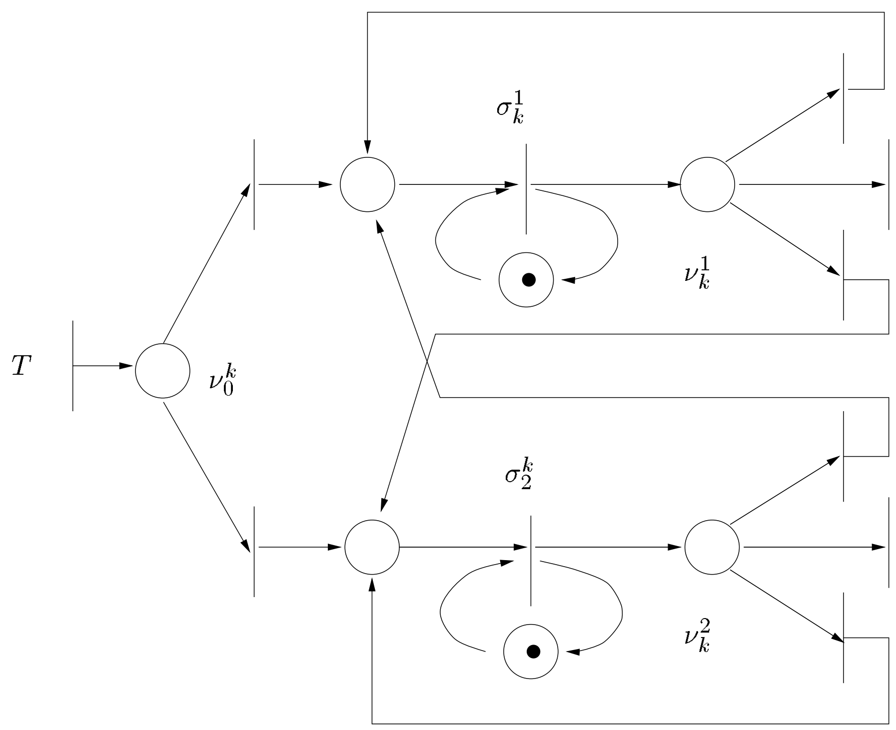
$$= \frac{q_k(s; x_0, x_1, \ldots, x_{k-1}) - q_k(s; x_1, x_2, \ldots, x_k)}{s} - e^{-sx_0}[p_k(x_1, x_2, \ldots, x_k) - p_k(x_0, x_1, \ldots, x_{k-1})],$$

p_k polynomials defined before

B + S. Hasenfuss + V. Schmidt, 1995

EXAMPLE OF FREE CHOICE NET

Flowshop – Jackson network



$\mathcal{C} \subset \mathcal{P}$ set of places with choice
$\mathcal{S} \subset \mathcal{T}$ set of transitions preceded by a ($\rightarrow$ exactly 1 – FC) $p \in \mathcal{C}$
Data: For all $q \in \mathcal{T}$, $\sigma_n^q \in I\!R$ For all $p \in \mathcal{C}$, $\nu_n^p \in p^\bullet$ routing

MAXIMAL EVENT GRAPH DECOMPOSITION

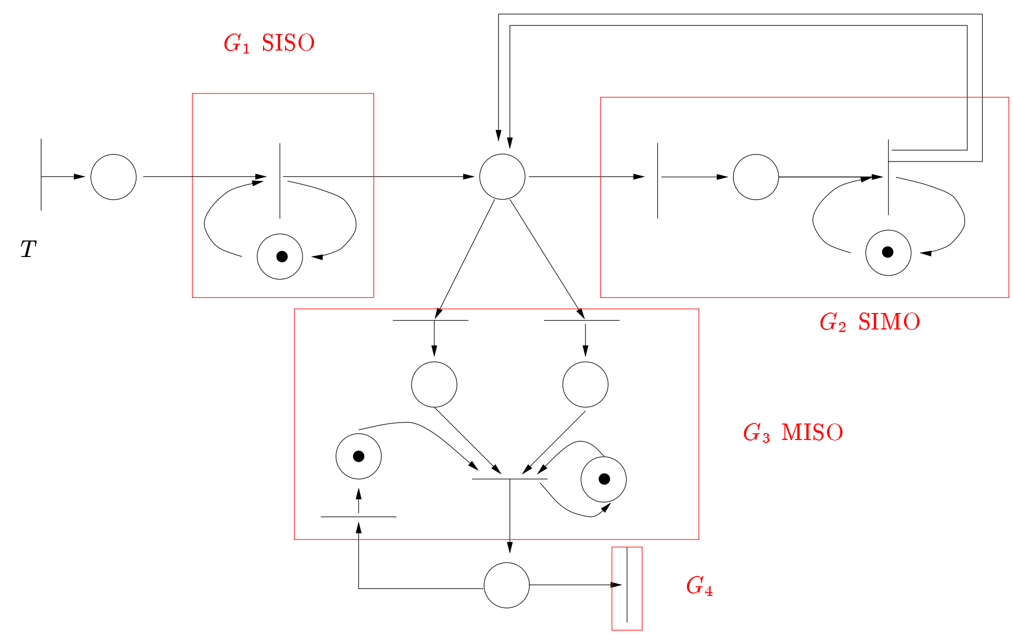
Serial place: p has at most one input t and at most one output t

