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Jeux stratégiques non-atomiques et applications aux réseaux

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Introduction

La théorie des jeux est l'étude de situations où plusieurs individus interagissent. Une interaction spécifie le comportement de chaque individu et donne à tous une utilité. Cette dernière se mesure par une fonction réelle, appelée fonction d'utilité ou fonction de paiement, que tout individu est supposé maximiser. Un individu est appelé joueur et un comportement stratégie. Formellement : I est l'ensemble des joueurs, S_i l'ensemble de stratégies du joueur $i \in I$ et $u_i : S = \Pi_{i \in I} S_i \rightarrow \mathbb{R}$ est sa fonction de paiement. Le triplet $(I, (S_i)_{i \in I}, (u_i)_{i \in I})$ constitue un *jeu* (stratégique), une interaction est décrite par un *profil de stratégies* $s = (s_i)_{i \in I} \in S$.

Un concept fondamental dans les jeux stratégiques est l'*équilibre de Nash* ; il s'agit d'un profil de stratégies tel qu'aucune déviation unilatérale ne soit profitable au joueur déviant. Explicitement, $s = (s_j, s_{-j})$ est un équilibre de Nash du jeu $(I, (S_i)_{i \in I}, (u_i)_{i \in I})$ si et seulement si

$$u_j\left(s)\right) \geq u_j\left(s'_j, s_{-j}\right), \quad \forall j \in I, s'_j \in S_j.$$

Lorque le mot équilibre est employé ci-après, il doit s'interpréter comme une variation de l'équilibre de Nash.

L'objet de cette thèse est double. D'une part il consiste en l'analyse de jeux avec un continuum de joueurs où la stratégie d'un de ceux-ci, quel qu'il soit, a une influence nulle sur la fonction de paiement de n'importe quel autre joueur. Ces jeux sont appelés jeux *non-atomiques*.

L'usage des jeux non-atomiques permet d'obtenir des résultats comme l'existence d'équilibre lorsque les ensembles de stratégies sont finis (autrement dit équilibre en stratégie pure, ce qui n'est généralement pas le cas pour les jeux avec un nombre fini de joueurs) et d'utiliser des outils d'analyse fonctionnelle.

Cependant, une infinité de joueurs (*a fortiori* un continuum) ne se rencontre pas "dans la réalité". Les jeux non-atomiques sont donc utilisés pour décrire des interactions avec un grand nombre d'individus et où le comportement de quiconque a une influence négligeable sur le paiement des autres. Nous appelons ces jeux *grands jeux*.

Nous montrons que les jeux non-atomiques sont de bons modèles pour analyser les grands jeux. Ceci est établi en considérant des suites de jeux avec un nombre fini de joueurs, où l'influence d'un joueur s'evanouit lorsque le nombre de joueurs augmente (dorénavant *hypothèse d'évanescence*). Puis, nous étudions diverses applications des jeux non-atomiques. Enfin, nous proposons un raffinement de l'équilibre que nous appelons *stabilité*. Ceci constitue la première partie de cet ouvrage.

L'autre volet de ce manuscrit considère des problèmes dans les domaines des télécommunications et de l'Internet, autrement dit des réseaux, dans une optique de théorie des jeux. Sont considérés des modèles où un grand nombre de paquets (dans le cadre du trafic routier, ceux-ci correspondent à des véhicules), modélisé par un continuum, doivent aller d'un point du réseau à un autre. Pour ce faire les paquets peuvent être délivrés via différents chemins (routes) possibles. Tout chemin présente un coût dépendant de sa fréquentation ou plus généralement de la répartition des paquets au travers des différents chemins. Les décisions de routage, c'est-à-dire le choix des routes pour les paquets, peuvent se faire de trois manières différentes :
(1) la décision centralisée : une entité organise la circulation des paquets afin de minimiser le coût total de transport,
(2) la décision groupée : les paquets sont groupés, chaque groupe ayant un décideur qui essaie de minimiser le coût de "son" groupe,
(3) la décision individuelle : chaque paquet possède son propre décideur qui choisit son chemin afin de minimiser son coût.
Le premier cas est un problème d'optimisation qui n'entre pas dans le cadre de la théorie des jeux et n'est donc pas considéré ici. Les deux derniers cas sont des jeux où les joueurs sont les décideurs et les fonctions de paiement sont les opposées des fonctions de coût. Le cas (2) correspond à un jeu avec un petit nombre de joueurs et le (3) à un jeu non-atomique.

Nous proposons et étudions tout d'abord un concept d'équilibre pour des modèles dans lesquels décision groupée et décision individuelle coexistent : certains paquets se routent seul, d'autres s'en remettent à une entité supérieure. Puis nous établissons la convergence de systèmes dynamiques vers des équilibres dans des jeux à deux joueurs présentant une architecture de réseaux spécifique. Finalement, nous modélisons une situation propre aux réseaux de communications où les paquets doivent aller d'une origine à plusieurs destinations.

PARTIE 1 : JEUX STRATÉGIQUES NON-ATOMIQUES

Le premier objectif de cette partie est de montrer que les modèles de jeux non-atomiques proposés par Schmeidler (1973) et Mas-Colell (1984) constituent de bons outils pour analyser des situations de jeux avec un grand nombre de joueurs satisfaisant l'hypothèse d'evanescence (grands jeux).

Un autre but est de présenter un cadre englobant diverses applications des jeux non-atomiques, tels les *jeux de routage*, les *jeux de foule* et les *jeux évolutionnaires* et d'y étendre un concept propre à ces derniers : la *stratégie evolutionnairement stable.*

Chapitre 1 : Deux modèles de jeux non-atomiques

Après avoir défini la non-atomicité, nous décrivons les jeux non-atomiques introduits par Schmeidler (1973) ($\mathcal{S}$*-jeux*) et ceux introduits par Mas-Colell (1984) ($\mathcal{M}$*-jeux*). Pour ces deux types de jeux, tous les joueurs ont le même ensemble de stratégies S, espace métrique compact.

Un $\mathcal{S}$-jeu est caractérisé par une fonction $\mathbf{u}$ qui associe à chaque joueur $t_0 \in T = [0,1]$ (muni de la mesure de Lebesgue, λ), une fonction de paiement u_{t_0}. La fonction u_{t_0} dépend de la stratégie de t_0 et d'une fonction mesurable f associant à chaque joueur $t \in T$ une stratégie $s_t \in S$. Une fonction f décrit une interaction et se nomme *profil de stratégies*.

Dans un $\mathcal{M}$-jeu, les fonctions de paiement dépendent de la stratégie du joueur concerné et de la répartition des stratégies au sein de la population. Plus précisément, un $\mathcal{M}$-jeu est décrit par une probabilité sur l'ensemble des fonctions réelles continues sur $S \times \mathcal{M}(S)$, noté $C(S \times \mathcal{M}(S))$ (où $\mathcal{M}(X)$ est l'ensemble des probabilités sur X, muni de la topologie faible) et les interactions sont modélisées par des probabilités sur l'ensemble produit $C(S \times \mathcal{M}(S)) \times S$.

Losque les $\mathcal{S}$-jeux et les $\mathcal{M}$-jeux sont définis sous les mêmes hypothèses, nous établissons les liens les unissant. Une des hypothèses pour les $\mathcal{S}$-jeux est que la fonction de paiement de tout joueur dépend d'un profil de stratégies f uniquement à travers sa distribution -ou mesure image- $f[\lambda] = \lambda \circ f^{-1} \in \mathcal{M}(S)$ (le jeu est alors dit *anonyme*). Nous montrons que
(i) la measure image $\mathbf{u}[\lambda]$ d'un $\mathcal{S}$-jeu anonyme $\mathbf{u}$ est un $\mathcal{M}$-jeu,
(ii) la measure image $(\mathbf{u}, f)[\lambda]$ de $\mathbf{u}$ et d'un de ses *équilibres* f est un *équilibre* pour le $\mathcal{M}$-jeu $\mathbf{u}[\lambda]$.

Réciproquement nous établissons la représentation des $\mathcal{M}$-jeux (et de leurs équilibres) par des $\mathcal{S}$-jeux anonymes. Ce dernier résultat est une application du théorème de représentation de Skorohod.

Chapitre 2 : Approximation des grands jeux par les jeux non-atomiques

Nous appelons *suite évanescente de jeux*, une suite de jeux avec un nombre fini de joueurs (dorénavant jeux finis), telle que l'influence de tout joueur sur le paiement des autres s'évanouit lorsque le nombre de joueurs augmente. Nous considérons toujours des jeux finis où tous les joueurs ont le même ensemble de stratégies S.

Il est montré ici que les jeux non-atomiques constituent un cadre adéquat pour analyser les grands jeux.

Pour ce faire, nous présentons trois résultats d'approximation correspondant aux trois formulations de jeux finis suivantes.
Forme adaptée [1] : où l'ensemble des joueurs est plongé dans l'intervalle $[0, 1] = T$. Les profils de stratégies y sont des fonctions f adaptées à une certaine partition $\mathcal{T}^n$ de T. Le jeu est alors représenté par une fonction $\mathbf{u}$ adaptée à $\mathcal{T}^n$.
Forme quasi-normale : $P^n = \{1, \ldots, n\}$ est l'ensemble des joueurs, un profil de stratégies est une fonction de P^n dans S. Le jeu est décrit par une fonction qui associe à chaque joueur i sa fonction de paiement $u_i : S^n \to \mathbb{R}$.
Forme probabiliste : les fonctions de paiement sont définies sur $S \times \mathcal{M}(S)$, et le jeu est donné par une probabilité atomique μ sur l'ensemble de ces fonctions.

A la forme adaptée, nous associons une convergence "uniforme", à la forme quasi-normale, la convergence en distribution utilisée par Hildenbrand (1974) et à la forme probabiliste, la convergence faible des mesures. Les deux premières approximations concernent les $\mathcal{S}$-jeux, la dernière les $\mathcal{M}$-jeux. Les résultats obtenus sont du type suivant.

[1]Etant donnée une partition mesurable $\mathcal{T}$ de T, une fonction φ ayant pour domaine de définition T et constante sur chaque élément de $\mathcal{T}$ est appelée fonction *adaptée* à la partition $\mathcal{T}$.

Soient une suite évanescente de jeux G^n qui converge vers un jeu non-atomique G et une suite d'interactions φ^n de G^n qui converge vers une interaction φ de G. Alors
(1) si φ^n est un équilibre de G^n, pour un nombre infini de n, alors φ^n est un équilibre de G et inversement,
(2) si φ est un équilibre de G alors pour n assez grand φ^n est "approximativement" un équilibre de G^n.

Chapitre 3 : Extensions et variations

Ce chapitre présente tout d'abord des résultats sur l'extension des $\mathcal{S}$-jeux anonymes établis par Rath (1992) et Khan-Sun (2002), puis étend les $\mathcal{M}$-jeux à des situations où
(i) l'ensemble des joueurs est partionné en un nombre fini de *populations*, chaque population ayant son propre ensemble de stratégies,
(ii) les fonctions de paiement dépendent des répartitions des stratégies au sein de chaque population.
Les jeux ainsi définis sont appelés *$\mathcal{C}$-jeux*. Nous montrons l'existence d'équilibre pour les $\mathcal{C}$-jeux de deux manières différentes. La première consiste en l'application classique d'un théorème de point-fixe à une correspondance de meilleures réponses (ses points fixes sont des équilibres). La seconde considère une fonction allant de l'ensemble des $\mathcal{C}$-jeux dans celui des $\mathcal{M}$-jeux et montre qu'à un équilibre d'un $\mathcal{M}$-jeu μ correspond un équilibre du $\mathcal{C}$-jeu dont μ est l'image par la fonction mentionnée ci-dessus.

Nous nous restreignons ensuite à une classe de $\mathcal{C}$-jeux où tous les joueurs d'une même population ont la même fonction de paiement. Dans ces jeux, appelés *jeux de population*, l'existence d'équilibre peut s'obtenir à l'aide d'inégalités variationnelles. Comme il est montré dans le chapitre 4, ces jeux englobent diverses applications des jeux non-atomiques.

Finalement, nous présentons une sous-classe des jeux de population, les *jeux de potentiel* introduits par Sandholm (2001). Ces jeux ont la propriété de posséder une fonction, appelée *fonction de potentiel*, dont les maxima sont des équilibres du jeu considéré. Sandholm (2001) établit un résultat d'approximation des jeux de potentiel avec un nombre fini de joueurs par les jeux de potentiel avec un continuum de joueurs. Nous montrons que ce résultat est plus restrictif que les approximations obtenues au chapitre 2.

Chapitre 4 : Applications des jeux non-atomiques

Ce chapitre est consacré à l'étude de trois types de jeux de population.

Les *jeux de routage* sont tout d'abord considérés. Le contexte est celui d'un trafic routier et un jeu modélise des situations où un grand nombre de véhicules doivent aller d'une origine à une destination. Pour ce faire ils ont plusieurs routes possibles et chacun désire minimiser son temps de trajet, temps qui dépend de la route choisie et de sa fréquentation. Nous sommes donc dans le cas de décision individuelle définie au début de l'introduction. La solution d'équilibre dans ces jeux remontent à Wardrop (1952). Nous présentons un résultat d'approximation des équilibres des jeux de routage finis (décision groupée) par l'équilibre de Wardrop dû à Haurie et Marcotte (1985).

Ensuite, sont étudiés les *jeux de foule*, introduits par Milchtaich (2000). Il s'agit d'une extension des jeux de congestion à n joueurs définis par Rosenthal (1973). Etant donné le choix d'une stratégie, la fonction de paiement d'un joueur dépend du "nombre" de joueurs ayant adopté cette stratégie et est continuement décroissante. Il s'agit donc d'un cas particulier de jeu de routage. Nous présentons un résultat de Milchtaich (2000) sur l'approximation de jeux finis par les jeux de foules.

Finalement, sont analysés les *jeux évolutionnaires*. Ils sont issus de la biologie évolutionnaire et remontent à Maynard Smith-Price (1973) et Maynard Smith (1974). Ils modélisent l'évolution des caractères au sein d'une espèce et abandonnent l'hypothèse de rationalité des joueurs, traditionnelle en théorie des jeux. Ces modèles considèrent des interactions entre individus d'une même espèce (ou population) qui se font de manière répétée ; chaque individu a une stratégie fixée (un caractère) et son taux de reproduction dépend du paiement qu'il obtient au cours des interactions. Une probabilité sur l'ensemble des stratégies représente donc une répartition des caractères au sein de l'espèce ou encore une *composition de la population*. Ainsi plus une caractéristique est efficace pour une composition de la population, plus elle aura tendance à être représentée. Un équilibre dans ces jeux correspond à une situation où chaque caractéristique a le même taux de reproduction : il détermine donc une composition "stable" de la population. Mais cette stabilité peut s'avérer très sensible aux petites perturbations de la population (comme les mutations). C'est pourquoi la théorie des jeux évolutionnaires s'intéresse au concept de *stratégie évolutionnairement stable* qui définit des compositions de population robustes aux changements lorsqu'ils sont suffisamment petits.

Chapitre 5 : Raffinement de l'équilibre : la stabilité

Ce chapitre conclut la première partie en étendant la notion de *stratégie évolutionnairement stable* aux $\mathcal{S}$-jeux définissant ainsi les *profils de stratégies stables*. Un profil de stratégies est stable si, pour toute déviation d'un ensemble de joueurs de mesure "faible", étant donné le nouveau profil, la moyenne des paiements de ces joueurs est plus faible que la moyenne des paiements obtenus s'ils n'avaient pas dévié individuellement. Formellent, f est stable pour le $\mathcal{S}$-jeu $\mathbf{u}$ s'il existe $\bar{\varepsilon} > 0$ tel que, pour tout sous-ensemble T_ε de T de measure $\varepsilon < \bar{\varepsilon}$ et pour tout profil g avec $g(t) = f(t)$ pour tout $t \in T \setminus T_\varepsilon$,

$$\int_{T_\varepsilon} u_t(f(t), g)\, d\lambda(t) > \int_{T_\varepsilon} u_t(g(t), g)\, d\lambda(t).$$

Nous montrons d'abord qu'un profil de stratégies stable dans un $\mathcal{S}$-jeu $\mathbf{u}$ est un équilibre. Puis, étant donnés un $\mathcal{S}$-jeu et une suite de jeux sous forme adaptée $\mathbf{u}^n$ convergeant vers $\mathbf{u}$, nous caractérisons l'ensemble des profils de stratégies stables de $\mathbf{u}$ comme les points limites de suites d'équilibres stricts f^n de $\mathbf{u}^n$. Ensuite, nous nous plaçons dans le cadre des jeux de potentiel afin d'étudier les relations entre stabilité et maxima des fonctions de potentiels. Finalement, nous présentons un processus de sélection de profils de stratégies dont les solutions sont stables.

PARTIE 2 : APPLICATIONS DE LA THÉORIE DES JEUX AUX RÉSEAUX

Cette partie est constituée de trois articles ayant trait aux problèmes de modélisation du routage dans des réseaux.

Un réseau est un graphe orienté fini dans lequel un continuum de paquets doivent circuler. A chaque paquet est associé un couple de sommets (origine-destination). Les paquets ont différents chemins à leur disposition et tout chemin a un coût dépendant de la répartition des paquets dans le graphe. Ce coût peut représenter par exemple un temps de transport ou une probabilité de perte. Le graphe constitue l'*architecture* du réseau.

Nous considérons deux types de décision définis au début de l'introduction : décision groupée et décision individuelle. Le premier type définit un jeu avec un nombre fini de joueurs où le concept de solution étudié est l'équilibre de Nash. Pour le deuxième (un jeu non-atomique), le concept de solution adopté est l'équilibre de Wardrop : tout chemin utilisé doit avoir un coût moindre que n'importe quel autre chemin (ayant même origine-destination).

Chapitre 6 : L'équilibre mixte dans les jeux de routage

Plaçons nous dans un contexte de trafic routier afin d'expliquer le problème modélisé dans ce chapitre.

Soit un réseau où de nombreux véhicules doivent aller d'une origine à une destination, tous voulant minimiser leur temps de trajet. Origines et destinations dépendent des véhicules mais sont en nombre fini. Un véhicule peut soit déterminer seul son itinéraire (décision individuelle), soit s'abonner à un système de guidage qui lui recommande un itinéraire (décision groupée). On suppose que les abonnés suivent la proposition conseillée et que tout système de guidage cherche à minimiser le temps de trajet moyen de ses clients.

Nous modélisons cette situation par un jeu composé de deux types de joueurs : des "gros" joueurs représentant les systèmes de guidage et des "petits" joueurs correspondant aux véhicules "'indépendants". L'ensemble des gros joueurs, $\mathcal{N}$, est fini, celui des petits joueurs, $\mathcal{W}$, est un continuum. Une stratégie d'un gros joueur $i \in \mathcal{N}$ est le choix d'une répartition de ses paquets au travers des differentes routes : S_i est donc le simplexe de $\mathbb{R}^{k_i}$ (k_i étant le nombre de chemins possibles pour i). Celle d'un petit joueur est le choix d'un itinéraire. Supposons pour alléger la présentation que tous les petits joueurs ont les mêmes origine, destination, ensemble de chemins S_w et fonction de coût u_w. S_w consiste en la base canonique de $\mathbb{R}^{k_w}$. Ce jeu peut s'écrire sous la forme $(\mathcal{N}, \mathcal{W}, (S_i)_{i\in\mathcal{N}}, S_w, (u_i)_{i\in\mathcal{N}}, u_w)$ où u_i est la fonction de coût du joueur $i \in \mathcal{N}$.

