COMPETITION FOR POPULARITY IN SOCIAL NETWORKS

Part 1: Theory

Game Theoretic Tools

1 Networking Games Examples and Classification

1.1 The Association Probelem

- To which WIFI shall we connect?

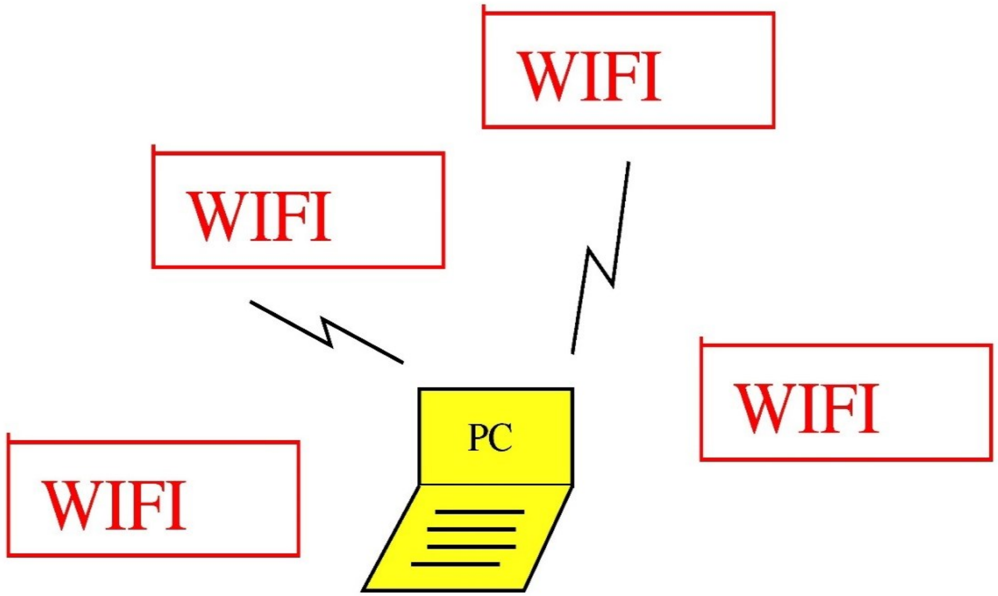


Figure 1: The Association Problem (1)

The man-machine interface

Figure 2: The association problem: the display

The Association Problem

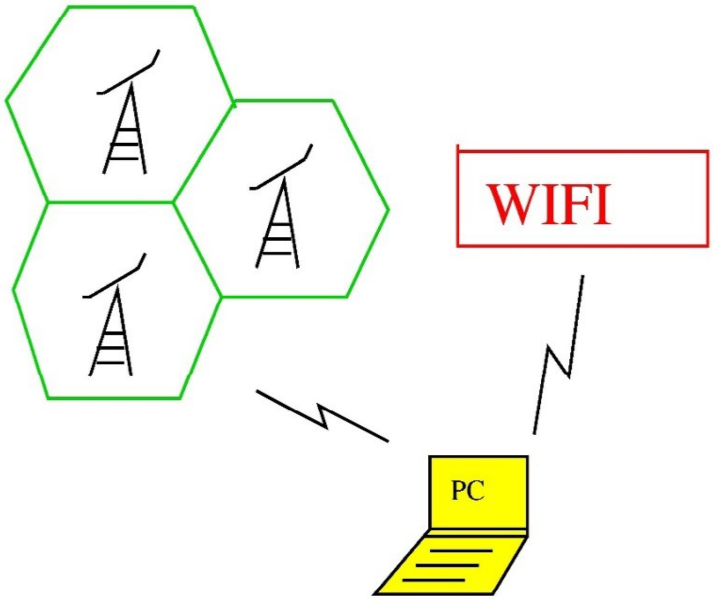
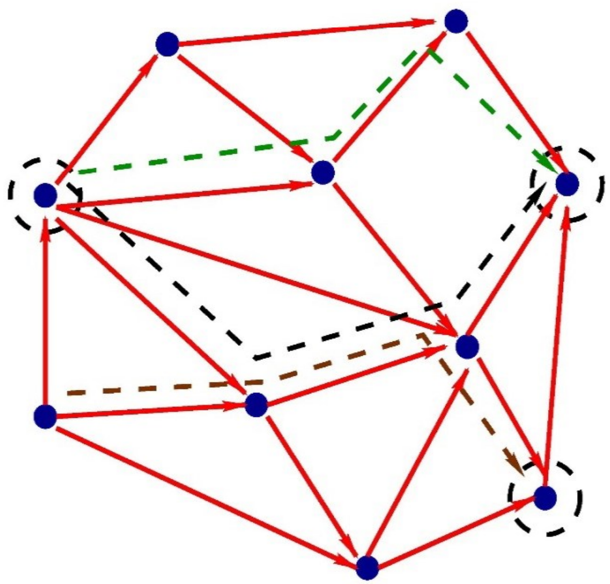