Equivalence relation on $\mathcal{T}$ $t\mathcal{R}s$ iff $\exists$ a non directed path between s and t with only serial places

Decomposition:

- $T_1, \ldots, T_n$: $\mathcal{R}$ equivalence classes
- $G_1, \ldots, G_n$: G_i event graph with transitions set T_i and places set $P_i = \{p \text{ serial} \mid {}^\bullet p,\ p^\bullet \in T_i\}$

Residual places: non serial routing places



COUNTERS AND DATERS

Data

– For all $q \in \mathcal{T}$, $\sigma_n^q \in I\!R$

– For all $p \in \mathcal{C}$, $\nu_n^p \in p^\bullet$ Routing

Routing functions For all $q \in \mathcal{S}$ $\pi^q : I\!N \to I\!N$

$$\pi^q(m) = \sum_{l=1}^{m} 1_{\{\nu_l^{\bullet q} = q\}}$$

Counters – Daters

– X_k^q daters of transition q

– Y_t^q # of firings initiated by q before time t

$$Y_t^q = \sum_{k \geq 1} 1_{X_k^q \leq t}$$

State vectors

$$\overline{Y}_t = (Y_t^q), \quad q \in \overline{\mathcal{S}} = \mathcal{T} - \mathcal{S}$$

preceded by serial places only

$$\widetilde{Y}_t = (Y_t^q), \quad q \in \mathcal{S}$$

preceded by one non – serial place

COUNTER EQUATIONS FOR FREE CHOICE NETS

Case with constant integer service times

Reference algebras

- $(I\!R^K_{min}, \oplus, \otimes)$: $\oplus = \min$, $\otimes = +$
- $(I\!R^K, +, \times)$

Evolution equations

$$\overline{Y}_t = \bigoplus_{l=1}^{M} \left[A_l \otimes \overline{Y}_{t-l} \oplus B_l \otimes \widetilde{Y}_{t-l}\right]$$

$$\widetilde{Y}_t = \Pi\left(\sum_{l=1}^{M} \left[P_l \times \widetilde{Y}_{t-l} + Q_l \times \overline{Y}_{t-l}\right] + R_t\right)$$

- $M \in I\!N$: maximal firing time
- $\Pi(Z) = (\pi^q(Z^q))$ (Π linear in expectation $\rightarrow$ polynomial evolution in fluid case)
- R^q_t: # arrivals of T_n in non-serial place ${}^\bullet q$ by time t
- $(A_l)_{ij} = c$ if $j \in {}^{\bullet\bullet}i$, $i, j \in \overline{\mathcal{S}}$, $\mu(j,i) = c$, $\sigma^j = l$, ε otherw.
- $(P_l)_{ij} = 1$ if $j \in {}^{\bullet\bullet}i$, $i, j \in \mathcal{S}$ and $\sigma^j = l$, 0 otherw.

B+ Foss & Gaujal 94, Cohen, Gaubert & Quadrat 95: fluid

EQUATION FOR TOTAL NUMBER OF EVENTS

Total number of events:

$$\overline{Y} = \lim_t \overline{Y}_t, \quad \widetilde{Y} = \lim_t \widetilde{Y}_t, \quad R = \lim_t R_t$$

$$\begin{aligned} \overline{Y} &= A \otimes \overline{Y} \oplus B \otimes \widetilde{Y} \\ \widetilde{Y} &= \Pi \left(P \times \widetilde{Y} + Q \times \overline{Y} + R \right) \end{aligned}$$

- $A = \oplus A_l, \; B = \oplus B_l$
- $P = \Sigma P_l, \; Q = \Sigma Q_l$

Liveness, deadlockfreeness are associated with routing, topology and initial marking, but not with timing variables σ

Maximal dater

$$Z = \max_q \{ X^q_{Y^q} + \sigma^q_{Y^q} \}$$

Time of last activity in the net

ERGODIC THEOREMS

SIMO input-connected case

Logical assumptions: promptness

1. The initial marking is a deadlock
2. If $R < \infty$, then $Y < \infty$ (promptness of input)

$(R_{[1,n]})_t$: restriction of the input to its first n points
$(Y_{[1,n]})_t$: associated counters
$(Z_{[1,n]})$: associated maximal dater

$$V_{[1,n]} = P \times \widetilde{Y}_{[1,n]} + Q \times \overline{Y}_{[1,n]} + R_{[1,n]}$$