Nous définissons un concept d'équilibre pour de tels jeux : l'*équilibre mixte*. Nous montrons son existence pour des architectures quelconques de deux manières différentes. La première consiste en la caractérisation de l'équilibre en termes d'inégalités variationnelles, du type

$$F(x) \cdot (y - x) \geq 0, \quad \forall y \in \mathcal{X} = \Pi_{i\in\mathcal{N}} S_i \times co(S_w) \subset \mathbb{R}^n,$$

où $F : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}^n$ et $co(S_w)$ est l'enveloppe convexe de S_w. L'existence d'une solution à ce problème est connue lorsque F est continue et $\mathcal{X}$ est non-vide, convexe et borné. La deuxième décompose, dans un premier temps, le jeu en deux sous-jeux, l'un, $(\mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$, composé des gros joueurs et l'autre, $(\mathcal{W}, S_w, u_w)$, des petits joueurs. Il est ensuite montré que le jeu non-atomique $(\mathcal{W}, S_w, u_w)$ admet une fonction de potentiel φ. Finalement, l'on construit un jeu à $\#\mathcal{N}+1$ joueurs pour lequel le $\#\mathcal{N}+1^{ème}$ joueur a pour ensemble de stratégies $co(S_w)$ et pour fonction de coût φ. Un équilibre de Nash de ce jeu est alors un équilibre mixte du jeu originel.

Ensuite, nous établissons des conditions pour l'unicité de l'équilibre mixte. L'unicité de l'équilibre de Nash étant rare, nous ne pouvons nous attendre à obtenir celle de l'équilibre mixte sans certaines restrictions. Nous l'obtenons pour
(i) des fonctions de coût spécifiques et des architectures quelconques,
(ii) des architectures simples et des fonctions de coût générales.

Chapitre 7 : Convergence vers l'équilibre de Nash dans un problème de systèmes distribués

Nous considérons des jeux de routage avec deux types d'architectures dans lesquelles tous les arcs possibles pointant vers un sommet spécifique d sont présents.

Dans la première architecture, les autres sommmets sont au nombre de deux et sont reliés par une arête (arc non orienté). Dans la deuxième, les autres sommets (n) forment un circuit orienté.

Dans les deux réseaux le sommet d est la destination commune à tous les joueurs. A chaque sommet différent de d est associé un joueur. Tout joueur a un continuum de paquets à router vers d. Nous sommes donc dans le cas de décisions groupées.

Pour chacun des jeux, nous montrons l'unicité de l'équilibre de Nash. Par la suite, nous étudions la convergence d'un algorithme de meilleure réponse où les joueurs actualisent leur stratégie à tour de rôle. La convergence est due, dans le premier jeu, à la monotonie de la suite engendrée par l'algorithme et, dans le deuxième, à la contraction des écarts entre deux pas consécutifs de l'algorithme. De plus, pour le premier jeu, nous étendons la convergence à deux autres types d'algorithme.

Chapitre 8 : Routage dans des réseaux de télécommunications multipoint

Ce dernier chapitre traite de la modélisation par la théorie des jeux d'une situation de communication multipoint : les paquets doivent aller d'une origine à plusieurs destinations. Pour ceci, les paquets peuvent se dédoubler à chaque noeud du réseau.

Nous modélisons cette situation par un jeu à n joueurs où la décision d'un joueur consiste en la répartition de ses paquets au travers de chemins multipoints virtuels représentés par des arbres. Dans un arbre, tout joueur envoie uniquement une copie de chaque paquet, le réseau dupliquant l'information aux sommets appropriés. Cette construction rend caduc les méthodes utilisées pour les communications de point-à-point.

Nous analysons trois types de fonctions de coût, puis nous définissons l'équilibre de Nash de ce jeu et étudions son unicité. Celle-ci est obtenue pour des fonctions de coût générales et des architectures spécifiques (et inversement). Nous établissons également la convergence vers l'équilibre d'une dynamique de meilleures réponses sous des conditions sur les coûts et les architectures. Une des fonctions de coût étudiées permet d'itentifier équilibre de Nash et équilibre de Wardrop. Ainsi dans ce cadre, l'unicité de l'équilibre de Nash et la convergence vers celui-ci de dynamiques du type Brown-Von Neumann (1950) sont obtenus.

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Part I

Nonatomic strategic games

Notations

For any metric space S, $\mathcal{B}(S)$ denotes the Borel σ-algebra on S, and $\mathcal{M}(S)$ the set of probabilities on $\mathcal{B}(S)$. By a probability μ on S we shall understand an element of $\mathcal{M}(S)$.
Let A be a finite set: $A = \{1, \ldots, m\}$. It is identified with the standard basis of $\mathbb{R}^m$ consisting of the coordinate axes: Each element $a \in A$ is send to the vector $\delta_a \in \mathbb{R}^m$, defined by
$\delta_a = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$
In other terms, δ_a is the Dirac measure on a. Thus the convex hull of A, denoted by $co(A)(=\mathcal{M}(A))$ is identified with the simplex $\Delta_m = \Delta$ of $\mathbb{R}^m$.
For any compact set X, the space a real-valued continuous functions on X is denoted by $C(X)$.
For a set B, 1_B denotes the indicator function of B.
The inner product in $\mathbb{R}^m$ is denoted by $\cdot$ and its euclidean norm by $\|\cdot\|$.
The notation $:=$ indicates a definition.
A superscript refers to the set of actions, whereas a subscript refers to the set of players.

When a reference is made to a lemma, a proposition, a theorem, ..., the first character refers to the chapter, for example Theorem 1.5 is located in Chapter 1, if the character is not a number but a letter, then the location is in the corresponding appendix for example Theorem A.15 is located in Appendix A.

Chapter 1

Two models of nonatomic games

In this chapter, first is defined *nonatomic measure space* a key notion of nonatomic games. Then two models of nonatomic games in which all players have same strategy set are described. They differ on the notion of strategy and on the set on which payoff functions are defined.

The first one is due to Schmeidler (1973): a strategy profile is a class of equivalence of measurable functions from the player set to the strategy set. The payoff function of a player, defined in a great generality, depends on his strategy and on the strategy profile played. We call these games $\mathcal{S}$*-games*.

The second one is due to Mas-Colell (1984)[1] : the main concern is not strategy profiles but probabilities over the product space (payoff function-strategy). Therefore payoff functions do not depend on the strategy profile played but only on the measure induced by this last. We call these games $\mathcal{M}$*-games*.

Section 1.2 presents $\mathcal{S}$-games and gives the proofs of equilibrium's existence in mixed and also in pure strategies. Section 1.3 deals with $\mathcal{M}$-games, proofs of equilibrium's existence are also given.

Finally Section 1.4 establishes relations between $\mathcal{S}$-games and $\mathcal{M}$-games. Namely, when both games satisfy common assumptions, it is shown that any $\mathcal{S}$-game induces an $\mathcal{M}$-game and that any $\mathcal{M}$-game can be represented by an $\mathcal{S}$-game and a correspondence between equilibria of $\mathcal{S}$-games and equilibria of $\mathcal{M}$-games is established.

1.1 The assumption of nonatomicity

We consider games with many players, where the strategy of any single player has a negligible influence on the payoff of the other players, but strategies of a group of players can affect any payoff functions. Thus, given a profile of players' strategies, if a player changes his strategy, he is the only one whose payoff may be concerned by this change. To describe such games we assume that the set of players is a nonatomic (or atomless) measure space:

1.1. Definition. Let $(T, \mathcal{T}, \lambda)$ be a measure space where T is a set, $\mathcal{T}$ is a σ-algebra

[1]All the references to Schmeidler and Mas-Colell will be to, respectively, Schmeidler (1973) and Mas-Colel (1984).

of subsets of T, and λ is a measure on $\mathcal{T}$; $(T, \mathcal{T}, \lambda)$ is *nonatomic* if

$$\forall E \in \mathcal{T} \text{ such that } \lambda(E) > 0, \quad \exists F \subset E \in \mathcal{T} \text{ such that } 0 < \lambda(F) < \lambda(E).$$

If $(T, \mathcal{T}, \lambda)$ is nonatomic, then T is uncountable. On the other hand, if T is an uncountable complete separable metric space, then there exists a nonatomic measure on the Borel σ-algebra of T (Parthasarathy, 1967, Theorem 8.1, p. 53).

We restrict our attention to the measure space of players $(T, \mathcal{B}(T), \lambda)$ where $T := [0, 1]$ and λ is Lebesgue measure; it is denoted $\underline{T} := (T, \mathcal{B}(T), \lambda)$.

1.2 $\mathcal{S}$-games

Let S be a set of strategies, S is a compact subset of $\mathbb{R}^m$.

1.2. Definition. A *strategy profile* is an equivalent class f of measurable functions from T to S. $\mathcal{F}_S$ denotes the set of strategy profiles.

1.3. Definition. An $\mathcal{S}$*-game* is defined by a triple $(\underline{T}, \mathcal{F}_S, (u_t)_{t \in T})$, where $\underline{T} = (T, \mathcal{B}(T), \lambda)$ is the measure space of players, and $u_t : S \times \mathcal{F}_S \to \mathbb{R}$ is the payoff function of player t, for each $t \in T$.

Schmeidler extended the notion of Nash equilibrium to $\mathcal{S}$-games as follows:

1.4. Definition. A strategy profile f of the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t \in T})$ is an *equilibrium strategy profile* if

$$\forall y \in S, \quad u_t(f(t), f) \geq u_t(y, f), \text{ for } \lambda\text{-a.e. } t \in T.$$

To prove the existence of an equilibrium strategy profile in an $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t \in T})$, we will define as usual a correspondence α whose fixed points will be equilibria. To apply fixed point theorems requires some assumptions on (i) the strategy set S, (ii) the set of strategy profiles $\mathcal{F}_S$ and (iii) the payoff functions u_t. We will present two sets of assumptions under which an $\mathcal{S}$-game has an equilibrium strategy profile.

1.2.1 Payoff functions defined on $S \times \mathcal{F}_S$

Let $A = \{1, \ldots m\}$ be a finite action set, it is identified with the standard basis of $\mathbb{R}^m$ (see page 17). Let S be the set of mixed strategies, then it is the simplex $\Delta_m = \Delta$ of $\mathbb{R}^m$, hence a convex, compact subset of $\mathbb{R}^m$. Moreover for all $t \in T$, the payoff function u_t can be defined for all $x \in \Delta$ and $f \in \mathcal{F}_\Delta$ as

$$u_t(x, f) := \sum_{a \in A} x^a u_t(a, f).$$

In other words, for all $t \in T$ and $f \in \mathcal{F}_\Delta$, $u_t(\cdot, f)$ is linear.

For $a \in A$, f^a is Lebesgue integrable. Then $\mathcal{F}_\Delta$ is a subset of $L_1(\lambda, \mathbb{R}^m)$ that we endow with the weak topology. For any subset $Z \in \mathcal{T}$, write $\int_Z f$ for the vector $\left(\int_Z f^a(t)\, d\lambda(t)\right)_{a\in A}$.

Under the following conditions of continuity and measurability:

(a) for all $t \in T$ and $a \in A$, $u_t(a, \cdot)$ is continuous on $\mathcal{F}_\Delta$,

(b) for all $f \in \mathcal{F}_\Delta$ and a, $b \in A$, the set $\{t \in T; u_t(a, f) > u_t(b, f)\}$ is measurable, Schmeidler shows the existence of an equilibrium strategy profile using Fan-Glicksberg fixed point theorem (Theorem A.1).

1.5 Theorem. *(Schmeidler, 1973, Theorem 1)*
An $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_\Delta, (u_t)_{t\in T})$ fulfilling conditions (a) and (b) has an equilibrium strategy profile.

The proof given follows the lines of (Khan, 1985, Theorem 4.1.)'s proof. It explicits two points of the original one. Namely,
(i) We show that $\mathcal{F}_\Delta$ is weakly compact, this is the next proposition.
(ii) We explain why a sequence of $\mathcal{F}_\Delta$ instead of a net can be used to prove the upper semicontinuity of a correspondence from $\mathcal{F}_\Delta$ to itself.

The specificity of these two points, compared to the proof of existence of Nash equilibrium for finite-player games, lays in the fact that $\mathcal{F}_\Delta$ is a subset of an infinite dimensional vector space.

1.6 Proposition. *$\mathcal{F}_\Delta$ is weakly compact.*

Proof: From Dunford-Pettis' theorem (Theorem A.2), to show that $\mathcal{F}_\Delta$ is weakly compact, it is enough to show that $\mathcal{F}_\Delta$ is weakly closed. Due to the convexity of $\mathcal{F}_\Delta$, this is equivalent to show, as stated by Mazur's theorem (Theorem A.3), that $\mathcal{F}_\Delta$ is closed in the norm topology. Let $\{f^n\}$ be a norm convergent sequence of $\mathcal{F}_\Delta$, and f be its limit, f belongs to $L_1(\lambda, \mathbb{R}^m)$. From (Dunford and Schwartz, 1988, Theorem III.3.6), $\{f^n\}$ converges in measure to f. Now apply (Dunford and Schwartz, 1988, Corollary III.6.3) to extract a subsequence $\{f^{n'}\}$ of the sequence $\{f^n\}$ which converges λ-almost everywhere to f. Finally note that Δ is closed to conclude that $f(t)$ belongs to Δ, λ-almost everywhere, so f belongs to $\mathcal{F}_\Delta$. ∎

Proof of Theorem 1.5: To establish the existence of an equilibrium we shall construct as usual a best reply correspondence and look for a fixed point of this correspondence.

For all $t \in T$ and $f \in \mathcal{F}_\Delta$, let $\mathrm{BR}_t(f)$ be the set of best reply strategies of t against f, that is $\mathrm{BR}_t(f) := \{x \in \Delta; u_t(x, f) \geq u_t(y, f), \forall y \in \Delta\}$.

Consider the correspondence $\alpha : \mathcal{F}_\Delta \to \mathcal{F}_\Delta$ with

$$\alpha(f) := \{g \in \mathcal{F}_\Delta; g(t) \in \mathrm{BR}_t(f), \text{ for } \lambda\text{-a.e. } t \in T\}.$$

A fixed point of α is an equilibrium strategy profile.

To apply Fan-Glicksberg fixed point theorem one needs to show that

1) $\mathcal{F}_\Delta$ is compact, nonempty and convex in a locally convex topological vector space.

2) For every strategy profile $f \in \mathcal{F}_\Delta$, the set $\alpha(f)$ is convex and nonempty.

3) The correspondence α is upper semicontinuous.

1) The nonemptyness and convexity of $\mathcal{F}_\Delta$ are straightforward, its (weak) compactness is established by Proposition 1.6. Moreover, from the definition of the weak topology, it follows that $L_1(\lambda, \mathbb{R}^m)$ is a locally convex topological vector space (Dunford and Schwartz, 1988, p. 419).

2) To prove the nonemptyness of $\alpha(f)$ we shall construct a strategy profile belonging to it. Consider a strategy profile $f \in \mathcal{F}_\Delta$. For any action $a \in A$, define the set

$$T^a := \{t \in T; u_t(a, f) \geq u_t(b, f), \forall b \in A\}.$$

This definition yields $T = \cup_{a \in A} T^a$ and for all players t in T^a, the vector a belongs to $\mathrm{BR}_t(f)$. Because of condition (b), T^a is measurable. Let

$$Z^1 := T^1 \text{ and } Z^a := T^a \setminus (\cup_{b=1}^{a-1} T^b).$$

The strategy profile g, defined by $g(t) := a$ for $t \in Z^a$, belongs to $\alpha(f)$.

The convexity of $\alpha(f)$ is due to the convexity of $\mathrm{BR}_t(f)$ for all $t \in T$.

3) Claim 3 is the hard task, first we shall show that BR_t is upper semicontinuous, then we shall proceed by contradiction to establish the upper semicontinuity of α.

The set Δ being compact, the upper semicontinuity of BR_t will follow if BR_t has a closed graph in $\mathcal{F}_\Delta \times \Delta$ (Proposition A.4). For all $a \in A$, the function $u_t(a, \cdot)$ is continuous (condition (a)), therefore the payoff function $u_t : \Delta \times \mathcal{F}_\Delta \to \mathbb{R}$ is jointly continuous, hence BR_t has a closed graph.

Similarly to show the upper semicontinuity of α is equivalent to show that it has a closed graph. Since the weak topology of an infinite dimensional normed vector space is not metrizable (Diestel, 1984, p. 10-11), one should have to consider a net to establish the closedness of the graph of α. Nevertheless the weak compactness of $\mathcal{F}_\Delta$ allows to consider a sequence instead of a net: Indeed, let $\{(f^\nu, g^\nu)\}$ be a net converging to (f^0, g^0), where $g^\nu \in \alpha(f^\nu)$ (we will have to show that $g^0 \in \alpha(f^0)$). The union of the net $\{(f^\nu, g^\nu)\}$ and (f^0, g^0) is is relatively weakly sequentially compact by virtue of Eberlein-Smulian's theorem (Theorem A.5). Now use Theorem A.6 to extract a sequence $\{(f^n, g^n)\}$ from $\{(f^\nu, g^\nu)\}$ that weakly converges to (f^0, g^0).

From Proposition 1.6's proof, it follows that $g^0(t)$ belongs to Δ, λ-almost everywhere.

We now show by contradiction that $g^0 \in \alpha(f^0)$. Assume that for a nonnull measurable subset Z of T, the strategies of players in S are not best replies against the strategy profile f^0, namely for all $t \in Z$, $g^0(t) \notin \mathrm{BR}_t(f^0)$. For each t, the set $\mathrm{BR}_t(f^0)$ is a convex hull of a subset of the set A. So there is a nonnull, measurable subset V of Z and a subset $\{j_1, \ldots, j_k\}$ of A such that, for each $t \in V$,

$$\mathrm{BR}_t(f^0) = co(\{j_1, \ldots, j_k\}) \text{ and } g^0(t) \notin \mathrm{BR}_t(f^0).$$

Hence there exists $y \in \Delta$ such that

$$y \cdot g^0(t) > 0 \text{ and } y \cdot j_l = 0,\ l = 1, \ldots, k.$$

For example let $y^a = 1/(m-k)$ for all $a \neq j_l$, $l = 1, \ldots, k$. Note that y does not depend on t.

So the inner product $y \cdot \int_V g^0$ is (strictly) positive, but for each $g \in \mathcal{F}_\Delta$ with $g(t) \in \mathrm{BR}_t(f^0)$ for $t \in V$,

$$y \cdot \int_V g = 0.$$

Now, $\{g^n\}$ weakly converges to g^0, therefore $\int_V g^0 = \lim \int_V g^n$.