Figure 4: The Association Problem (3)

1.2 Routing



•Objectives:

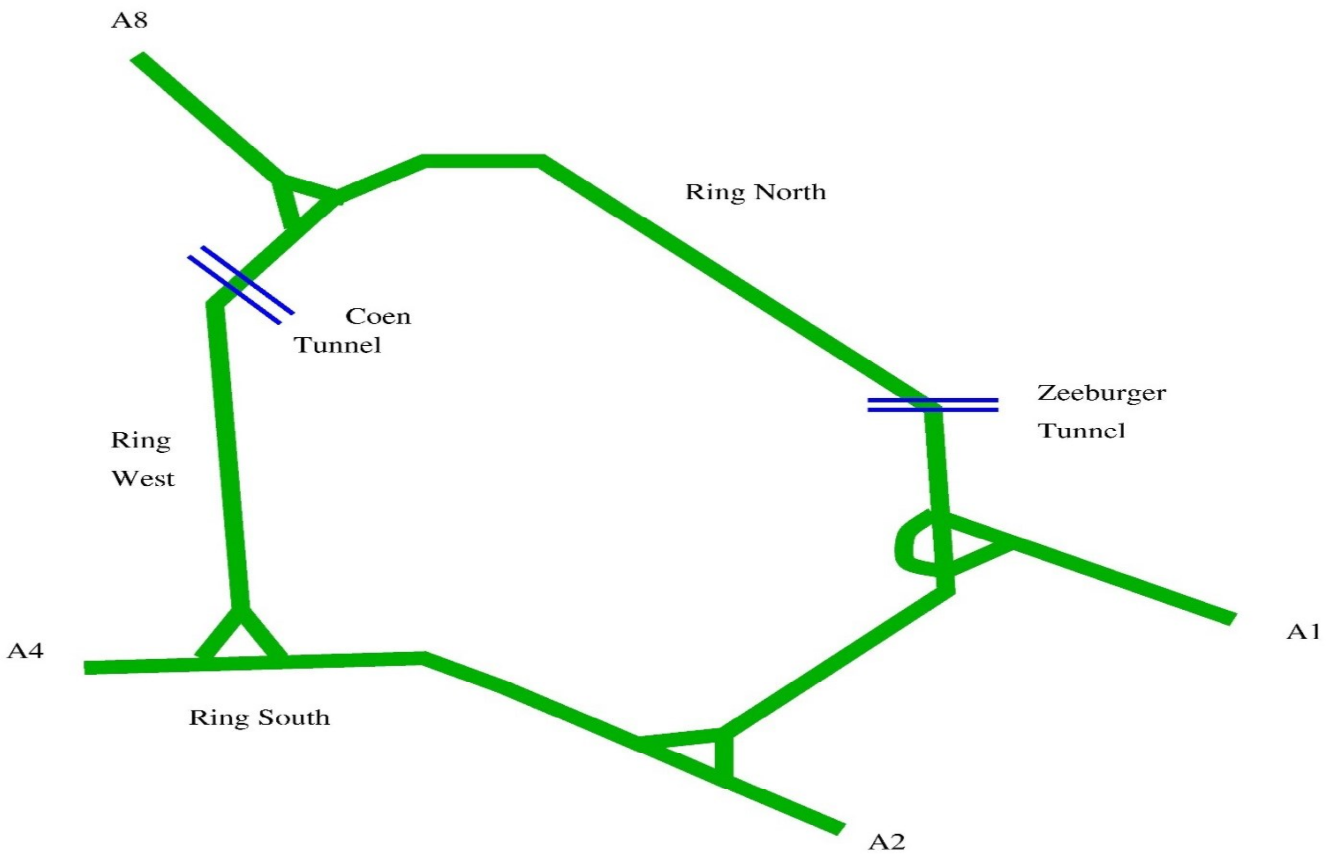
- **Cooperative case:** minimize global cost.
- **non-cooperative case:** minimize individual cost.