$V^q_{[1,n]}$: total # tokens ever to enter ${}^\bullet q$, $q \in \mathcal{S}$, for $R_{[1,n]}$

Stochastic assumptions

1. Compatibility the sequences

   $-\{\nu^{\bullet q}_{k+V^q_{[1,n]}},\ q \in \mathcal{S},\ k \in I\!N^\star\}$ routing after $V_{[1,n]}$

   $-\{R_{t+T_n} - R_{T_n},\ t \in I\!N\}$ arrivals after T_n

   are jointly stationary and ergodic (e.g. i.i.d. routing)
2. Integrability $Z_{[1,1]}$, $\tau_n = T_{n+1} - T_n$ integrable

FIRST AND SECOND ORDER ERGODIC THEOREMS

SIMO input-connected case

Constants

– γ_i: maximal $I\!R_{max}$ Lyapunov exponent of G_i

– $b_i = E\left[Y_{[1,1]}^{(i)}\right]$ total # firings of any $q \in G_i$

– $\beta = \max_i \gamma_i b_i$

Again using Kingman's subadditive ergodic theorem

Under above assumptions,

1. (first order theorem)

$$\exists \lim_{n\to\infty} \frac{Z_{[1,n]}}{n}, \quad \lim_{n\to\infty} \frac{EZ_{[1,n]}}{n} = \gamma \quad P\ a.s.$$

   for some constant γ (extensions for nets with at least one prompt transition)

2. (second order theorem)

   – If $\lambda\beta < 1$, then $\gamma = \lambda^{-1}$ and the marking process admits a minimal stationary regime

   – If $\lambda\beta > 1$, then the marking process is unstable

B-Foss 93, B-Foss-Gaujal 94

ONGOING RESEARCH

– Linear Nets

1. Minimal representation Gaubert, Gaujal, Jean-Marie, Mairesse
2. Identification Cuninghame Greene
3. Optimization Proth, Xie, Wardi
4. Computational Problems Chang, Jean-Marie, Keane, Liu, Nain, Olsder
5. Control Cofer, Garg, Kamen, Olsder, Wagneur

– Non-Linear Nets

1. Fluid Networks Gaubert, Cohen, Quadrat
2. Stochastic Networks Baccelli, Foss, Gaujal, Mairesse
3. Stability Gunawardena, Keane, Olsder

BIBLIOGRAPHY

– F. B, G. Cohen, G.J. Olsder, J.P. Quadrat , *Synchronization and Linearity*, Wiley 1992.

– F. B and A. Makowski, “ Queueing Models of Systems with Synchronization Constraints”, Proceedings of the IEEE, 1989.

– F. B and M. Canales, “Parallel Simulation of Stochastic Petri Nets using Recurrence Equations”, ACM Tomacs, **1** (3), January 1993.

– F. B and S. Foss, “On the Saturation Rule for the Stability of Queues”, INRIA Report #2015, *Journal of Applied Probability*, June 1995.

– F. B, S. Foss and B. Gaujal, “Structural, timed and stochastic properties of free choice Petri nets”, INRIA Report #2411, *IEEE Trans. Automatic Control*, to appear, 1995.

– F. B and V. Schmidt, “Taylor Series Expansions for Poisson Driven $(\max,+)$-Linear Systems”, Rapport INRIA #2494, March 1995, Submitted to *Annals of Applied Probability*

– G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. "A linear system-theoretic view of discrete event processes." *Proceedings of the 22-nd Conference on Decision and Control*. IEEE, 1983.

– G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. "A linear system-theoretic view of discrete event processes and its use for performance evaluation in manufacturing." *IEEE Transactions on Automatic Control*, AC-30:210–220, 1985.

– G. Cohen, S. Gaubert, J.P. Quadrat. "Asymptotic Throughput of Continuous Timed Petri Nets", CDC, 1995.

– S. Gaubert and J. Mairesse, "Task Resource Models and (max,+) Automata", Idempotency, Cambridge University Press, 1995.

– B. Gaujal, A. Jean-Marie and J. Mairesse, "Minimal Representation of Uniform Recurrence Equations". INRIA Report #2568, June 1995.

– A. Jean-Marie, "Analytical computation of Lyapunov exponents in stochastic event graphs", In O. Boxma and G. Koole, editors, Third QMIPS Workshop, CWI Tracts #106, 1994.

– J. Mairesse, "A graphical approach of the spectral theory in the (max,+) algebra", *IEEE Trans.*

Automatic Control, September 1995. To appear.

– J. Mairesse, "Products of irreducible random matrices in the (max,+) algebra", Technical Report #1939, INRIA, Sophia Antipolis, France, 1993. To appear in *JAP*.