Let

$$\int_V \limsup\{g^n(t)\} := \left\{ \int_V g; \text{for } \lambda\text{-a.e. } t \text{ in } V,\, g(t) \text{ is a limit point of } \{g^n(t)\} \right\}.$$

Then (Aumann, 1965, Proposition 4.1) (Theorem A.9) [2], yields

$$\lim \int_V g^n \subset \int_V \limsup\{g^n(t)\}.$$

But each limit point of $\{g^n(t)\}$ belongs to $\mathrm{BR}_t(f^0)$ (BR_t is upper semicontinous). Hence $y \cdot \int_V g^0 = y \cdot \lim \int_V g^n = 0$, a contradiction.

The conditions of Fan-Glicksberg fixed point theorem are satisfied for α, thus there exists a strategy profile $f \in \mathcal{F}_\Delta$ such that $f \in \alpha(f)$, in other words f is an equilibrium of $(\underline{T}, \mathcal{F}_\Delta, (u_t)_{t\in T})$. ∎

Note that to prove the non-emptyness of α, we used the finiteness of the set of extremal points of Δ, and the linearity of $u_t(\cdot, f)$. In other words, the fact that players act in mixed strategies is essential to obtain an equilibrium strategy profile when u_t is defined on $\Delta \times \mathcal{F}_\Delta$. Nevertheless Schmeidler establishes also the existence of a *pure equilibrium strategy profile* f if for every $a \in A$ and λ-a.e. $t \in T$, $u_t(a, \cdot)$ depends on f only through $\int_T f$. This will lead to consider the second set of assumptions. Before to do this we reformulate the previous theorem.

Let $E(S)$ denote the set of extremal points of S. An $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ where $u_t : S \times \mathcal{F}_S \longrightarrow \mathbb{R}$ has an equilibrium strategy profile if it satisfies the conditions **A.1-5** below.
A.1 S is a convex hull of a finite number of points $\{p_1, \ldots, p_n\}$ of $\mathbb{R}^m$.
A.2 $\mathcal{F}_S$ is endowed with the weak topology of $L_1(\lambda, \mathbb{R}^m)$.
A.3 $u_t(x, \cdot)$ is continuous for all $t \in T$ and all $x \in S$.
A.4 $u_t(\cdot, f)$ is linear for all $t \in T$ and all $f \in \mathcal{F}_S$.
A.5 For all $f \in \mathcal{F}$ and $p_k, p_l \in E(S)$, the set $\{t \in T; u_t(p_k, f) > u_t(p_l, f)\}$ is measurable.

[2] This proposition is an analogue of Fatou's lemma in several dimensions.

1.2.2 Payoff functions defined on $S \times co(S)$

Consider the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$, where u_t is defined on $S \times co(S)$. In this framework a strategy profile $f \in \mathcal{F}_S$ is an equilibrium of $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ if

$$\forall y \in S, \quad u_t\left(f(t), \int f\right) \geq u_t\left(y, \int f\right), \text{ for } \lambda\text{-a.e. } t \in T.$$

1.7. Remark. *On* $\int f$.
When S represents a finite set of actions A, then $\int f^a$ is the weight of players choosing action a, and $\int f = \left(\int f^a\right)_{a\in A}$ gives the distribution of actions across the player set.

When S represents a convex set of actions (for example a player is a firm and an element of S is a quantity of output), then $\int f$ is an average.

The next theorem due to Rath (1992) establishes that the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ has an equilibrium strategy profile if
B.1 S is a compact subset of $\mathbb{R}^m$.
B.2 u_t is jointly continuous on $S \times co(S)$, where S and $co(S)$ are endowed with the euclidean metric.
B.3 The function $t \mapsto u_t$ is measurable, where the space of continuous functions on $S \times co(S)$ is endowed with the supremum norm.

1.8 Theorem. *(Rath, 1992, Theorem 2)*
Let $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ *be an* $\mathcal{S}$*-game satisfying assumptions* **B.1-3**. *Then* $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ *has an equilibrium strategy profile.*

Proof: The proof is presented only for the case where S is the basis of coordinate axes of $\mathbb{R}^m$, $S = A$. The convex hull of A is seen as the set of integrals of pure strategy profiles, $co(A) = \Delta = \{\int f; f \in \mathcal{F}_A\}$.

The set of best reply strategies of player t is now defined for a probability $x \in \Delta$ and is $\mathrm{BR}_t(x) := \{a \in A; u_t(a, x) \geq u_t(b, x), \forall b \in A\}$. The best reply correspondence is $\alpha : \Delta \to \Delta$,

$$\alpha(x) := \int_T \mathrm{BR}_t(x)\, d\lambda(t) \left(:= \left\{\int f; f(t) \in \mathrm{BR}_t(x),\ \lambda\text{-a.e.}\right\}\right).$$

As in Theorem 1.5, one has to establish claims 1, 2 and 3, the only difference being that Δ belongs to some finite euclidean space and this simplifies the proof.

1) Δ is the unit simplex in $\mathbb{R}^m$, thus a nonempty, convex and compact subset of $\mathbb{R}^m$.

2) The nonemptiness of $\mathrm{BR}_t(x)$ comes from the finiteness of the strategy set A. The same construction as in Theorem 1.5's proof establishes that α is nonempty. The convexity of α follows from the nonatomicity of λ (Lyapounov's theorem, see, e.g. (Aliprantis and Border, 1999, Theorem 12.33)).

3) The last claim is that α has a closed graph. In fact, since
(i) $0 \leq f^a(t) \leq 1$ for every $t \in T$, $a \in A$ and $f \in \mathcal{F}_A$ and
(ii) $\mathrm{BR}_t(\cdot)$ has a closed graph (due to the continuity of $u_t(a, \cdot)$ for all $a \in A$),

it follows from (Aumann 1976) (Theorem A.13) that α has a closed graph (integration preserves upper semicontinuity).

It remains to apply Kakutani's fixed point theorem [3] (see, e.g., (Berge, 1965, p. 183)) to α. ∎

1.9. Remark. *On the assumptions of measurability.*
Rath (1995) shows that assumption **B.3** implies assumption (b). The converse is not true as shown in Appendix B.

1.10. Example. *$\mathcal{S}$-games and equilibrium strategy profiles.*
(A) Consider the $\mathcal{S}$-game with three actions a, b and c, where each player has payoff functions

$$u(a,f) = \left(\int f^a\right)^2 + \int f^b,\ u(b,f) = \int f^a + \left(\int f^b\right)^2 \text{ and } u(c,f) = \int f^c.$$

The equilibrium strategy profiles of this $\mathcal{S}$-game are
(i) the three constant pure strategy profiles: $f \equiv a$, $g \equiv b$ and $h \equiv c$,
(ii) the constant mixed strategy profiles $f_\varepsilon \equiv ((1-\varepsilon)/2, (1+\varepsilon)/2, 0)$, where $-1 < \varepsilon < 1$ and
(iii) all the strategy profiles f such that $\int f^a = (1-\varepsilon)/2$, $\int f^b = (1+\varepsilon)/2$ and $\int f^c = 0$, for some $-1 < \varepsilon < 1$.

(B) Consider an action set $A = \{a, b\}$ and define the payoff function of player t by $u_t(c, f) = \|t\delta_c + \int f\|^2$ ($\|\cdot\|$ is here the euclidean norm of $\mathbb{R}^2$, and $\delta_c \in \mathbb{R}^2$ the Dirac measure on c), for $c \in A$ $f \in \mathcal{F}_\Delta$ and $t \in T$. In the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_\Delta, (u_t)_{t\in T})$, any player $t \neq 0$ strictly prefers the action the most played. Thus the pure equilibrium strategy profiles are
(i) the constant functions $f \equiv a$ and $g \equiv b$ and
(ii) those where half of the players chooses action a and the other half action b.

Are also equilibria
(iii) the constant mixed strategy profile where all players select the mixed strategy $(1/2, 1/2)$ and
(iv) all the strategy profiles f that satisfy $\int f^a = \int f^b$ (for example $f(t) = (1/4, 3/4)$, for $t \in [0, 1/2)$ and $f(t) = (3/4, 1/4)$, for $t \in [1/2, 1]$).

$\mathcal{S}$-games constitute an exhaustive framework to analyse interactions with a large number of individuals. Indeed, a pair $(f, (u_t)_{t\in T})$ gives t's strategy and t's payoff function, moreover this payoff function may depend on the entire strategy profile. First this exhaustiveness may be unrealistic in many game's situation (e.g., congestion games, market games, population games) where (i) players are affected by a strategy profile only through the measure it induces on the strategy set as in section 1.2.2 and (ii) only this measure is known and not each individual's behavior. Second from a mathematical point of view, the set of strategy profiles $\mathcal{F}_S$ is very big, and thus is not easily tractable. That leads to consider the model of Mas-Colell where

[3] Equivalent to Fan-Glicksberg fixed-point theorem in a finite dimensional euclidean space.

the relevant parameter is distribution of payoff functions and of strategies across the players (instead of strategy profiles).

1.3 $\mathcal{M}$-games

Mas-Colell considers nonatomic games where (i) players' identity is not known, that is the set of players is implicit and (ii), which is an implication of (i), strategy profiles are not specified, the only relevant ingredient being probabilities on the product space (payoff function, strategy). We call these games *$\mathcal{M}$-games*. First, an example to distinguish $\mathcal{S}$-games from $\mathcal{M}$-games is given. Then the model is described and results on existence of equilibria are stated.

1.11. Example. *Distinction between $\mathcal{S}$-games and $\mathcal{M}$-games.*
Let $\{T_1, T_2\}$ be a partition of T such that $\lambda(T_1) = \lambda(T_2) = 1/2$, and assume that each player has only two possible actions a or b. Consider the two following strategy profiles:
- f: players in T_1 play a and players in T_2 play b;
- g: players in T_1 play b and players in T_2 play a.

Considering an $\mathcal{S}$-game, a player choosing action (for example) a may have payoffs depending on the profile ($u_t(a, f) \neq u_t(a, g)$). In an $\mathcal{M}$-game, since f and g induce the same measure: $\lambda(\{t \in T; g(t) = a\}) = \lambda(\{t \in T; f(t) = a\}) = 1/2$, they lead to the same payoff.

The strategy set S is a non-empty, compact, metric space. $\mathcal{M}(S)$ is endowed with the weak topology, hence compact and metrizable.

Let $C_S := C(S \times \mathcal{M}(S))$ be the space of real-valued continuous functions on $S \times \mathcal{M}(S)$ endowed with the supremum norm (denoted by $\|\cdot\|_\infty$), C_S is a complete metric space (Dunford and Schwartz, 1988, p. 261) and is separable (Aliprantis and Border, 1999, 3.85).

1.12. Definition. An *$\mathcal{M}$-game* is defined by a probability μ on C_S.

The probability μ represents the distribution of payoff functions across the players. The distribution of strategies across the payoff functions is described by a probability on the product space $C_S \times S$.

The equilibrium notion of $\mathcal{M}$-games is the equilibrium probability:

1.13. Definition. Let μ be an $\mathcal{M}$-game. A probability σ on $C_S \times S$ is an *equilibrium* (or *equilibrium probability*) of μ if, denoting by σ_C, σ_S the marginals of σ on C_S and S respectively,

(i) $\sigma_C = \mu$ and

(ii) $\sigma(\{(u, s); u(s, \sigma_S) \geq u(s', \sigma_S), \forall s' \in S\} = 1$.

In other words, σ has to respect the original measure on the space of payoff functions (condition (i)) and the set of pairs (u, s) where s does not maximize u is of σ-measure 0 when the measure on the strategy set is given by σ_S (condition (ii)).

1.14. Remark. *On nonatomicity.*
Given an $\mathcal{M}$-game μ, any probability σ on $C_S \times S$ whose marginal on C_S is μ

describes an interaction of μ. This entails the nonatomicity of $\mathcal{M}$-games.

Mas-Colell establishes the existence of (1) an equilibrium probability for any $\mathcal{M}$-game μ using the same fixed point theorem as Schmeidler, and (2) an equilibrium probability where players with the same payoff function play the same strategy under the conditions that μ is atomless and the strategy set S is finite.

1.15 Theorem. *(Mas-Colell, Theorem 1)*
Any $\mathcal{M}$-game μ has an equilibrium probability.

Proof: Let $\Sigma \subset \mathcal{M}(C_S \times S)$ be the set of probabilities whose marginal on the first component is μ ($\sigma_C = \mu$, condition (i)). Consider, for all $\sigma \in \Sigma$, the set

$$\mathrm{BR}_\sigma := \{(u,s); u(s,\sigma_S) \geq u(s',\sigma_S),\ \forall s' \in S\} \subset C_S \times S.$$

Then a fixed point of the correspondence $\alpha : \Sigma \to \Sigma$, with

$$\alpha(\sigma) := \{\tilde{\sigma} \in \Sigma; \tilde{\sigma}(\mathrm{BR}_\sigma) = 1\}$$

is an equilibrium probability of μ. So we shall prove claims 1, 2 and 3 in the proof of Theorem 1.5.

1) The set Σ is nonempty, convex; endowed with the weak convergence topology Σ is compact (Since S is compact, $\mathcal{M}(S)$ is compact (Parthasarathy, 1967,Theorem 6.4, p. 45)).

2) The set BR_σ is nonempty (each $u \in C_S$ is continuous on a compact set), then α maps each point of Σ to a nonempty convex subset of Σ.

3) To show the upper semicontinuity of α, we shall first show that BR_σ is closed. By contradiction, suppose that the sequence $\{(u^n,s^n)\}$, $(u^n,s^n) \in \mathrm{BR}_\sigma$, converges to some (u,s) not in BR_σ. Then there exist $\varepsilon > 0$ and $s' \in S$ such that

$$u(s,\sigma_S) < u(s',\sigma_S) - \varepsilon.$$

Since u is continuous, there exist $N \in \mathbb{N}$ and $\varepsilon_N > 0$ such that for all $n \geq N$,

$$u(s^n,\sigma_S) < u(s',\sigma_S) - \varepsilon_N.$$

Now from the convergence of $\{u^n\}$ to u with respect to the supremum norm, it follows that there exist $N_1 \in \mathbb{N}$ and $\varepsilon_{N_1} > 0$ such that for all $n \geq N_1$,

$$u^n(s^n,\sigma_S) < u^n(s',\sigma_S) - \varepsilon_{N_1}.$$

This contradicts $(u^n,s^n) \in \mathrm{BR}_\sigma$.

Assume that α is not upper semicontinuous, that is there exists a sequence of probabilities $\{(\sigma^n,\tilde{\sigma}^n)\}$, satisfying $\tilde{\sigma}^n(\mathrm{BR}_{\sigma^n}) = 1$, that weakly converges to the pair $(\sigma,\tilde{\sigma})$ with $\tilde{\sigma}(\mathrm{BR}_\sigma) < 1$. Since BR_σ is closed, there exists an open set $U \times V \subset C_S \times S$, such that $(U \times V) \cap \mathrm{BR}_\sigma = \emptyset$ and $\tilde{\sigma}(U \times V) > \varepsilon$ for some $\varepsilon > 0$. We claim that for some $N \in \mathbb{N}$ and all $n \geq N$, $\tilde{\sigma}^n(U \times V) > \varepsilon$ for some $\varepsilon > 0$.

To see this, denote C the largest closed subset included in $U \times V$. Then by Urysohn's lemma (Parthasarathy, 1967, Theorem 1.6, p. 4) there exists a continuous function φ which takes the values 1 on C and 0 on $(C_S \times S) \setminus (U \times V)$. The inequality

$$0 > \tilde{\sigma}(U \times V) = \sup\{\tilde{\sigma}(F) : F \subset U \times V, F \text{ closed }\}^4$$

yields $\int \varphi \, d\tilde{\sigma} > 0$. It remains to invoke the weak convergence of the sequence $\{\tilde{\sigma}^n\}$ to $\tilde{\sigma}$, to assert the existence of $N \in \mathbb{N}$ such that for all $n \geq N$, $\int \varphi \, d\tilde{\sigma}^n > 0$ and therefore $\tilde{\sigma}^n(U \times V) > \varepsilon$ for some $\varepsilon > 0$.

Finally the continuity of u (with respect to its second coordinate) implies that $(U \times V) \cap \mathrm{BR}_{\sigma^n} = \emptyset$. A contradiction. Hence α is upper semicontinuous.

The conditions of Fan-Glicksberg fixed point theorem are fulfilled, thus there exists $\sigma \in \Sigma$ such that $\sigma \in \alpha(\sigma)$, namely σ satisfies condition (ii), hence is an equilibrium probability of μ. ∎

1.16. Definition. A probability σ for μ is *symmetric* if there exists a measurable function $h : C_S \to S$ such that $\sigma(\text{graph } h) = 1$, in other words players with the same payoff function play the same strategy. A *symmetric equilibrium probability* is an equilibrium probability which is symmetric.

Assume that S is finite, therefore players are not allowed to randomize, and consider the game where a payoff function represents a player. Then a symmetric probability of μ corresponds to a pure strategy profile of the new game.

1.17 Theorem. *(Mas-Colell, 1984, Theorem 2)*
An $\mathcal{M}$-game μ has a symmetric equilibrium probability whenever S is finite and μ atomless.

Proof: The set S is the representation of $A = \{1, \ldots, m\}$ in $\mathbb{R}^m$. Thus the set of probabilities on S, $\mathcal{M}(S)$ is the simplex Δ of $\mathbb{R}^m$.

We shall first define a correspondence whose fixed points yield symmetric equilibrium probabilities, then we shall establish the existence of a fixed point for this correspondence.

Let $\mathcal{S}$ be the support of the probability μ and define the correspondence ϕ : $\mathcal{S} \times \Delta \to \mathbb{R}^m$ by

$$\phi(u, \nu) = \{a \in A; u(a, \nu) \geq u(b, \nu), \ \forall b \in A\}.$$

It is non-empty valued and upper semicontinuous. Now let $\phi : \Delta \to \Delta$ be defined by

$$\phi(\nu) = \int \phi(u, \nu) \, d\mu(u) := \left\{ \int_{\mathcal{S}} g(u) \, d\mu(u); g : \mathcal{S} \to S \text{ and } g(u) \in \phi(u, \nu), \ \mu\text{-a.e.} \right\}.$$

Let $\bar{\nu}$ be a fixed point of ϕ. By definition of the integral $\int \phi(u, \nu) \, d\mu(u)$, there is a measurable function $g : \mathcal{S} \to S$ such that $u(g(u), \bar{\nu}) \geq u(b, \bar{\nu})$ and $\bar{\nu} = g[\mu]$ $(= \mu \circ g^{-1})$, for all $b \in A$ and μ-almost every $u \in \mathcal{S}$. Define σ by $\sigma(B) = \mu(\{u \in$

[4] Any measure on a metric space is regular (Parthasarathy, 1967, Theorem 1.2, p. 27).

$\mathcal{S}; (u, g(u)) \in B\})$ for every Borel set B of $C_S \times S$. The probability σ is a symmetric equilibrium of μ.

To prove the existence of a fixed point of ϕ, Δ being a subset of $\mathbb{R}^m$, we shall establish claims 1, 2 and 3 as in the proof of Theorem 1.8.

1) The set Δ is a nonempty, convex and compact subset of $\mathbb{R}^m$.

2) Due to the nonemptyness of $\phi(u, \nu)$, $\phi(\nu)$ is also nonempty. Since μ is atomless, it follows from (Aumann, 1965, Theorem 1) (Theorem A.10) that ϕ is convex valued.