Example: The highway around Amsterdam

European project DRIVE II:

The vehicles have to decide the direction on the ring.

Objective: signalisation



Games with finite number of actions

2 Introduction to non-cooperative games

2.1 Example: The Prisoner's Dilemma

•Emil and France are suspected of a crime.

•If both admit, they get 1 year in prison each.

•If both do not admit, they get 10 year in prison each.

•If one admits, he gets 15 years and the other is freed.

EMIL \ FRANCE	1	2
I	(1,1)	(15,0)
II	(0,15)	(10,10)

Figure 5: Prisoner's Dilemma

•The solution (1,1) is unstable: each prisoner may try to improve by deviating. Only stable solution is (10,10).

Optimality Concepts

•**Nash Equilibrium:** A "stable" solution: no player has an incentive to deviate unilaterally.

•**Pareto Equilibrium:** A cooperative "dominating solution": there is no way to improve the performance of one player without harming another one.

2.2 Inefficiency of Nash equilibria

FRANCE

EMIL	1	2
I	(1,1)	(15,0)
II	(0,15)	(10,10)

Pareto Equilibrium

Nash Equilibrium

•Nash Equilibria may be very inefficient!

•(10,10) is called an equilibrium in **pure strategies**

2.3 Non-monotonicity of Nash equilibria

Assume we change the "0" to "5": a prisoner that does not admit is always punished (we may have prior information that the crime was committed together).

FRANCE

EMIL	1	2
I	(1,1)	(15,5)
II	(5,15)	(10,10)

(1,1) is Nash equilibrium and unique Pareto solution!

2.4 Pure and Mixed Nash equilibria

		Player II L	Player II R
Pl. I	T	0 (0)	1 (−1)
Pl. I	B	1 (−1)	0 (0)

•Matrix U: the utility for player 1 is one if his action is different than that of player j. Else it is zero. Player 2 wants to minimize that utility: zero-sum game.

•**No pure Nash equilibrium!** Consider mixed actions: $p = (p_T, p_B)$ and $q = (q_L, q_R)$ are probabilities to choose actions by the players. The utility for player 1 is $\mathcal{U}(p, q) = pUq^T$.

•$p = q = (1/2, 1/2)$ is an equilibrium. With mixed policies, **every matrix game has an equilibrium**

Definitions

Let a be a given action vector and let J(i;a) be the utility function of player i

Define a(-i) the vector obtained from a by omitting a(i).

Define [a(-i) , b(i)] to be the action vector obtained from a by replacing the i-th component a(i) of a by b(i).

a(i) is said to be a best response action for player i at a(-i) if it maximises

J(i;[a(-i) , b(i)]) over all possible b(i)

An action vetor a is an equilibrium if for all player i, a(i) is a best response at a(-i)

Concave Games with Continuum strategy set

Theorem [Rosen]

Assume

- Action space of each player i is a convex compact subet of real valued vectors of finite dimension n(i)
- Assume that the utility J(i;a), a=(a(1),…,a(I)) of player i is concave in a(i) and continuous in a(j) for j ≠ i.

Then the game has a Nash equilibrium in pure strategies.

Example: power control

- Assume each of I mobile terminals controles its transmission power p(i) to a common base station
- Assume symmetric Gaussian channels with no attenuation such that p(i) is also the received power from terminal i
- Shannon Theorem states that the throughput that can be achieved by mobile i is

$$\text{Thp}(i) = \log(\ 1\ +\ SINR(i)\ \ )$$

where

$$\text{SINR}(i) = \frac{p(i)}{N + \sum_{j \neq i} p(j)}$$

and where N is some constant (thermal noise at the BS)

The equilibrium

The utility for player i is given by

$$J(i;p) = Thp(i) - g\,p(i)$$

It is concave in p(i) and continuous in p(j) so a pure equilibrium exists

Equivalent game

- The value of a(i) that maxmises the utiliy function J(i;a) of player i is unchanged if we add to J(i;a) a constant.
- It is unchanged if we add to J(i;a) a function of the actions of players j ≠ i
- It is also unchaned if we replace it with h(J(i;a)) where h is any strictly monotone increasing function