3) For all $\nu \in \Delta$, the correspondence $\phi(\cdot, \nu)$ is upper semicontinuous, and for all $x \in \phi(u, \nu)$, $\|x\| \leq 1$. Therefore it follows from (Aumann 1976) (Theorem A.13) that ϕ is upper semicontinuous.

Hence, by Kakutani's fixed point theorem, there exists a fixed point $\bar{\nu}$ of ϕ which induces a symmetric equilibrium probability. ∎

Theorems A.10 and A.13 require the finiteness of S. The proof of Theorem 1.15 slightly differs from the original one. Precisely, claim 3 is here detailed.

We conclude this section with some examples of $\mathcal{M}$-games.

1.18. Example. *$\mathcal{M}$-games and equilibrium probabilities.*
(A) Consider the $\mathcal{S}$-game in Example 1.10(A). It defines an $\mathcal{M}$-game through the Dirac measure μ on the real-valued payoff function $u : A \times \Delta$ with $A = \{a, b, c\}$, defined by

$$u(d, x) = 1_{d=a}[(x^a)^2 + x^b] + 1_{d=b}[x^a + (x^b)^2] + 1_{d=c}x^c.$$

The equilibria of μ are the probabilities corresponding to the pure constant strategy profiles f, g and h, $\sigma_f(u, a) = 1$, $\sigma_g(u, b) = 1$ and $\sigma_h(u, c) = 1$ (these three probabilities are symmetric) and the probabilities σ_ε with $\sigma_\varepsilon(u, a) = (1 - \varepsilon)/2$, $\sigma_\varepsilon(u, b) = (1 + \varepsilon)/2$ and $\sigma_\varepsilon(u, c) = 0$, where $-1 < \varepsilon < 1$.

(B) Let $S = [0, 1]$ and $\mathbf{u}$ be a measurable function from T to the space C_S such that each function $u_t := \mathbf{u}(t)$ is constant with respect to its second variable and defined by $u_t(s, x) = -|t - s|$. The probability μ on C_S defined by $\mu := \lambda \circ \mathbf{u}^{-1} = \mathbf{u}[\lambda]$ is an $\mathcal{M}$-game. To define the equilibria of μ, identify the function mapping (s, x) to $-|t - s|$ with the point $t \in [0, 1]$. Let σ be the probability such that $\sigma(\{[a, b], [c, d]\}) = \beta - \alpha$, where $1 \geq b \geq a \geq 0$, $1 \geq d \geq c \geq 0$ and $[a, b] \cap [c, d] = [\alpha, \beta]$. The probability σ is the only equilibrium of μ.

1.4 Relations between $\mathcal{S}$-games and $\mathcal{M}$-games

To establish relations between $\mathcal{S}$-games and $\mathcal{M}$-games, we have to work under assumptions common to both kind of games. Namely, the strategy set S is a compact subset of $\mathbb{R}^m$ and any player's payoff function is defined on $S \times \mathcal{M}(S)$ and is continuous. $\mathcal{S}$-games satisfying this last assumption are called *anonymous*.

1.19. Definition. Let $(X, \mathcal{X}, \rho)$ be a measure space and φ be a measurable function from X to a measurable space $(M, \mathcal{B})$, the *image measure* of ρ under φ, $\varphi[\rho]$, is the measure ν on $(M, \mathcal{B})$ defined by $\nu(E) = \varphi[\rho](E) = \rho(\{t \in T : \varphi(t) \in E\})$, for all

$E \in \mathcal{B}$. When ρ is clear from the context, we will omit it and call ν the *image measure* under φ.

1.20. Definition. A payoff function u is *anonymous* if it depends on a strategy profile f only through its image measure on the set S, $f[\lambda] \in \mathcal{M}(S)$. An $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ such that, for all $t \in T$, u_t is anonymous, is *anonymous*.

1.21. Notation. Given a strategy set S, an anonymous $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ is described by the map $\mathbf{u} : t \in T \to u_t \in C(S \times \mathcal{M}(S)) = C_S$. In the remainder of the section, we assume that $\mathbf{u}$ is a measurable function (condition **B.3** in Section 1.3).

Given a measurable space $(X, \mathcal{X})$, we will consider in Section 1.4.1 the map $\mathcal{L}_X$ which sends a measurable function φ from T to X to its *image measure* $\varphi[\lambda]$ on X. We will show that
(i) if $X = C_S$, the map $\mathcal{L}_X$ (henceforth $\mathcal{L}_C$) sends *anonymous $\mathcal{S}$-games* to $\mathcal{M}$-games,
(ii) if $X = C_S \times S$, then $\mathcal{L}_X$ maps a pair $\mathcal{S}$-game-strategy profile $(\mathbf{u}, f)$ to a probability σ on $C_S \times S$ such that its marginal to C_S be the $\mathcal{M}$-game $\mathcal{L}_C(\mathbf{u})$ and
(iii) $\mathcal{L}_X$ preserves equilibria: namely if f is an equilibrium strategy profile of the $\mathcal{S}$-game $\mathbf{u}$ then $\mathcal{L}_X(\mathbf{u}, f)$ is an equilibrium of the $\mathcal{M}$-game $\mathcal{L}_C(\mathbf{u})$.

Then, in Section 1.4.2, we will study the inverse correspondence $\Gamma_X = \mathcal{L}_X^{-1}$ which assigns to a probability τ on X the set of its *representations*: that is measurable functions φ from T to X such that their image measure on X be τ, $\mathcal{L}_X(\varphi) = \tau$. When X is a complete and separable metric space, Skorohod's theorem entails the non-emptyness of Γ_X. As we will see, this result will imply, when S is compact, that
(i) any $\mathcal{M}$-game on C_S has an $\mathcal{S}$-game as representation,
(ii) for any probability σ on $X = C_S \times S$, there exists a pair $(\mathbf{u}, f) \in \Gamma_X(\sigma_C)$ where $\mathbf{u}$ is an $\mathcal{S}$-game and f a strategy profile.
Moreover we will establish that Γ_X preserves equilibria. Finally, we will see the limits of the "representation" of $\mathcal{M}$-games.

1.4.1 From $\mathcal{S}$-games to $\mathcal{M}$-games

Given a measurable space $(X, \mathcal{X})$, the map $\mathcal{L}_X$ sends a measurable function φ from T to X to its image measure $\varphi[\lambda]$ on X.

We will first consider $\mathcal{L}_X$ when X is the space C_S, it will follow that the image of an $\mathcal{S}$-game $\mathbf{u}$ by $\mathcal{L}_C$ is an $\mathcal{M}$-game. Then, we will consider $\mathcal{L}_X$, when X is the space $C_S \times S$. We will show that (i) the image by $\mathcal{L}_X$ of a pair $\mathcal{S}$-game-strategy profile is a probability on $C_S \times S$ whose marginal on C_S is the $\mathcal{M}$-game $\mu = \mathcal{L}_C(\mathbf{u})$ induced by $\mathbf{u}$ and (ii) $\mathcal{L}_X$ preserves equilibria.

1.22 Lemma. *Let* $\mathbf{u}$ *be an anonymous $\mathcal{S}$-game. Then $\mu = \mathcal{L}_C(\mathbf{u})$ is an $\mathcal{M}$-game. It is the unique $\mathcal{M}$-game* induced by $\mathbf{u}$.

Proof: Straightforward. ∎

The following lemma establishes that the image measure of a pair $(\mathbf{u}, f)$, where f is a strategy profile, is a probability whose restriction on the space of payoff functions is the $\mathcal{M}$-game induced by $\mathbf{u}$, $\mu = \mathcal{L}_C(\mathbf{u})$.

1.23 Lemma. *Let* $\mathbf{u}$ *be an anonymous* $\mathcal{S}$*-game. Let* $X = C_S \times S$*. Then for all* $f \in \mathcal{F}_S$*, the probability* $\sigma = \mathcal{L}_X(\mathbf{u}, f)$ *satisfies:*

$$(1)\ \sigma_C = \mathcal{L}_C(\mathbf{u}) \ \ \textit{and} \ \ (2)\ \sigma_S = \mathcal{L}_S(f).$$

Proof: The space $X = C_S \times S$ is a complete (and separable) metric space. Thus the map $\mathcal{L}_X$ is well defined. The function $(\mathbf{u}, f); T \to X = C_S \times S$ defined by $(\mathbf{u}, f)(t) := (u_t, f(t))$ is a measurable function. Hence $\sigma = \mathcal{L}_X(\mathbf{u}, f) \in \mathcal{M}(C_S)$.

The probability σ satifies conditions (1) and (2): Indeed, for any Borel set $U \in C_S$,

$$\begin{aligned} \sigma_C(U) &= \sigma(U \times S) = (\mathbf{u}, f)[\lambda](U \times S) \\ &= \lambda(\{t \in T; (u_t, f(t)) \in U \times S\}) = \lambda(\{t \in T; u_t \in U\}) \\ &= \mu(U). \end{aligned}$$

Hence $\sigma_C = \mu$.

For any Borel set $Z \in S$,

$$\begin{aligned} \sigma_S(Z) &= \sigma(C_S \times Z) = (\mathbf{u}, f)[\lambda](C_S \times Z) \\ &= \lambda(\{t \in T; (u_t, f(t)) \in C_S \times Z\}) = \lambda(\{t \in T; f(t) \in Z\}) \\ &= f[\lambda](Z). \end{aligned}$$

∎

We now establish that the map $\mathcal{L}_X$ preserves equilibria.

1.24 Proposition. *Let* $\mathbf{u}$ *be an anonymous* $\mathcal{S}$*-game. If* $f \in \mathcal{F}_S$ *is a strategy profile in equilibrium then* $\mathcal{L}_X(\mathbf{u}, f)$ *is an equilibrium probability of the* $\mathcal{M}$*-game* $\mathcal{L}_C(\mathbf{u})$*.*

Proof: Condition (1) in Lemma 1.23 is identical to condition (i) in Section 1.3: $\sigma_S = \mu$. Hence only condition (ii) has to be shown.

When f is played, player t's payoff is $u_t(f(t), f[\lambda])$. Let

$$T_0 := \{t \in T; u_t(f(t), f[\lambda]) \geq u_t(s, f[\lambda]), \forall s \in S\}.$$

Since f is in equilibrium, $\lambda(T_0) = 1$ and

$$\sigma\left((\mathbf{u}, f)(T_0)\right) = (\mathbf{u}, f)[\lambda]((\mathbf{u}, f)(T_0)) = \lambda(T_0) = 1.$$

∎

We illustrate the previous results with an example.

1.25. Example. $\mathcal{S}$*-game induces* $\mathcal{M}$*-game.*
Examples 1.10(A) and 1.18(A) show that the $\mathcal{S}$-game with pure strategies $(\underline{T}, \mathcal{F}_A, u)$ induces the $\mathcal{M}$-game μ on C_A. Consider now the case of mixed strategies. The

$\mathcal{M}$-game corresponding to the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_\Delta, u)$ is now the Dirac measure μ on the payoff function u on $\Delta \times \mathcal{M}(\Delta)$ defined by

$$u(x,\sigma) = \sum_{a \in A} x^a u\left(a, \int_\Delta y \, d\sigma(y)\right).$$

The probability σ is an equilibrium of μ if

$$\sigma(x) > 0 \Longrightarrow \forall y \in \Delta,\ u(x,\sigma) \geq u(y,\sigma).$$

Since $\int_\Delta y \; df[\lambda](y) = \int_T f(t) \; d\lambda(t)$, it is trivial to check that the probability $f[\lambda]$ where f is any strategy profile in equilibrium in the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_\Delta, u)$ is an equilibrium of μ.

1.4.2 From $\mathcal{M}$-games to $\mathcal{S}$-games

Given a measurable space $(X, \mathcal{X})$, consider the correspondence $\Gamma_X = \mathcal{L}_X^{-1}$ which assigns to a probability τ on X the set of its *(Skorohod) representations*: that is measurable functions φ from T to X such that its image measure on X be τ, $\mathcal{L}_X(\varphi) = \tau$.

We will first establish the conditions in order that Γ_X be non-empty. Then we will obtain, as corollaries, the existence of representation for $\mathcal{M}$-games and probabilities on $C_S \times S = X$. We will establish that Γ_X preserves equilibria. We will conclude this section with results and examples to give the limits of the "representation" of $\mathcal{M}$-games.

The following proposition is a direct implication of Skorohod's theorem (Theorem A.14).

1.26 Proposition. *Let X be a complete and separable metric space. Then $\Gamma_X \neq \emptyset$.*

1.27 Corollary. *Let S be a compact metric space and μ be an $\mathcal{M}$-game on C_S. Then (i) for any $\mathcal{M}$-game μ on C_S, $\Gamma_C(\mu) \neq \emptyset$ and (ii) for any probability σ on $C_S \times S = X$, $\Gamma_X(\sigma) \neq \emptyset$.*

Proof: Since S is a compact metric space, the metric space C_S is complete and separable. ∎

Assertion (i) of the previous corollary states that any $\mathcal{M}$-game has an $\mathcal{S}$-game as representation and assertion (ii) that any "probabitity" of an $\mathcal{M}$-game can be represented by a pair $\mathcal{S}$-game-strategy profile. Note that assertion (ii) in the previous corollary implies that for any probability σ on $C_S \times S$, there exists a pair $(\mathbf{u}, f) \in C_S \times S$ such that
(i) $\mathbf{u} \in \Gamma_C(\sigma_C)$, and
(ii) $f \in \Gamma_S(\sigma_S)$.
Moreover, as established in the next proposition, if σ is an equilibrium (of σ_C), then f is an equilibrium of $\mathbf{u}$.

1.28 Proposition. *Let $X = C_S \times S$. The correspondence Γ_X preserves equilibria.*

Proof: Let σ be an equilibrium probability of an $\mathcal{M}$-game μ and $h = (\mathbf{u}, f) \in \Gamma_X(\sigma)$, which exists by Corollary 1.27.

Assume that f is not an equilibrium strategy profile of the $\mathcal{S}$-game $\mathbf{u}$. Then there exists a nonnull subset T_0 of T such that

$$\forall t \in T_0, \exists s_t \in S, \quad u_t(f(t), f[\lambda]) < u_t(s_t, f[\lambda]).$$

Since $f[\lambda] = \sigma_S$ (Lemma 1.23), this contradicts condition (ii) (of Section 1.3):

$$\sigma(\{(u,s); u(s,\sigma_S) \geq u(s',\sigma_S), \forall s' \in S\} = 1.$$

Hence σ is not an equilibrium of μ. ∎

1.29. Example. *Representation of an $\mathcal{M}$-game.*
Consider Example 1.18(B), the $\mathcal{S}$-game $\mathbf{u}$ is a representation of μ; the strategy profile $f : t \mapsto t$ is in equilibrium and "corresponds" to the probability σ ($\sigma = (\mathbf{u}, f)[\lambda]$).

A question remains: Given an $\mathcal{M}$-game μ, a representation $\mathbf{u}$ of μ ($\mathbf{u} \in \Gamma_C(\mu)$), and a probability σ on $C_S \times S$ whose marginal on C_S is μ, does there exist a strategy profile $f \in \mathcal{F}_S$ such that $(\mathbf{u}, f) \in \Gamma_X(\sigma)$?

The following example gives a negative answer.

1.30. Example. *A $\mathcal{S}$-game $\mathbf{u}$ and a probability σ but no strategy profile f such that $(\mathbf{u}, f) \in \Gamma_X(\sigma)$.*
Assume that S contains two (pure) strategies 1 and 2. Let $\mathbf{u} : T \to C_S$ be the function which sends t to the payoff function mapping (a, x) in $S \times \Delta$ to $t + x^a$ in $\mathbb{R}$. The function $\mathbf{u}$ constitutes an $\mathcal{S}$-game and is continuous. Finally, consider the probability σ on $C_S \times S$ which satisfies (i) $\sigma_C = \mathbf{u}[\lambda]$, (ii) for every measurable set U of C_S, $\sigma(U \times 1) = \sigma(U \times 2)$.

There exists no measurable function f from T to S that satisfies $\sigma = (\mathbf{u}, f)[\lambda]$. Otherwise, for any subinterval $[a, b]$ of $[0, 1]$, the set $\{t \in [a, b] : f(t) = 1\}$ would have λ-measure $(1/2)(b - a)$; but there is no Lebesgue-measurable set E on the real line such that for every Lebesgue-measurable set F in $[a, b]$, $\lambda(E \cap F) = (1/2)\lambda(F)$ (Schmeidler, 1973, Remark 3).

Nevertheless Rath (1995) gives a solution to this problem when (i) equilibria are considered and (ii) the strategy set is finite. He does not deal with the set of equilibrium probabilities but with the set of equivalent classes of equilibria probabilities: σ and σ' belong to a same class if $\sigma_S = {\sigma'}_S$.

1.31 Theorem. *(Rath, 1995, Theorem 8)*
Let S be a finite set. Let μ be an $\mathcal{M}$-game on C_S and $(\underline{T}, \mathcal{F}_S, \mathbf{u})$ be any representation of μ. Let σ be any equilibrium probability of μ. There exists a strategy profile f of $(\underline{T}, \mathcal{F}_S, \mathbf{u})$ such that (1) f is an equilibrium of $(\underline{T}, \mathcal{F}_S, \mathbf{u})$, (2) $\sigma' = (\mathbf{u}, f)[\lambda]$ is an equilibrium of μ, and (3) $\sigma_S = \sigma'_S$.

This theorem is not true when S is uncountable as shown below.

1.32. Example. *Failure of Theorem 1.31 when $S = [-1, 1]$.*
(Khan et al., 1997, Section 2) construct an anonymous $\mathcal{S}$-game with an uncountable strategy set $S = [-1, 1]$ which has no strategy profile in equilibrium. The $\mathcal{S}$-game is defined by the function

$$u_t(s, \nu) = h(s, \nu) - |t - |s||,$$

where $h : S \times \mathcal{M}(S) \to \mathbb{R}$ is a jointly continuous function which is equal to 0 when the second component is λ^*, the uniform measure on $[-1, 1]$.

The non-existence of a strategy profile in equilibrium comes from a measurability condition. The function h is defined in such a way that any strategy profile whose image measure is not λ^* cannot be in equilibrium in $\mathbf{u}$. So the only candidate for an equilibrium in $\mathbf{u}$ is a selection of the best response correspondence $t \mapsto \{t, -t\}$ ($u_t(s, \lambda^*) = -|t-|s||$). But there is no measurable selection from this correspondence which induces the measure λ^*: In fact suppose that f is such a measurable selection; let $E := \{t \in [0, 1] : f(t) \in [0, 1]\}$. Then $\lambda^*(E) = f[\lambda](E) = \lambda(E)$ (the first, respectively the second, equality follows from the definition of f, respectively E). Since $\lambda^* = (1/2)\lambda$, $\lambda(E) = 0$, and hence $\lambda^*([-1, 0]) = f[\lambda]([-1, 0]) = \lambda(\{t \notin E\}) = 1$, a contradiction.

Nevertheless the $\mathcal{M}$-game $\mu = \mathbf{u}[\lambda]$ satisfies the conditions of the model presented in Section 1.3 and therefore has an equilibrium. It follows that Theorem 1.31 is false when S is uncountable.

With the same example Rath et al. (1995) establish that the $\mathcal{M}$-game $\mu = \mathbf{u}[\lambda]$ has no symmetric equilibrium (due to the same measurability property). Is there an equivalence between symmetric equilibrium probability of an $\mathcal{M}$-game μ and equilibrium pure strategy profile of any of its representation? We shall prove that the existence of a symmetric equilibrium probability implies the existence of an equilibrium pure strategy profile and give an example to show that the converse is not true.

1.33 Lemma. *Let S be a compact metric set, $X = C_S \times S$. Let μ on C_S be an $\mathcal{M}$-game and $\mathbf{u} \in \Gamma_X(\mu)$ be a representation of μ. To any symmetric equilibrium of μ corresponds an equilibrium pure strategy profile of the $\mathcal{S}$-game $\mathbf{u}$.*

Proof: Let $\mathbf{u}$ be a measurable function from T to C_S with image measure μ and σ be a symmetric equilibrium of μ.

By definition, there exists a measurable function $h : C_S \to S$ such that $\sigma(graph(h)) = 1$. The function $h \circ \mathbf{u} : T \to S$ is a strategy profile of $\mathbf{u}$ and $(h \circ \mathbf{u})[\lambda] = \lambda(\mathbf{u}^{-1} \circ h^{-1}) = \sigma$.

We claim that $h \circ \mathbf{u}$ is in equilibrium, namely $u_t(h \circ u_t, \sigma_S) \geq u_t(s, \sigma_S)$ for every $s \in S$ and λ-almost every $t \in T$. If not, there exists a nonempty subset of T, say T_0, of dissatisfied players. Let $U_0 := \mathbf{u}(T_0)$, hence $\mu(U_0) > 0$ and for all $u \in U_0$, there exists $s \in S$ that satisfies $u(h(u), \sigma_S) < u(s, \sigma_S)$, which contradicts the hypothesis that σ is an equilibrium of μ. ∎

We now present an example extracted from Rath *et al.* (1995) to show that the converse of the previous lemma does not hold.

1.34. Example. *An $\mathcal{M}$-game with no symmetric equilibria whose representation has pure strategy profiles in equilibrium.*
Consider a set consisting of two actions $A = \{1, 2\}$ and define the payoff function $u \in C_A$ by $u(1, \nu) = 1/2$ and $u(2, \nu) = 1 - \nu(2)$, for all $\nu \in \mathcal{M}(A)$. Define the $\mathcal{M}$-game μ by the Dirac measure on u, $\mu(u) = 1$. Hence, if the probability σ on $C_A \times A$ is an equilibrium of μ, $\sigma_A(1) = \sigma_A(2) = 1/2$. It follows that σ cannot be symmetric. However the representation of μ where all players have payoff function u has as equilibrium any pure strategy profile f such that $f[\lambda](1) = f[\lambda](2) = 1/2$.

The results of this section are extensions of those of Rath (1995) which deals with a finite strategy set.

Chapter 2

Approximation of large games by nonatomic games

A *large game* is a game with a large (but finite) number of players where the influence of a player's behavior on the other players' payoff is "small". Large games considered will have a unique strategy set S. A large game is *anonymous* if any player's payoff function is anonymous.

The aim of this chapter is to show that $\mathcal{S}$-games and $\mathcal{M}$-games are useful conceptual representations of large games and anonymous large games, respectively. For this purpose, we consider convergence of sequences of games such that
(i) the number of players grows, and
(ii) the influence of a player is decreasing with respect to the number of players.
We call such a sequence a *vanishing sequence*. To define convergence of vanishing sequences we express a finite game $(P^n, S, (u_i)_{i\in P^n})$ in three different ways:
The *adapted form* which consists in an embedding of the set of players to a partition $\mathcal{T}^n$ of T and in the description of the game through functions adapted[1] to $\mathcal{T}^n$. Then a game is represented by a function $\mathbf{u}^n$ adapted to $\mathcal{T}^n$ with values in a set of functions.
The *quasi-normal form*: strategy profiles are functions from T^n to S and a large game is described by a function from T^n to $C(S \times \mathcal{M}(S))$.
The *probability form*: payoff functions are defined on $S \times \mathcal{M}(S)$ and a game is given by an atomic probability (with atoms of size k/n, with $k \in \mathbb{N}$) on this set of functions.

To the adapted form corresponds a "uniform" convergence to $\mathcal{S}$-games. To the quasi-normal form corresponds the convergence in distribution to anonymous $\mathcal{S}$-games. Finally, to the probability form corresponds the weak convergence of probabilities to $\mathcal{M}$-games.

For each form, it is shown that the limit of a convergent sequence of equilibria of large games is an equilibrium of the "limit" nonatomic game and that an equilibrium of the nonatomic game is "approximatively" an ε-equilibrium of the corresponding large games.

The terms introduced above are now formally defined.

[1]Let I be a finite set of indices and $\mathcal{T} = \{T_i\}_{i\in I}$ be a measurable partition of T. A function h constant (λ-a.e.) on each set T_i is called *adapted* to the partition $\{T_i\}_{i\in I}$.

2.1. Definition. A *vanishing sequence of games* (with a unique strategy set) is a sequence of games $\{(P^n, S, (u_i^n)_{i\in P^n})\}_{n\in\mathbb{N}}$ such that $\exists K \in \mathbb{R}$, $\forall i, j \in P^n$, $i \neq j$,

$$|u_j(s) - u_j(s'_i, s_{-i})| \leq K/n, \qquad \forall s = (s_1, \dots s_n) \in S^n,\ \forall s'_i \in S.$$

where $P^n = \{1, \dots, n\}$ is the set of players and $u_i^n : S^n \to \mathbb{R}$ is player i's payoff function, $i \in P^n$.

A *large game* is a game with a large number of players that belongs to a vanishing sequence of games. A large game $(P^n, S, (u_i)_{i\in P^n})$ is *anonymous* if for all $i \in P^n$, $u_i : S \times \mathcal{M}(S) \to \mathbb{R}$ and u_i depends on i's strategy and on the distribution of strategies across the set of the other players.

2.1 $\mathcal{S}$-games and λ-convergence

The first step is to describe finite-player games in a form "close" to $\mathcal{S}$-game. It is called the *adapted form*. It consists in an embedding of the set of players to a partition of the set $T = [0, 1]$ and in the description of the game through functions adapted to the partition. The second step is to define a convergence for a vanishing sequence of games to an $\mathcal{S}$-game, where finite-player games are described by the adapted form and $\mathcal{S}$-game by a function $\mathbf{u} : T \to C(S \times \mathcal{F}_S)$. These two steps allow to establish results of approximation.

A The adapted form of a large game

2.2. Definition. The *uniform partition* $\mathcal{T}^n = \{T_1^n, \dots, T_n^n\}$ of T is defined by $T_i^n = \left[\frac{i-1}{n}, \frac{i}{n}\right)$, for $1 \leq i < n$ and $T_n^n = \left[\frac{n-1}{n}, 1\right]$.

2.3. Notation. Given a set S and the uniform partition of n elements, denoted by $\mathcal{T}^n$, let $\mathcal{F}_S^n$ be the set of equivalence classes of functions from T to S adapted to $\mathcal{T}^n$.

$$\mathcal{F}_S^n = \{f \in \mathcal{F}_S : f(t) = f(t'), \text{ for } \lambda\text{-a.e. } t, t' \in T_i^n,\ i = 1, \dots, n\}.$$

Clearly there is a one-to-one function from S^n onto $\mathcal{F}_S^n$. Thus we are now in position to define the adapted form.

2.4. Definition. The *adapted form* of a game $(P^n, S, (u_i)_{i\in P^n})$ is a function $\mathbf{u}^n$ from T to the set of real-valued functions on $S \times \mathcal{F}_S^n$, adapted to $\mathcal{T}^n$ and defined by $\mathbf{u}^n : t \mapsto u_t^n$ with

$$u_t^n(s, f) = u_i(f_1, \dots, f_{i-1}, s, f_{i+1}, \dots, f_n),$$

where $t \in T_i^n$ and f_j is the value that f takes on T_j^n.

To define Nash equilibrium for a game in adapted form $\mathbf{u}^n$, we introduce a notation.

2.5. Notation. Given a function $f \in \mathcal{F}_S^n$, an element $s \in S$ and a set $T_i^n \in \mathcal{T}^n$, the function that coincides with f except on T_i^n where it is worth s, is denoted by $f^{(i \to s)}$:

$$f^{(i \to s)}(t) = \begin{cases} f(t) & \text{for } t \in T \setminus T_i^n, \\ s & \text{otherwise.cr} \end{cases}$$

The function $f^{(i \to s)}$ belongs to $\mathcal{F}_S^n$.

2.6. Definition. A *Nash equilibrium* of the game $\mathbf{u}^n$ is a function $f \in \mathcal{F}_S^n$ such that for all $i \leq n, t \in T_i^n$ and $s \in S$,

$$u_t^n(f(t), f) \geq u_t^n\left(s, f^{(i \to s)}\right).$$

Obviously this definition coincides with the definition of Nash equilibium for the game in normal form, that is s is a Nash equilibrium of $(P^n, S, (u_i)_{i \in P^n})$ if and only if f is a Nash equilibrium of $\mathbf{u}^n$, where $\mathbf{u}^n$ is the adapted form of $(P^n, S, (u_i)_{i \in P^n})$ and for all $i \in \{1, \ldots, n\}$ and λ-a.e $t \in T_i^n$, $f(t) = s_i$.

B Topology on a set of functions of functions

We now construct a topology on a set of functions $\mathbf{u}^n$ and $\mathbf{u}$ defined above. This topology will allow to define convergence of vanishing sequences of finite-player games $\mathbf{u}^n$ to $\mathcal{S}$-games $\mathbf{u}$. We begin with a definition.

2.7. Definition. Given two partitions $\mathcal{P}$, $\mathcal{Q}$ of a set X, $\mathcal{P}$ *refines* $\mathcal{Q}$ if

$$\forall P \in \mathcal{P}, \exists Q \in \mathcal{Q} \text{ such that } P \subset Q.$$

In notation: $\mathcal{P} \subset \mathcal{Q}$.

Let $\{\mathcal{T}^{k_n}\}_n$ be a sequence of uniform partitions such that $\mathcal{T}^{k_{n+1}}$ refines $\mathcal{T}^{k_n}$. From now on, we write n for k_n.

We will first define a net of functions which compare any two real-valued functions defined on different spaces $S \times \mathcal{F}_S^p$.

Let $p, q \in \mathbb{N} \cup \{\infty\}$ and let $z = \min\{p, q\}$. For any $t \in T$ and any functions $u^p : S \times \mathcal{F}_S^p \to \mathbb{R}$ and $u^q : S \times \mathcal{F}_S^q \to \mathbb{R}$ (with $\mathcal{F}_S^\infty = \mathcal{F}_S$), define a "distance" for t between u^p and u^q by

$$D_t(u^p, u^q) = \sup_{s \in S, f \in \mathcal{F}_S^z} |u^p(s, f) - u^p(s, f)|.$$

D_t is not neither a metric ($D_t(u, v) = 0 \not\Rightarrow u = v$), nor a pseudometric (the triangle inequality is not satisfied).

Consider the set X of measurable functions $\mathbf{u}$ such that either $\mathbf{u}$ maps T to the space $C(S \times \mathcal{F}_S)$ or there exists $n \in \mathbb{N}$ such that $\mathbf{u}$ maps T to the set of real-valued functions on $S \times \mathcal{F}_S^n$ and $\mathbf{u}$ is adapted to $\mathcal{T}^n$. Formally,

$$X = \mathbf{U} \bigcup \cup_n \mathbf{U}^n,$$

where
$$\mathbf{U} = \{\mathbf{u} : t \in T \mapsto C(S \times \mathcal{F}_S),\ \text{measurable}\}$$
and
$$\mathbf{U}^n = \{\mathbf{u}^n : t \in T \mapsto (u^n_t : S \times \mathcal{F}^n_S \to \mathbb{R}),\ \text{adapted to } \mathcal{T}^n\}.$$

For any $\varepsilon > 0$ and any $\mathbf{u} \in \mathbf{U}$, define the set $B_\varepsilon(\mathbf{u})$ by
$$B_\varepsilon(\mathbf{u}) = \{\mathbf{v} \in X : D_t(u_t, v_t) < \varepsilon,\ \text{for } \lambda\text{-a.e. } t \in T\}.$$
Consider the topology generated by $\{B_\varepsilon(\mathbf{u}) : \varepsilon > 0, \mathbf{u} \in \mathbf{U}\}$ and call it the λ-*topology.*

We now state the convergence of functions $\mathbf{u}^n$.

2.8. Definition. Let $\mathbf{u} \in \mathbf{U}$ and $\mathbf{u}^n \in \mathbf{U}^n$. The sequence $\{\mathbf{u}^n\}$ λ-*converges* to $\mathbf{u}$ (in notation $\mathbf{u}^n \xrightarrow{\lambda} \mathbf{u}$) if for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, $\mathbf{u}^n \in B_\varepsilon(\mathbf{u})$.

In other terms, $\{\mathbf{u}^n\}$ λ-*converges* to $\mathbf{u}$ if
$$\lim_{n\to\infty}\left(\inf\{M : \sup_{f\in\mathcal{F}^n_S} |\mathbf{u}(f(t), f) - \mathbf{u}^n(f(t), f)| \leq M,\ \text{for } \lambda\text{-a.e. } t \in T\}\right) = 0.$$

Note that λ-convergence is λ-almost everywhere a uniform convergence.

2.9. Example. λ-*convergence.*
Consider the n-player game with a unique strategy set $S = \{\delta_1, \delta_2\}$ and payoff functions $u^n_i : S^n \to \mathbb{R}$ defined by
$$u^n_i(x_1, \dots, x_n) = \frac{1}{n^2}\left\| i x_i - \sum_{j<i} x_j \right\|^2.$$
The game $(P^n, S, (u_i)_{i\in T_n})$ in adapted form is described by the function $\mathbf{u}^n$ where for all i, for all $t \in T^n_i$ and all $f \in \mathcal{F}^n_S$.
$$u^n_t(\delta_a, f) = \left\| \frac{i-1}{n}\delta_a - \int_{[0,(i-1)/n)} f \right\|^2.$$
Hence, a player is interested in choosing the strategy the least used by the players "preceding" him.

We claim that the vanishing sequence of finite-player games $\{\mathbf{u}^n\}$ λ-converges to the $\mathcal{S}$-game $\mathbf{u}$ defined by $u_t(\delta_a, f) = \|t\delta_a - \int_{[0,t)} f\|^2$. This is easily seen. Indeed, for all n, t in $[i - 1/n, i/n)$ and $f \in \mathcal{F}^n_S$ with $f(t) = \delta_a$,
$$\begin{aligned} |u^n_t(\delta_a, f) - u_t(\delta_a, f)| \quad = \quad & t^2 - \left(\frac{i-1}{n}\right)^2 - 2\left(t - \frac{i-1}{n}\right)\int_{[\frac{i-1}{n},t)} f^a \\ & + \sum_{c=a,b}\left(\int_{[\frac{i-1}{n},t)} f^c\right)^2 \\ & \qquad +2\left(\int_{[\frac{i-1}{n},t)} f^c\right)\left(\int_{[0,\frac{i-1}{n})} f^c\right). \end{aligned}$$
Since f is bounded, the sequence $\{\mathbf{u}^n\}$ λ-converges to $\mathbf{u}$.

C Approximation

In this paragraph, S is assumed to be the standard basis of $\mathbb{R}^m$: $S = A$. Hence players are supposed to act in pure strategies.

2.10 Theorem. *Let* $\mathbf{u}^n$ *be a* n*-player game and* $\mathbf{u}$ *be an* $\mathcal{S}$*-game such that* $\{\mathbf{u}^n\}$ λ*-converges to* $\mathbf{u}$. *Let* $\{f^n\}$, $f^n \in \mathcal{F}^n_S$, *be a sequence that* weakly converges *to* f. *If, for infinitely many* n, f^n *is a Nash equilibrium of* $\mathbf{u}^n$, *then* f *is an equilibrium of* $\mathbf{u}$.

Proof: The limit f belongs to $\mathcal{F}_S$. Assume that f is not an equilibrium of $\mathbf{u}$. Then there exists a nonnull subset T_0 of T for which for all $t \in T_0$, $f(t)$ is not a best reply against f:

$$\forall t \in T_0,\ \exists s_t \in S,\ u_t(f(t), f) < u_t(s_t, f) - \varepsilon, \quad \text{for some } \varepsilon > 0.$$

We claim that there exist a subset T_0' of positive measure such that f is constant on T_0' and an open set $O \subset \mathcal{F}_S$ such that for some $\varepsilon > 0$, for all $g, h \in O$,

$$\exists s \in S,\ \forall t \in T_0',\ u_t(s', g) < u_t(s, h) - \varepsilon, \tag{2.1}$$

where s is the value of f on T_0'. To see this, note that S is finite and T_0 uncountable. Thus there exists a nonnull subset T_0' of T_0 such that s_t can be chosen uniformly ($s_t = s$) and f is constant on T_0' with value s':

$$\exists s \in S,\ \forall t \in T_0',\ u_t(s', f) < u_t(s, f) - \varepsilon, \quad \text{for some } \varepsilon > 0.$$

Equation (2.1) follows from the continuity of u_t (for all t).

Now, for all n, there exists $1 \le i \le n$, such that $\lambda(T_0' \cap T_i^n) > 0$ (the partition $\mathcal{T}^n$ has a finite number of elements), let $T_0^n := T_0' \cap T_i^n$. Since f^n weakly converges to f, there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$,
(i) for all $t \in T_0^n$, $f^n(t) = s'$ (f^n constant on T_i^n),
(ii) $f^n \in O$ and
(iii) $f^{n(i \to s)} \in O$ (since $\lambda(T_i^n) \to_n 0$, we have $\int_T (f^n - f^{n(i\to s)}) g \, d\lambda = \int_{T_i^n} (f^n - f^{n(i\to s)}) g \, d\lambda \to_{n\to\infty} 0$, for all $g \in L_\infty$).

Therefore from (2.1) and (i)-(iii), it follows that for some $\varepsilon > 0$ and all $n \ge N_0$,

$$\exists s \in S,\ \forall t \in T_0^n,\ u_t(f^n(t), f^n) < u_t\left(s, f^{n(i\to s)}\right) - \varepsilon.$$

Finally invoke the λ-convergence of $\{\mathbf{u}^n\}$ to $\mathbf{u}$ to assert that there exists $N_1 \ge N_0$ such that for all $n \ge N_1$,

$$\exists s \in S,\ \forall t \in T_0^n,\ u_t^n(f^n(t), f^n) < u_t^n\left(s, f^{n(i\to s)}\right) - \varepsilon, \quad \text{for some } \varepsilon > 0.$$

But this is impossible, since, for all N, there exists $n \ge N$ such that f^n is a Nash equilibrium of $\mathbf{u}^n$. ∎

In order to prove a converse property, we introduce a new definition.