Example: the power control game

- Let |p| := $\sum_{j=1}^{I} p(j)$. Then for given p(j), j ≠ i,

p(i) maximises Thp(i) - g p(i) if and only if it maximises

Thp(i) - g |p|

Potential

Thp(i) = log(1 + $\frac{p(i)}{N+\sum_{j\neq i} p(j)}$)

= log (N + $\sum_{j=1}^{I} p(j)$) – log (N + $\sum_{j\neq i} p(j)$)

Note that the second term does not depend on p(i). Thus p(i) maximises J(i;p) if and only if it maximises

V(|p|) := log (N + |p|) - g |p|

V is the same for all players. Any optimal solution of max V is an equilibrium for the original game. V is called a potential.

Potential Games

- V is a potential if for every i and every vector a and action b(i)

J(i ; a) - J(i ; [a(-i),b(i)]) = V(a) - V ([a(-i),b(i)])

Equivalently, for all i

$$\frac{\partial V(a)}{\partial\, a(i)} = \frac{\partial J(i;a)}{\partial\, a(i)}$$

Convergence to equilibrium

- Define a best responsee sequence as a couple (t(n), i(n)) where t(n) is a strictly increasing seqence of times such that at time t(n) player i(n) updates its action using a best response to the current actions of the other players. We assume that each player appears infinitely often in i(n)
- A change of a policy by a player results in higher utility if and only if it results in a higher potential.
- This implies that any sequence of best responses of the players converges to a local maximum of the potential in finite time
- If the potential is strictly concave then it has a uniqe local maximum which is a global maximum
- Therefore there is a unique equilibrium and any sequence of best responses of the players converges to it in finite time

Congestion games over parallel links [Rosenthal]

- There are M parallel links between a source S and a destination D.
- Each of I players have one unit flow to ship. It has to decide through which link to ship that flow.
- Let x(m) be the total flow sent to link m under some action vector a
- The cost to ship a unit of flow through link m is f(m , x(m))
- Assume payer i decides to change from link k to n. Then

  C(i;a) – C(i ; [a(-i) , n]) = f(k , x(k)) - f(n , x(n)+1)

- Define

$$V(x) = \sum_{m=1}^{M} v(m,x) \quad where \quad v(m,x) := \sum_{j=1}^{x(m)} f\,(m,j)$$

Then V is a potential.

Conclusions and comments

- There exists a pure equilibrium
- Each local minimum of the potential is an equilibrium.
- If f(m,x) are integer convex then there is a unique local minimum of V and thus a unique equilibrium

Application to a content provider (CP) game

- There are M types of contents and I CPs.
- Each CP has to decide in what content to specialise
- A utility of a CP i if it specializes in coontent type m is a function f(m;x(m)) of the total number x(m) that specialize in m. It is assumed to be strictly concave decreasing.
- Indeed, the more CPs specialize in a content m, the less is the revenue of those that specialize in that content.

- Then there is a unique equilibrium in pure strategies to which any best response sequence converges

Generalized Kelly mechanism

- Some resource, C is to be split between I players.
- Each player i makes a bid a(i)
- The fraction of C received by a player is proportional to its bid
- It has to pay some cost proportional to its bid.
- The utility of player i

$$J(i;a) = U\ (i, d(i)) - a(i)$$

$$d(i) := \frac{a(i)}{\sum_{j=1}^{I} a(j)} C \text{ if } |a| > 0 \quad \text{and is}$$

otherwise 0

- U(i,.) is a utility that may depend on the player i.

A potential game

- Define $\mathrm{V}(a) = \sum_{i=1}^{I} v(i,a)$ where

$$v(i,a) = \left(1 - \frac{d(i)}{C}\right) U\big(i, d(i)\big) + \frac{d(i)}{C}\left(\frac{1}{d(i)}\int_{0}^{d(i)} U(i,z)dz\right)$$

V is a potential for the game and an equilibrium is obtained by maximizing V over nonnegative vectors d satisfying |d| ≤ C. Indeed

$$\frac{\partial V(a)}{\partial\, a(i)} = \frac{\partial U(i,d(i))}{\partial\, a(i)}$$

If U are concave continuous then the equilibrium is unique

References

- B Hajek and Gopalakrishnan
- Johari and Tsitsiklis, MOR, 2004