2.11. Definition. Let $\mathbf{u}^n$ be a n-player game. A strategy profile f^n is a (δ, ε)-*Nash equilibrium* if there exists a subset B of T that satisfies $\lambda(T \backslash B) < \delta$ and for all $t \in B$, $f^n(t)$ is an ε-best reply strategy against the strategy profile f^n: namely, if $t \in T_i^n \cap B$, then

$$u_t^n(f^n(t), f^n) \geq u_t^n(s, f^{n(i \to s)}) - \varepsilon, \quad \forall s \in S.$$

The following theorem asserts that if f is an equilibrium of an $\mathcal{S}$-game $\mathbf{u}$ then for any large game $\mathbf{u}^n$ close to $\mathbf{u}$ (with respect to the topology introduced in B), any strategy profile f^n of $\mathbf{u}^n$ close to f (with respect to λ-almost everywhere convergence) is "almost" an equilibrium.

2.12 Theorem. *Let* $\mathbf{u}^n$ *be a* n*-player game and* $\mathbf{u}$ *be an* $\mathcal{S}$*-game such that* $\{\mathbf{u}^n\}$ λ*-converges to* $\mathbf{u}$*. Let* $\{f^n\}$*,* $f^n \in \mathcal{F}_S^n$*, be a sequence that converges* λ-almost everywhere *to an* equilibrium f *of* $\mathbf{u}$*. Then for all* δ*,* $\varepsilon > 0$*, there exists* N *such that for all* $n \geq N$*,* f^n *is a* (δ, ε)*-Nash equilibrium of* $\mathbf{u}^n$*.*

Proof: The strategy profile f is an equilibrium of $\mathbf{u}$, hence

$$u_t(f(t), f) \geq u_t(s, f), \quad \forall s \in S, \text{ for } \lambda\text{-a.e. } t \in T.$$

The continuity of u_t yields for any $\varepsilon > 0$ the existence of an open set $O \ni f$ such that for all $g, h \in O$

$$u_t(f(t), g) \geq u_t(s, h) - \varepsilon, \quad \forall s \in S, \text{ for } \lambda\text{-a.e. } t \in T.$$

But the λ-almost everywhere convergence of $\{f^n\}$ implies that there exists an integer N such that for all $n \geq N$, f^n and $f^{n(i \to s)}$ belong to O for $1 \leq i \leq n$ and $s \in S$. Thus

$$u_t(f(t), f^n) \geq u_t(s, f^{n(i \to s)}) - \varepsilon, \quad 1 \leq i \leq n, \forall s \in S, \text{ for } \lambda\text{-a.e. } t \in T. \tag{2.2}$$

Invoke now Egoroff's theorem (Dunford and Schwartz, 1988, III.6.12) (Theorem A.15) to assert that for all $\delta > 0$, there is a subset $B_\delta \in \mathcal{B}(T)$ that satisfies $\lambda(T \backslash B_\delta) < \delta$ and $\{f^n\}$ converges uniformly to f on B_δ.

Fix $\delta > 0$, and let $B = B_\delta$. Then, for all $\varepsilon > 0$, there exists N_ε such that for all $t \in B$ and all $n \geq N_\varepsilon$,

$$\|f(t) - f^n(t)\| \leq \varepsilon.$$

The finiteness of S and the previous equation leads to the existence of an integer N_1 such that for all $n \geq N_1$ and for any $t \in B$, $f(t) = f^n(t)$. Hence equation (2.2) gives

$$u_t(f^n(t), f^n) \geq u_t(s, f^{n(i \to s)}) - \varepsilon, \quad 1 \leq i \leq n, \forall s \in S, \text{ for } \lambda\text{-a.e. } t \in T.$$

Moreover by definition of the λ-convergence, for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N$,

$$D_t(u_t, u_t^n) \leq \varepsilon, \text{ for } \lambda\text{-a.e. } t \in T.$$

Therefore, λ-almost everywhere in B and for all $n \geq \max\{N_1, N_2\}$,

$$u_t^n(f^n(t), f^n) \geq u_t^n(s, f^{n(i \to s)}) - 3\varepsilon, \quad \forall s \in S.$$

We conclude that for any $\delta > 0$ and $\varepsilon > 0$, there exists N such that for all $n \geq N$, f^n is a (δ, ε)-Nash equilibrium of $\mathbf{u}^n$. ∎

It is possible to refine this theorem when the family of functions $\{u_t\}$ is equicontinuous to obtain that f^n is an ε-Nash equilibrium of $\mathbf{u}^n$. This constitutes the following corollary. We define first ε-Nash equilibrium.

2.13. Definition. An *ε-Nash equilibrium* of a game $\mathbf{u}^n$ is a (δ, ε)-Nash equilibrium where $\delta = 0$.

2.14 Corollary. *Let $\mathbf{u}^n$ be a n-player game and $\mathbf{u}$ be an $\mathcal{S}$-game such that $\{\mathbf{u}^n\}$ λ-converges to $\mathbf{u}$. Assume that there exists a compact subset C of $C(S \times \mathcal{F}_S)$ such that $u_t \in C$, for all $t \in T$. Let f be a strategy profile in equilibrium of the $\mathcal{S}$-game $\mathbf{u}$. There exists a sequence $\{f^n\}$, $f^n \in \mathcal{F}_S^n$, that converges* λ-almost everywhere *to f and satisfies: for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, f^n is an ε-Nash equilibrium of $\mathbf{u}^n$.*

Proof: From Theorem 2.12, it follows that for all sequences $\{f^n\}$ that converges λ-almost everywhere to f, there exists, for all $\delta, \varepsilon > 0$, a number $N \in \mathbb{N}$ such that for all $n \geq N$, the strategy profile f^n is a (δ, ε)-Nash equilibrium of $\mathbf{u}^n$.

For all $n \geq N$, consider the strategy profile $\tilde{f}^n$ where players not in B (in other words, those who may not play a best reply against f^n) play a best reply against f^n and those in B, play their strategy induces by f^n. We shall show that for all $\varepsilon > 0$, there exists $N_\varepsilon \geq N$ such that for all $n \geq N_\varepsilon$, the strategy profile $\tilde{f}^n$ is an ε-Nash equilibrium of $\mathbf{u}^n$.

The range of $\mathbf{u}$ is included in a compact subset C of $C(S \times \mathcal{F}_S)$, then Arzelà-Ascoli's theorem (Theorem A.17) implies that the family of functions u_t, $t \in T$, is equicontinuous: for all $s \in S$ and $f \in \mathcal{F}_S$, for all $\varepsilon > 0$, there exists a neighborhood O of (s, f) such that

$$\sup_{t \in T} \sup_{(s', g) \in O} |u_t(s, f) - u_t(s', g)| < \varepsilon.$$

In particular, there exists an integer N_1 such that, for all $s \in S$,

$$\sup_{t \in T} |u_t(s, f^n) - u_t(s, \tilde{f}^n)| < \varepsilon.$$

Since $\{\mathbf{u}^n\}$ λ-converges to $\mathbf{u}$, there exists $N_2 > N_1$ such that, for all $n \geq N_2$ and λ-almost everywhere in T,

$$|u_t^n(s, f^n) - u_t^n(s, \tilde{f}^n)| < 3\varepsilon.$$

To conclude recall that $\mathbf{u}^n$ is function adapted to $\mathcal{T}^n$. ∎

We illustrate the results of this section with some examples.

2.15. Example. *Sequences of equilibria and convergent sequences to equilibria.*
(A) Consider the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, u)$ of Example 1.10(A). It is described by the constant function $\mathbf{u} \equiv u$, where u defined by

$$u(a, f) = (\int f^a)^2 + \int f^b, \; u(b, f) = \int f^a + (\int f^b)^2 \text{ and } u(c, f) = \int f^c.$$

Consider the sequence of finite-player games $\{\mathbf{u}^n\}$ where $u^n_t(f(t), f) = u_t(f(t), f)$ for all $f \in \mathcal{F}^n_S$. Obviously, $\{\mathbf{u}^n\}$ λ-converges to $\mathbf{u}$. Consider the sequence of (pure) strategy profiles $\{f^n\}$, $f^n \equiv a$. For all n, f^n is a Nash equilibrium of $\mathbf{u}^n$ and the limit of f^n is the constant strategy profile $f \equiv a$ which is in equilibrium in $\mathbf{u}$. Consider now a partition $\{T_1, T_2\}$ of T, such that $\lambda(T_1) = \lambda(T_2) = 1/2$ and define the strategy profile g of $\mathbf{u}$ by $g(t) = a$ if $t \in T_1$, $g(t) = b$ otherwise. The strategy profile g is in equilibrium in $\mathbf{u}$, but, for any n, g is not a Nash equilibrium of $\mathbf{u}^n$: g gives a payoff of $3/4$ to every player (choosing strategy a or b), if a player deviates in $\mathbf{u}^n$, say from a to b, then he has a payoff of $3/4 + 1/n^2$, strictly greater than the original one. Nevertheless, g is a $(1/n^2)$-Nash equilibrium of $\mathbf{u}^n$, for all n.

(B) Consider the $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_S, (u_t)_{t\in T})$ of Example 1.10(B). It is described by the function $\mathbf{u}$ ($u_t(c, f) = \|t\delta_c + \int f\|^2$ for $c \in \{a, b\}$, $f \in \mathcal{F}_S$ and $t \in T$). Define the n-player game $\mathbf{u}^n$ by

$$u^n_t(c, f) = \left\|\frac{i-1}{n}\delta_c + \int f\right\|^2, \quad \forall t \in T^n_i, \; f \in \mathcal{F}^n_S, \text{ with } f(t) = e^c.$$

The sequence of games $\{\mathbf{u}^n\}$ λ-converges to the $\mathcal{S}$-game $\mathbf{u}$. Indeed for all $\varepsilon > 0$ and all t, there exists n such that for all $n \geq N$, $t \in T^n_i$ for some $i = 1, \ldots n$ and

$$D_t(u^n_t, u_t) \leq |(\frac{i-1}{n} - t)(\frac{i-1}{n} + t + 2)| < \varepsilon.$$

Consider the equilibrium strategy profile h of $\mathbf{u}$ where players in $[0, 1/2)$ choose strategy a and those in $[1/2, 1]$ choose b. For all n, define the strategy profile h^n by

$$h^n(t) = \begin{cases} a & \text{if } t < i/n \leq 1/2, \\ b & \text{otherwise.} \end{cases}$$

The sequence (h^n) converges λ-almost everywhere to h, nevertheless for any n, h^n is not a Nash equilibrium of $\mathbf{u}^n$: assume that n is even, so that half of the players selects a and the other half b. The payoff of player i at h^n is $(i-1/n)^2 + (i-1/n) + (1/2)$, if he deviates he obtains $(i-1/n)^2 + (i-1/n) + (1/2) + i(2/n^2)$, h^n is a $(4/n^2)$-Nash equilibrium. On another hand, the strategy profile $f \equiv a$ is a Nash equilibrium for the games $\mathbf{u}^n$, for any n, and is also in equilibrium in the $\mathcal{S}$-game $\mathbf{u}$.

(C) Consider the sequence $\mathbf{u}^n$ of Example 2.9 and its λ-limit $\mathbf{u}$. For any n, the strategy profile f^n defined by $f^n(t) = a$ if $t \in T^n_i$ for an odd i and $f^n(t) = b$ otherwise, is a Nash equilibrium: if i is odd, then $\int_{[0,i/n)} f^{na} = \int_{[0,i/n)} f^{nb}$ and the two actions give to player i the same payoff; if i is even, then $\int_{[0,i/n)} f^{na} = i/2 >$

$i/2 - 1 = \int_{[0,i/n)} f^{nb}$, so that strategy b gives to player i a strictly better payoff than strategy a. However the $\mathcal{S}$-game $\mathbf{u}$ has no pure strategy profile in equilibrium (Schmeidler, 1973, Remark 3) (also Example 1.25). The point is that the sequence $\{f^n\}$ does not converge.

2.2 $\mathcal{S}$-games and convergence in distribution

Consider a sequence of measurable functions with values in a common metric space, but which may be defined on different measure spaces. To consider convergence of such sequences, we focus on the sequence of their image measures (or distributions).

2.16. Definition. Let φ^n be a measurable function from $(\Omega^n, \mathcal{A}^n, \nu^n)$ to $(X, \mathcal{X})$. The sequence $\{\varphi^n\}$ *converges in distribution* to a measurable function φ (in notation $\varphi^n \xrightarrow{D} \varphi$) from $(\Omega, \mathcal{A}, \nu)$ to X if the sequence of measures $\{\varphi^n[\nu^n]\}$ weakly converges to $\varphi[\nu]$.

A consequence of the previous definition is: if payoff functions of finite-player games and of $\mathcal{S}$-games are defined on a common space, then we can define convergence in distribution of a sequence of finite-player games to an $\mathcal{S}$-game.

In the following, S is a compact metric space.

Before to go in further details, the quasi-normal form is defined and the concept of anonymity for large games is refined.

2.17. Definition. The *quasi-normal form* of an anonymous n-player game $(P^n, S, (u_i)_{i\in P^n})$ is a function $G^n : i \in P^n \to u_i \in C(S \times \mathcal{M}(S))$ which maps a players to his payoff function. A *strategy profile* of the anonymous n-player large game is a function $s : P^n \to S$.

2.18. Definition. A n-player game $(P^n, S, (u_i)_{i\in P^n})$ is
- *anonymous* if $u_i \in C(S \times \mathcal{M}(S))$,
- *e-anonymous* if it is anonymous and u_i depends on i's strategy and on the distribution of strategies across the *entire* set of players
- *p-anonymous* if it anonymous and u_i depends on i's strategy and on the distribution of strategies across the (partial) set of the other players.

2.19. Remark. *Anonymity*
When the set of players is nonatomic e-anonymity and p anonymity are equivalent. For n-player games, the difference between these two definitions are substantial as long as n is not so large and vanishes when n tends to infinity. Thus the choice of e-anonymity or p-anonymity is done to simplify proofs. In this section we focus on e-anonimity, in the next one on p-anonymity.

A sequence of anonymous finite-player games $\{G^n\}$ (where n tends to infinity) is necessarily a vanishing sequence.

For all n, define the probability λ^n by $\lambda^n(E) = \frac{\#E}{\#P^n}$, for all $E \subset P^n$.

2.20. Definition. A sequence of e-anonymous games $\{G^n\}$ *converges in distribution* if the sequence of function $\{G^n\}$ converges in distribution.

2.21. Definition. A $\mathcal{S}$-game $\mathbf{u}$ is a *nonatomic representation* of the converging sequence $\{G^n\}$ (in notation $G^n \xrightarrow{D} \mathbf{u}$) if there exists a sequence $\{\alpha^n\}$ of measurable functions α^n from T to P^n such that
(i) $\alpha^n[\lambda](E) = \lambda^n(E)$, for all measurable sets $E \in P^n$ and
(ii) $\{\tilde{G}^n = G^n \circ \alpha^n\}$ converges λ-almost everywhere to $\mathbf{u}$

Note that $\tilde{G}^n$ is a n-player game in adapted form, it corresponds to the game (P^n, S, G^n) up to a permutation of the players.

Converging sequences of games and nonatomic representations are the analogue of purely competitive sequences of simple economies and continuous representation in general equilibrium theory, see (Hildenbrand, 1974, Ch 2.1).

The following proposition states the existence of a nonatomic representation for any converging sequences of finite player games and links their strategy profiles. It follows from (Hildenbrand, 1974, Proposition 2, p. 139) (only slight modifications are needed in the proof) and corresponds to an application of Skorohod's theorem.

2.22 Proposition. *Let the sequence of e-anonymous games $\{G^n\}$ converge in distribution.*
(1) The sequence $\{G^n\}$ has a nonatomic representation.
(2) Let $\{s^n\}$, s^n strategy profile of G^n, be a sequence that converges in distribution (in S). There exist a subsequence $\{G^n\}_{n\in Q}$, $Q \subset \mathbb{N}$, and a nonatomic representation $\mathbf{u}$ with mappings $\alpha^n, n \in Q$, such that the sequence $\{\tilde{f}^n\}_{n\in Q}$, $\tilde{f}^n = s^n \circ \alpha^n$, converges λ-almost everywhere to a strategy profile f of the $\mathcal{S}$-game $\mathbf{u}$.

2.23. Example. *Convergence in distribution and nonatomic representation.*
Let $I = \{1, \ldots, I\}$ be a finite set of types and $\mathcal{T} := \{T_i\}_{i\in I}$ the uniform partition of T. For $i \in I$, let $u_i \in C_S$. Define the e-anonymous $\mathcal{S}$-game $\mathbf{u}$ by $u_t = u_i$, for $i \in I$ and all $t \in T_i$. We shall approach $\mathbf{u}$ by a "sequence of replica".

For all n, consider the e-anonymous game $(P^{nI}, S, (u_k)_{k\in I^n})$, where there is n players of each type, that is

$$P^{nI} = \{1_1, \ldots 1_n, 2_1, \ldots, I_n\}$$

and $u_{i_j} = u_i$. From now on, write n for nI. Let G^n be its quasi-normal form. Given a strategy profile $f^n : P^n \to S$, the payoff of player i_j is $u_i(f^n(i_j), f^n[\lambda^n])$.

The sequence of games $\{G^n\}$ is called a *sequence of replica*. We claim that $G^n \xrightarrow{D} \mathbf{u}$.

To see this, note that the probabilities $G^n[\lambda^n]$ $(n \in \mathbb{N})$ and $\mathbf{u}[\lambda]$ are atomic and coincide: indeed $G^n[\lambda^n](u_i) = \mathbf{u}[\lambda](u_i)$, for all n and $i \leq n$.

On the other hand the sequence $\{G^n\}$ has the $\mathcal{S}$-game $\mathbf{u}$ as nonatomic representation. To see this, consider the uniform partition $\mathcal{T}^n$ of T. Define the mappings $\alpha^n : T \to P^n$ $(n \in \mathbb{N})$ by $\alpha^n(t) = i_j$ if $t \in T_{i_j}$. Conditions (i) and (ii) of Definition 2.21 are satisfied by the sequence of mappings $\{\alpha^n\}$.

Next, we state results similar to those of Section 2.1 for convergence in distribution.

2.24 Proposition. *Let the sequence of e-anonymous games $\{G^n\}$ converge in distribution and s^n be a Nash equilibrium of G^n. If the sequence $\{s^n\}$ converges in distribution (in S), then there exists a strategy profile f such that a subsequence $\{s^{n_k} \circ \alpha^{n_k}\}$ converges λ-almost everywhere to f which is a Nash equilibrium of the nonatomic representation $\mathbf{u}$.*

The proof differs slightly of the one of Theorem 2.10, hence is postponed to Appendix C.

2.25. Definition. Let G^n be an e-anonymous n-player game. A strategy profile s^n is a (δ, ε)-*Nash equilibrium* if there exists a subset B of P^n that satisfies $\lambda^n(P^n \backslash B) < \delta$ and for all $i \in B$, $s^n(i)$ is an ε-best reply strategy against the strategy profile s^n: namely

$$u_i^n(s^n(i), s^n[\lambda^n]) \geq u_i^n(s_i', \hat{s}^n[\lambda^n]) - \varepsilon, \ \forall s_i' \in S,$$

where $\hat{s}^n(i) = s_i'$ and $\hat{s}^n(j) = s^n(j)$ for $j \neq i$

2.26 Proposition. *Let $\{G^n\}$ be a converging sequence of e-anonymous games with nonatomic representation $\mathbf{u}$; let $\{s^n\}$, s^n strategy profile of G^n, be a sequence that converges in distribution (in S). If f is a strategy profile in equilibrium of the $\mathcal{S}$-game $\mathbf{u}$, then for all $\delta, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N, n \in Q$ (Q refers to assertion (2) of Proposition 2.22), s^n is a (δ, ε)-Nash equilibrium of G^n.*

2.27 Corollary. *Let $\{G^n\}$ be a converging sequence of e-anonymous games with nonatomic representation $\mathbf{u}$. Assume that there exists a compact subset C of C_S such that, for all n, the range of G^n is included in C. Let f be a strategy profile in equilibrium of the $\mathcal{S}$-game $\mathbf{u}$. There exists a sequence $\{s^n\}$, s^n strategy profile of G^n, that converges in distribution to f and satisfies: for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N, n \in Q$, s^n is an ε-Nash equilibrium of G^n.*

The proof of Proposition 2.26 (resp. Corollary 2.27) is similar to the proof of Theorem 2.12 (resp. Corollary 2.14).

2.3 $\mathcal{M}$-games and weak convergence

Convergence in distribution is only concerned with weak convergence of measures. Hence it seems that the results previously obtained may be adapted to $\mathcal{M}$-games. To do this we define the probability form of anonymous games.

The strategy set S is a compact metric space.

2.28. Definition. The *probability form* of an anonymous n-player game is an atomic probability μ^n on C_S where each atom has a rational weight k/n, k integer; in other words μ^n can be written as $\sum_{i=1}^n (1/n)\delta_{u_i}$, for some family $(u_1, \ldots, u_n)$ of payoff functions (recall that δ_x denotes the Dirac measure on x). A play of μ^n is is an atomic probability σ^n on $C_S \times S$, where each atom has weight k/n and whose marginal on the set of payoff functions is μ^n. Given a n-player large game μ^n, its set of plays is denoted Σ^n.

Given a measurable space $(X, \mathcal{X})$, let $\mathcal{L}^n_X$ be the function which maps a measurable function φ from T^n to X to its image measure $\varphi[\lambda^n]$. Let G^n be the quasi-normal form of an anonymous game and write C for C_S then $\mathcal{L}^n_X(G^n) = \mu^n$. (The image measure of λ^n under an anonymous n-player game is a n-player game.) Now let X be the set $C_S \times S$, then $\mathcal{L}^n_X(G^n, s^n) = \sigma^n$.

From now on, μ^n is always a p-anonymous game.

2.29. Notation. Given an atomic probability τ^n on S with atoms of weight k/n, if $\tau^n(s) > 0$, then $\tau^{n(-s)}$ denotes the probability on S given by τ^n where we remove a weight of $1/n$ to strategy s: $\tau^{n(-s)} := (n\tau^n - \delta_s)/(n-1)$.

2.30. Definition. A probability σ^n is an *equilibrium probability* of μ^n if (i) its marginal on the space of payoff functions C_S, denoted by σ^n_C, coincides with μ^n and (ii) for all $(u,s) \in C_S \times S$ with $\sigma^n(u,s) > 0$,

$$u(s, \sigma_S^{n(-s)}) \geq u(s', \sigma_S^{n(-s)}),\ \forall s' \in S,$$

where σ^n_S is the marginal of σ^n on the strategy set S.

The set of plays of an $\mathcal{M}$-game μ is denoted Σ,

$$\Sigma := \{\sigma \in \mathcal{M}(C_S \times S); \sigma_C = \mu\}.$$

2.31. Definition. A sequence of games $\{\mu^n\}$ on C_S *weakly converges* to an $\mathcal{M}$-game μ on C_S (in notation $\mu^n \xrightarrow{w} \mu$) if $\{\mu^n\}$ weakly converges to μ.

2.32 Theorem. *Let $\{\mu^n\}$ be a sequence of games that weakly converges to an $\mathcal{M}$-game μ; let $\{\sigma^n\}$, $\sigma^n \in \Sigma^n$, be a sequence that weakly converges to σ. If, for infinitely many n, σ^n is an equilibrium of μ^n, then σ is an equilibrium of μ.*

Proof: Obviously $\sigma \in \mathcal{M}(C_S \times S)$ and $\sigma_C = \mu$. We shall prove by contradiction that it is an equilibrium of μ.

Define the set

$$B_\sigma := \{(u,s); u(s,\sigma_S) \geq u(s', \sigma_S)\ \forall s' \in S\} \subset C_S \times S.$$

Assume that σ is not an equilibrium of μ. Hence $\sigma(B_\sigma) < 1$. Since B_σ is closed (proof of Theorem 1.15), there exists an open set $O \subset C_S \times S$, such that $O \cap B_\sigma = \emptyset$ and $\sigma(O) > \varepsilon$, for some $\varepsilon > 0$. We claim that for some $N \in \mathbb{N}$ and for all $n \geq N$, $\sigma^n(O) > \varepsilon$, for some $\varepsilon > 0$.

To see this, denote C the largest closed subset included in O. From Urysohn's lemma (Parthasarathy, 1967, Theorem 1.6, p. 4), there exists a continuous function φ which takes the values 1 on C and 0 on $C_S \times S \setminus O$. The inequality

$$0 > \sigma(O) = \sup\{\sigma(C); C \subset O, C \text{ closed }\}^2$$

[2]See footnote 4, page 28.

yields $\int \varphi \, d\sigma > 0$. It remains to invoke the weak convergence of the sequence $\{\sigma^n\}$ to σ, to assert that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $\int \varphi \, d\sigma^n > 0$ and therefore $\sigma^n(O) > \varepsilon$, for some $\varepsilon > 0$.

Now, σ^n being an atomic probability, there exists a pair $(u_0, s_0) \in O$ with strictly positive measure. For $(u_0, s_0) \in O$, there exists some $s' \in S$ such that $u_0(s_0, \sigma_S) < u_0(s', \sigma_S) - \varepsilon$ for some $\varepsilon > 0$.

Appeal once more to the weak convergence of $\{\sigma^n\}$ to σ and remark that $p(\sigma^n_S, \sigma^{n(-s_0)}_S) \leq 1/n$ [3] to assert that, by continuity of u in its second component, for all n enough large,

$$u_0(s_0, \sigma^{n(-s_0)}_S) < u_0(s', \sigma^{n(-s_0)}_S).$$

But $\sigma^n(u_0, s_0) > 0$, hence for all N, there exists $n \geq N$ such that σ^n is not an equilibrium of μ^n. ∎

In order to obtain a converse statement, we introduce the following.

2.33. Definition. Let μ^n be a n-player game. A probability σ is a (δ, ε)-*equilibrium* of μ^n if (i) $\sigma_C = \mu^n$ and
(ii) $\sigma(\{(u, s); u(s, \sigma^{-s}_S) \geq u(s', \sigma^{-s}_S) - \varepsilon, \ \forall s' \in S\}) \geq 1 - \delta$.

In the next theorem, S is either a finite set or a convex and compact set of a finite dimensional euclidean space.

2.34 Theorem. *Let $\{\mu^n\}$ be a sequence of games that weakly converges to an $\mathcal{M}$-game μ; let $\{\sigma^n\}$, $\sigma^n \in \Sigma^n$, be a sequence that weakly converges to σ. If σ is an equilibrium of μ, then for all δ, $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, σ^n is an (δ, ε)-equilibrium of μ^n.*

Proof: By Skorohod's theorem (Theorem A.14), since $\{\sigma^n\}$ weakly converges to σ, there exist measurable functions h^n $(n = 1, 2, \ldots)$ and h from T to $C_S \times S$ such that $\{h^n\}$ converges λ-almost everywhere to h. From Egoroff's theorem (Theorem A.15), it follows that for all $\delta > 0$, there exists a subset B of T, such that $\lambda(T \setminus B) < \delta$ and $\{h^n\}$ converges uniformly to h on B.

Write $h^n = (\mathbf{u}^n, f^n)$ and $h = (\mathbf{u}, f)$, where $\mathbf{u}^n$ and $\mathbf{u}$ (resp. f^n and f) are measurable functions from T to C_S (resp. S). The uniform convergence of $\{(\mathbf{u}^n, f^n)\}$ implies, for all $\varepsilon > 0$, the existence of a number N_ε such that for all $t \in B$,

$$\|u_t - u^n_t\|_\infty \leq \varepsilon, \tag{2.3}$$

and

$$d(f(t), f^n(t)) \leq \varepsilon, \tag{2.4}$$

where $d(\cdot, \cdot)$ is the metric on S.

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[3] $p(\cdot, \cdot)$ is the Prohorov metric:

$$p(\tau_1, \tau_2) = \inf\{\varepsilon > 0; \tau_1(E) \leq \tau_2(B_\varepsilon(E)) + \varepsilon \text{ and } \tau_2(E) \leq \tau_1(B_\varepsilon(E)) + \varepsilon, \text{ for all } E \in \mathcal{B}(S)\}.$$

Typically it suffices to establish only one of these inequalities, see the details in (Dudley, 1989, p. 309).

The function $\mathbf{u}$ defines an anonymous $\mathcal{S}$-game and, by Proposition 1.28, h can be chosen such that f be a strategy profile in equilibrium of $\mathbf{u}$. Thus λ-almost everywhere, we have

$$u_t(f(t), \sigma_S) \geq u_t(s, \sigma_S), \quad \forall s \in S. \tag{2.5}$$

Now, it follows from (2.3) and (2.5), that λ-almost everywhere in B and for all $n \geq N_\varepsilon$, we have

$$u_t^n(f(t), \sigma_S) \geq u_t^n(s, \sigma_S) - 2\varepsilon, \quad \forall s \in S. \tag{2.6}$$

The sequence $\{\sigma^n\}$ weakly converges to σ, hence for all $\varepsilon_\sigma > 0$, there exists $N_\sigma \in \mathbb{N}$ such that

$$\forall n \geq N_\sigma,\ p(\sigma_S, \sigma_S^n) \leq \varepsilon_\sigma. \tag{2.7}$$

We can also assume that

$$\forall n \geq N_\sigma,\ p(\sigma_S, \sigma_S^{n(-s)}) \leq \varepsilon_\sigma, \tag{2.8}$$

for all s, $s \in S$ with $\sigma_S^n(s) > 0$.

For all $t \in T$, u_t^n is continuous (on $S \times \mathcal{M}(S)$, where S has the metric topology, and $\mathcal{M}(S)$ the weak topology). Therefore it follows from (2.4), (2.7) and (2.8), that for all $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$\left| u_t^n(f(t), \sigma_S) - u_t^n\left(f^n(t), \sigma_S^{n(-f^n(t))}\right) \right| \leq \varepsilon$$

and

$$\left| u_t^n(s, \sigma_S) - u_t^n\left(s, \sigma_S^{n(-f^n(t))}\right) \right| \leq \varepsilon.$$

Thus, λ-almost everywhere in B and for all $n \geq N$, the following holds

$$u_t^n\left(f^n(t), \sigma_S^{n(-f^n(t))}\right) \geq u_t^n\left(s, \sigma_S^{n(-f^n(t))}\right) - 4\varepsilon, \quad \forall s \in S.$$

In other words,

$$\begin{aligned} \lambda(\{t \in T; u_t^n(f^n(t), \sigma_S^{n(-f^n(t))}) \\ \geq u_t^n(s, \sigma_S^{n(-f^n(t))}) \quad - \quad 4\varepsilon,\ \forall s \in S\}) \geq 1 - \delta \end{aligned}$$

or equivalently

$$\sigma^n\left(\left\{(u, s); u(s, \sigma_S^{n(-f^n(t))}) \geq u(s', \sigma_S^{n(-f^n(t))}) - 4\varepsilon,\ \forall s' \in S\right\}\right) \geq 1 - \delta.$$

The probability σ^n is an (δ, ε)-equilibrium. ∎

2.35 Corollary. *Let $\{\mu^n\}$ be a sequence of games that weakly converges to an $\mathcal{M}$-game μ. Assume that there exists a compact subset C of C_S such that, for all n, the support of μ^n is included in C. Let $\sigma \in \mathcal{M}(C_S \times S)$ be an equilibrium of μ. There exists a sequence $\{\sigma^n\}$, $\sigma^n \in \Sigma^n$, that weakly converges to σ and satisfies, for all $\varepsilon > 0$ and some number N, σ^n is an ε-equilibrium of μ^n, for all $n \geq N$.*

Proof: Follows from Corollary 2.27 and Theorem 2.34. ∎

We conclude this section with an example.

2.36. Example. *Sequences of equilibrium probabilities.*
Consider the $\mathcal{M}$-game of Example 1.18(B). For any n, define the n-player game μ^n by $\mu^n(u_i^n) = 1/n$, for $i = 1, \ldots, n$, where $u_i^n(s,x) = -|i/n - s|$. Identify, as in Example 1.18(B), the function mapping (s,x) to $-|t-s|$ with the point $t \in [0,1]$. It follows from the inequality $\mu^n(B_{1/n}(U)) \geq \mu(U)$, for any Borel set U of $[0,1]$, that $p(\mu^n, \mu) \leq 1/n$. Thus the sequence $\{\mu^n\}$ weakly converges to μ. The only equilibrium $\sigma^n \in \mathcal{M}(C_S \times S)$ of the game μ^n is defined by $\sigma^n((u_i^n, i/n)) = 1/n$, for all $i = 1, \ldots, n$. The sequence $\{\sigma^n\}$ weakly converges to $\sigma \in \mathcal{M}(C_S \times S)$, the unique equilibrium of μ.

Chapter 3

Extensions and variations

This chapter is devoted to extensions of $\mathcal{S}$-games and $\mathcal{M}$-games. First, Section 3.1 presents models of anonymous $\mathcal{S}$-games due to Rath (1992) and Khan and Sun (2002). Then, Section 3.2 considers a model of nonatomic games based on probabilities where (i) players are gathered in classes, (ii) a player may be affected differently by strategy distributions of different classes and (iii) to different classes may be associated different strategy sets, the existence of equilibria for such games, called *$\mathcal{C}$-games*, is established. Section 3.3 deals with *population games* a restriction of $\mathcal{C}$-games where all players of a same class have same evaluation function. Population games are easier to handle than $\mathcal{C}$-games, their equilibria are characterized by variational inequalities which yields a specific proof for equilibrium's existence. Finally, Section 3.4 studied specific population games, called *potential games*.

3.1 Extensions of $\mathcal{S}$-games

Recall that the original model of nonatomic games defined by Schmeidler deals with a finite action set. Rath (1992) gives an extension where the action set is a compact subset of $\mathbb{R}^m$. He establishes the existence of an equilibrium strategy profile in the case where players act in pure strategies and evaluation functions depend on strategy profiles f only through their integral (Theorem 1.8). This result cannot be extented to general evaluation functions, indeed Schmeidler gives an example of an $\mathcal{S}$-game with two actions and no pure equilibrium strategy profile (Example 2.15(C)).

Another extension of $\mathcal{S}$-game with equilibrium pure strategy profiles is given by Khan and Sun (2002). It deals with payoff functions which depend on the average behavior of classes of players. Explicitely, let $\{T_i\}_{i\in I}$ be a partition of T. Define the probability λ_i by $\lambda_i(B) = \lambda(B)/\lambda(T_i)$, for any measurable set $B \subset T_i$. Let S be a countable compact metric space.

3.1 Theorem. *(Kahn and Sun, 2002, Theorem 2)*
Let $\mathbf{u}$ be a measurable function from T to $C(S\times\mathcal{M}(S)^I)$ and write u_t for $\mathbf{u}(t)$. Then there exists a measurable function $f : T \to S$ such that

$$u_t(f(t), f_1[\lambda_1], \ldots, f_I[\lambda_I]) \geq u_t(s, f_1[\lambda_1], \ldots, f_I[\lambda_I]), \quad \forall s \in S,\ \lambda\text{-a.e.}$$

where f_i is the restriction of f to T_i.

If the set S is not countable, the previous theorem does not hold. Khan and Sun (2002) give two examples of games where the strategy set is $S = [-1, 1]$ and payoff functions are defined on $S \times \mathcal{M}(S)$ without pure equilibrium strategy profiles. The failure of existence is due, in both case, to "the fact that it is impossible to choose from the correspondence on the Lebesgue unit interval defined by $t \to \{t, -t\}$; a measurable selection that induces the uniform measure on $[-1, 1]$" (Khan and Sun, 2002, p. 1777). One of this example is reproduced in Example 1.32.

In the three examples mentioned above, the non-existence of pure equilibrium strategy profiles in $\mathcal{S}$-games is due to measurability requirements. Hence we can hope that considering probabilities rather than strategy profiles will help solving the problem.

3.2 Extension of $\mathcal{M}$-games

In an $\mathcal{M}$-game, a player's payoff function depends on his own strategy and on the image measure on the strategy set under the strategy profile. A framework where payoff functions depend on several probabilities would be useful to modelize situations where (i) players are gathered in classes, (ii) a player may be affected differently by strategy distributions of different classes (as in Theorem 3.1) and (iii) to different classes may be associated different strategy sets. We extend $\mathcal{M}$-games to such a frawework. We call these games *$\mathcal{C}$-games.*

Let I be a finite set of classes, $I = \{1, \ldots, I\}$. For each i, the set S_i is the compact metric space of strategies of a player in class i. Let $C_i = C(S_i \times \Pi_{j\in I}\mathcal{M}(S_j))$

3.2. Definition. A *$\mathcal{C}$-game* is defined by a finite set of classes I, a family of strategy sets $(S_i)_{i\in I}$ and a family of probabilities $(\mu_i)_{i\in I}$, where μ_i is a probability on C_i. It is denoted by $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$.

3.3. Example. *$\mathcal{C}$-games.*
(A) *Atomic probabilities on the space of payoff functions.*
Consider a set with two classes of players, $I = \{1, 2\}$. Each player, whatever be his class, has two actions, $A_1 = A_2 = \{a, b\} =: A$. Given any action, the payoff of the players in class i increases proportionaly with (1) the measure of the players in the same class selecting the same action as him and (2) the measure of the players in the other class selecting the other action. Moreover, for any probability profile $(x_1, x_2) \in \Delta^2$, half of the players in class i prefers action a to action b.

Let u_{ia} (resp. u_{ib}) represent the payoff of players in class i who prefer action a (resp. b) to action b (resp. a). Let α_i and β_i be positive real numbers. Define the payoff functions as follows.

$$u_{ia}(a, x_1, x_2) = 1 + \alpha_i x_i^a + \beta_i x_j^b,\ u_{ia}(b, x_1, x_2) = \alpha_i x_i^b + \beta_i x_j^a,$$

and

$$u_{ib}(a, x_1, x_2) = \alpha_i x_i^a + \beta_i x_j^b,\ u_{ib}(b, x_1, x_2) = 1 + \alpha_i x_i^b + \beta_i x_j^a.$$

Let $\mu_i(u_{ia}) = \mu_i(u_{ib}) = 1/2$. The game $(\{1, 2\}, \{a, b\}, (\mu_1, \mu_2))$ is a $\mathcal{C}$-game.

(B) *Nonatomic probability.*
We modify the previous example in order that the probabilities μ_i be nonatomic. For all $t \in [0,1]$ and all $i \in I$ define the payoff functions

$$u_{tia}(a, x_1, x_2) = t + \alpha_i x_i^a + \beta_i x_j^b,\ u_{tia}(b, x_1, x_2) = \alpha_i x_i^b + \beta_i x_j^a,$$

and

$$u_{tib}(a, x_1, x_2) = \alpha_i x_i^a + \beta_i x_j^b,\ u_{tib}(b, x_1, x_2) = t + \alpha_i x_i^b + \beta_i x_j^a.$$

For $i \in I$, $c \in A$ and $t_0 \in [0,1]$, let V_{t_0ic} be the set of payoff functions $\{u_{tic}; t \leq t_0\}$. Define the probabilities μ_i as $\mu_i(V_{t_0ic}) := (1/2)t_0$ and $\mu_i(U) := \mu_i(U \cap (V_{t_0ia} \cup V_{t_0ib}))$, for all measurable sets U of $C(S_i \times \Pi_{j\in I}\mathcal{M}(S_j))$. The game $(\{1,2\}, \{a,b\}, (\mu_1, \mu_2))$ is a $\mathcal{C}$-game.

A probability σ_i on $C_i \times S_i$ describes the distribution of (payoff functions, strategies) across class i. Let σ_{i,S_i} denotes the marginal of σ_i on S_i. A probability profile $(\sigma_1, \ldots, \sigma_I)$ with σ_i in $\mathcal{M}(C_i \times S_i)$ describes a play of the game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$, where $\mu_i = \sigma_{i,S_i}$. The set of probability profiles is denoted by Σ_I.

3.4. Definition. An *equilibrium probability profile* $(\sigma_1, \ldots, \sigma_I)$ of a $\mathcal{C}$-game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$ is such that for all $i \in I$, the marginal of σ_i on $C(S_i \times \Pi_{j\in I}\mathcal{M}(S_j))$ is μ_i and

$$\begin{aligned} \sigma_i(\{(u,a) \ &\in \ C_i \times S_i; \forall b \in S_i \\ & u(a, \sigma_{1,A_1}, \ldots, \sigma_{I,A_I}) \geq u(b, \sigma_{1,A_1}, \ldots, \sigma_{I,A_I})\}) = 1. \end{aligned} \tag{3.1}$$

In other words, a probability profile is an equilibrium of a $\mathcal{C}$-game if it corresponds to the $\mathcal{C}$-game and if the set of players who have a profitable deviation is of null measure.

The following theorem states the existence of equilibrium probability profile. It is proven in two different ways. The first proof uses a fixed point theorem and follows the lines of the one of Theorem 1.15. The second uses the result of Theorem 1.15 to construct an equilibrium probability profile.

3.5 Theorem. *Any $\mathcal{C}$-game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$ has an equilibrium probability profile.*

Proof 1: An equilibrium probability profile is a fixed point of the correspondence $\alpha : \Sigma_I \to \Sigma_I$ with

$$\alpha(\sigma_1, \ldots, \sigma_I) := \{(\sigma'_1, \ldots, \sigma'_I); \sigma'_i(\mathrm{BR}^i_\sigma) = 1,\ \forall i \in I\},$$

where $\mathrm{BR}^i_\sigma := \{(u,s); u(s, \sigma_{1,S_1}, \ldots, \sigma_{I,S_I}) \geq u(s', \sigma_{1,S_1}, \ldots, \sigma_{I,S_I}),\ \forall s' \in S_i\} \subset C_i \times S_i$.

We use the same arguments than in the proof of Theorem 1.15 to establish that:
(i) Σ_I is compact (product of compact sets).
(ii) For all $i \in I$ and $\sigma \in \Sigma_I$, BR^i_σ is closed.

(iii) α is nonempty, convex-valued and upper semicontinuous.

It follows from Fan-Glicksberg fixed point theorem that α has a fixed point σ^*; σ^* is an equilibrium probability profile. ∎

Proof 2: We associate to any $\mathcal{C}$-game an $\mathcal{M}$-game such that to an equilibrium probability of the $\mathcal{M}$-game corresponds an equilibrium probability profile of the $\mathcal{C}$-game.

Consider the $\mathcal{C}$-game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$. Construct an action set $\tilde{S}$, where an element of $\tilde{S}$ is a pair (i, s) with $s \in S_i$, the set $\tilde{S}$ is the disjoint union $\cup_{i\in I} (\{i\} \times S_i)$. Denote d_i the metric on S_i and define the metric d on $\tilde{S}$ by

$$d((i,s),(j,s')) = \begin{cases} d_i(s,s') & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

$\tilde{S}$ is a compact metric space.

For any $i \in I$, denote by π_i the projection from $\mathcal{M}(\tilde{S})$ to $\mathcal{M}(S_i)$; let $\tilde{x} \in \mathcal{M}(\tilde{S})$ and $x_i := \pi_i(\tilde{x})$, then

$$x_i(B) = \frac{\tilde{x}(\{i\} \times B)}{\tilde{x}(\{i\} \times S_i)},$$

for every measurable set $B \in \mathcal{B}(\tilde{S}_i)$.

For any $i \in I$, define a function ϕ_i from the space C_i to a set of real-valued functions on $\tilde{S} \times \mathcal{M}(\tilde{S})$ which maps any $u_i \in C_i$ to the function $\phi_i(u_i)$ such that

$$\phi_i(u_i)((j,s),\tilde{x}) = \begin{cases} u_i(s, \pi_1(\tilde{x}), \ldots, \pi_I(\tilde{x})) & \text{if } j = i, \\ K_{u_i} - 1 & \text{otherwise,} \end{cases}$$

where K_{u_i} is a lower bound of the function u_i.

The functions $\phi_i(u_i)$ are continuous, thus ϕ_i maps C_i to $C_{\tilde{S}}$ $(= C(\tilde{S} \times \mathcal{M}(\tilde{S}))$. Finally, define an $\mathcal{M}$-game $\tilde{\mu}$ on $C_{\tilde{S}}$ by

$$\tilde{\mu}(\phi_i(U)) = \frac{1}{I}\,\mu_i(U), \qquad \text{for all } U \in \mathcal{B}(C_i).$$

By Theorem 1.15, the $\mathcal{M}$-game $\tilde{\mu}$ has an equilibrium probability $\tilde{\sigma}$. Obviously $\tilde{\sigma}$ satisfies

$$\tilde{\sigma}(\{\mu_i(U) \times i \times S_i; i \in I, U \in \mathcal{B}(C_i)\}) = 1. \tag{3.2}$$

Define the vector $\sigma = (\sigma_i)$ by

$$\sigma_i(\phi_i^{-1}(U) \times B) = \frac{\tilde{\sigma}(U \times i \times B)}{\tilde{\sigma}(C_{\tilde{S}} \times i \times S_i)}$$

for any measurable set $U \in \mathcal{B}(C_{\tilde{S}})$ and any measurable set $B \in \mathcal{B}(S_i)$. It follows from equation (3.2) that σ is a probability profile of $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$. The fact that σ is an equilibrium of the $\mathcal{C}$-game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$ is straightforward. ∎

We define the analogue of symmetric probability for $\mathcal{C}$-games.

3.6. Definition. A probability profile of a $\mathcal{C}$-game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$ is *symmetric* if for all $i \in I$, there exists a measurable function $h_i : C_i \to S_i$ such that $\sigma_i(\text{graph } h_i) = 1$, in other words players with the same payoff function play the same strategy. A *symmetric equilibrium probability profile* is an equilibrium probability profile which is symmetric.

The next result follows from Theorem 1.17 and the construction in Proof 2 of Theorem 3.5.

3.7 Theorem. *A symmetric equilibrium probabilty profile for a $\mathcal{C}$-game $(I, (S_i)_{i\in I}, (\mu_i)_{i\in I})$ exists whenever, for all $i \in I$, the set S_i is finite and the probability μ_i is nonatomic.*

We conclude this section with examples of equilibrium probabilty profile.

3.8. Example. *Equilibrium probability profiles.*
(A) Let $(\{1,2\}, \{a,b\}, (\mu_1, \mu_2))$ be the $\mathcal{C}$-game of Example 3.3(A). In this $\mathcal{C}$-game, all equilibria are symmetric. In fact, assume by contradiction that it has a non-symmetric equilibrium, denote x_1, x_2, the marginals on A of this probability profile, there exists i such that $u_{ia}(a, x_1, x_2) = u_{ia}(b, x_1, x_2)$ and $u_{ib}(a, x_1, x_2) = u_{ib}(b, x_1, x_2)$. These equalities lead to

$$1 + \beta_i - \alpha_i = 2(\beta_i x_j^a - \alpha_i x_i^a) \quad \text{and} \quad 1 + \alpha_i - \beta_i = 2(\alpha_i x_i^a - \beta_i x_j^a).$$

But summing these two equalities yields $2 = 0$, hence a contradiction.

Two examples of equilibrium probability profiles are:
(i) $\sigma_{1A}(a) = 1$ and $\sigma_{2A}(b) = 1$, if $\alpha_i + \beta_i \geq 1$, and
(ii) $\sigma_{1A}(b) = \sigma_{2A}(b) = 1$, if $\alpha_i \geq 1 + \beta_i$.

(B) Let $(\{1,2\}, \{a,b\}, (\mu_1, \mu_2))$ be the $\mathcal{C}$-game of Example 3.3(B). All the equilibria of this $\mathcal{C}$-game are obviously symmetric. The probability profiles (i) and (ii) of Example (A) above are also equilibria of this $\mathcal{C}$-game.

3.3 Population games

This section deals with a specific kind of $\mathcal{C}$-games, called *population games* which is useful in the study of many applications. In a population game, the payoff functions are no more player-dependent but only class-dependent, explicitely two players belonging to a same class have the same payoff function. Considering population games allows for a simple proof of existence of equilibria.

First we introduce a specific terminology for population games.

3.9. Definition. A *population* i is a collection of players, called also *individuals* or *members* such that all members of i have same finite action set A_i and same evaluation function U_i. Unless stated otherwise, the set of members of a population is assumed to be a nonatomic measure space. A *population state* x_i is a probability on A_i, x_i^a represents the average frequency of action a in population i. Given a finite set of populations I, a *populations state* x is a family of population state indexed by I, $x = (x_i)_{i \in I}$.

Until now, we have defined nonatomic games with strategy sets and real-valued payoff functions. The aim was to obtain a presentation as clear as possible. Now we will deal with action sets and vector-valued *evaluation functions*, since population games are more tractable in this form.

Given a set $A_i = \{1, \ldots, m_i\}$ identified with the canonical basis of $\mathbb{R}^{m_i}$, its convex hull is the simplex $\Delta_{m_i} = \Delta_i$ in $\mathbb{R}^{m_i}$. We are now in position to define *population games.*

3.10. Definition. A *population game* is a triple $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$, where I is a finite set of populations, $A_i = \{1, \ldots, m_i\}$ is the finite action set of i's members and $U_i : \Pi_{j \in I} \Delta_j \to \mathbb{R}^{m_i}$ is the (continuous) evaluation function of i's members.

To my knowledge, the first who introduced the termilogy of population games as defined above is Sandholm (2001).

We have asserted above that population games are a special class of $\mathcal{C}$-games. When players act in pure strategies, this is trivial: In fact any evaluation function $U_i : \Pi_{j \in I} \Delta_j \to \mathbb{R}^{m_i}$ induces a real-valued function u_i on $A_i \times \Pi_{j \in I} \Delta_j$ defined by $u_i(a, \cdot) = U_i^a(\cdot)$. Moreover the continuity of U_i implies that u_i belongs to $C(A_i \times \Pi_{j \in I} \Delta_j)$. It follows that the $\mathcal{C}$-game $(I, (A_i)_{i \in I}, (\mu_i)_{i \in I})$ where μ_i is the Dirac probability on u_i corresponds to the population game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$, where players act in pure strategies.

When players act in mixed strategies, to see that a population game is a $\mathcal{C}$-game requires a little more work. Indeed, the $\mathcal{C}$-game corresponding to a population game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$, where players act in mixed strategies would have been a triple $(I, (\Delta_i), (\mu_i))$ where μ_i is Dirac probability on a payoff function u_i "corresponding" to U_i. But u_i is defined on $\Delta_i \times \Pi_{j \in I} \mathcal{M}(\Delta_j)$, whereas U_i is defined on $\Pi_{j \in I} \Delta_j$. So it is necessary to define, for all i, a continuous function p_i which assigns to a probability ν_i in $\mathcal{M}(\Delta_i)$ a probability $p_i(\nu_i)$ in Δ_i in order that

$$u_i(x, \nu_1, \ldots, \nu_I) = x \cdot U_i(p_1(\nu_1), \ldots, p_I(\nu_I)).$$

This constitutes the next lemma.

3.11 Lemma. *Let Δ be the simplex in $\mathbb{R}^m$. There exists a continuous function p from $\mathcal{M}(\Delta)$ to Δ.*

Proof: Given a probability ν on Δ (that is $\nu \in \mathcal{M}(\Delta)$), Skorohod's theorem (Theorem A.14) implies that there exists a *representation* $f : T \to \Delta$ of ν, that is f is a measurable function from T to Δ whose image measure under λ is ν. Define the function p by $p(\nu) = \int_T f \, d\lambda$. Note that if g is another representation of ν, then $\int_T g \, d\lambda = \int_T f \, d\lambda$, to assert that p is well defined.

Moreover Skorohod's theorem yields the continuity of p. Indeed, consider a sequence $\{\nu^n\}$ of probabilities in $\mathcal{M}(\Delta)$ that weakly converges to ν, then Skorohod's theorem asserts that there exists a sequence $\{f^n\}$, $f^n : T \to \Delta$, that satisfies $\nu^n = f^n[\lambda]$, for all n and that converes λ-almost everywhere to f. Now, since

$\int_T g\ d\lambda = \int_T f\ d\lambda$ for any two representations of a same measure, it follows that $\{p(\nu^n)\}$ converges (with respect to the euclidean norm in $\mathbb{R}^m$) to $p(\nu)$. ∎

Due to the previous lemma, we can assert that the population game $(I, (A_i), (U_i))$ where players act on mixed strategies and the $\mathcal{C}$-game $(I, (\Delta_i), (\mu_i))$ where μ_i is Dirac probability on the function u_i defined by

$$u_i(x, \nu_1, \ldots, \nu_I) = x \cdot U_i(p_1(\nu_1), \ldots, p_I(\nu_I))$$

are equivalent.

The only relevant information in a probability profile $(\sigma_i)_{i\in I}$ where $\sigma_i \in \mathcal{M}(C_i \times A_i)$ for a population game is its marginals over action sets. This leads to substitute the following definition for equilibrium probability profile.

3.12. Definition. A populations state $x = (x_1, \ldots, x_I) \in \Pi_{i\in I}\Delta_i$ of $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ is an *equilibrium* (or *equilibrium state*) if, for all $i \in I$ and $a \in A_i$,

$$x_i^a > 0 \Longrightarrow U_i^a(x) \geq U_i^b(x), \quad \forall b \in A_i. \tag{3.3}$$

The following results are well known in routing games, see, e.g., Altman and Kameda (2001).

For population games we have the following characterization of equilibrium.

3.13 Lemma. *Let $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ be a population game. $x^* = (x_1^*, \ldots, x_I^*) \in \Pi_{i\in I}\Delta_i$ is an equilibrium state if and only if*

$$\forall i \in I, \quad (x_i^* - x_i) \cdot U_i(x^*) \geq 0, \quad \forall x_i \in \Delta_i. \tag{3.4}$$

The proof is straightforward and therefore omitted.

Since a population game is a $\mathcal{C}$-game, it follows from Theorem 3.5 that any population game $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ has an equilibrium state. But the characterization (3.4) allows us to have a simple proof which differs from the previous one; it is based on variational inequality methods and given below.

3.14 Theorem. *Any population game $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ has an equilibrium state.*

Proof: Let $m := \sum_{i\in I} m_i$. Let U the vector-valued function from $\Pi_{i\in I}\Delta_i$ to $\mathbb{R}^m$ defined by $U := (U_i)_{i\in I}$. The equilibrium condition (3.4) is equivalent to

$$(x^* - x) \cdot U(x^*) \geq 0, \quad \forall x \in \Pi_{i\in I}\Delta_i. \tag{3.5}$$

Since U is continuous and $\Pi_{i\in I}\Delta_i$ is compact and convex, it follows from (Kinderlehrer and Stampacchia, 2000, I, Theorem 3.1) that (3.5) has a solution. ∎

3.15. Remark. (Kinderlehrer and Stampacchia, 2000, I, Theorem 3.1) uses Brouwer fixed point theorem: *Let $K \subset \mathbb{R}^m$ be compact and convex and let $F : K \to K$ be continuous. Then F admits a fixed point.*

Assume that (3.5) has two solutions $\bar{x}$, $\hat{x}$. Then

$$(\bar{x} - \hat{x}) \cdot U(\bar{x}) \geq 0 \qquad \text{and} \qquad (\hat{x} - \bar{x}) \cdot U(\hat{x}) \geq 0,$$

to sum these two inequalities gives

$$(\bar{x} - \hat{x}) \cdot U(\bar{x}) - U(\hat{x}) \geq 0.$$

Consequently, if

$$(\bar{x} - \hat{x}) \cdot (U(\bar{x}) - U(\hat{x})) < 0, \quad \forall \bar{x}, \hat{x} \in \Pi_{i \in I} \Delta_i,\ \bar{x} \neq \hat{x}, \tag{3.6}$$

then (3.5) has a unique solution.

Condition (3.6) with the reverse inequality is known as strict monotony.

3.16. Definition. A function F from a nonempty, closed and convex subset X of $\mathbb{R}^n$ to $\mathbb{R}^n$ is *strictly monotone* if

$$(F(x) - F(y)) \cdot (x - y) > 0, \quad \forall x, y \in X,\ x \neq y.$$

The next result is deduced from the above discussion.

3.17 Theorem. *Let $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$ be a population game; let $U = (U_i)_{i \in I}$. If $-U$ is strictly monotone, then there is a unique equilibrium state.*

3.4 Potential games

Rosenthal (1973) defines a class of games where there are n players which want to minimize their cost function and k primary factors. Player i's set of pure strategies contains m_i elements, an element of this set being a set of primary factors. To each primary factor is associated a cost which depends only on the number of players choosing it. The cost function of a player is the sum of the cost of the primary factor he has chosen.

These games have the property that the same (unilateral) change from one action to another for any two players will involve the same change in payoff for these players. Rosenthal (1973) shows that these games admits pure Nash equilibria.

Monderer and Shapley (1996) generalize the previous property, which leads them to define *finite potential games*. A *finite potential game* is a n-player game such that there exists a real-valued function which measures the difference in the payoff that occurs to a player if he unilaterally changes his choice. A formal definition is the following.

3.18. Definition. A *finite potential game* is a n-player game $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$, where A_i is the finite action set of player i, which admits a function P from $\Pi_{i \in N} A_i$ to $\mathbb{R}$ such that: $\forall i \in N$, $\forall a_i, b_i \in A_i$, $\forall a_{-i} \in \Pi_{j \neq i} A_j$,

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = P(a_i, a_{-i}) - P(b_i, a_{-i}).$$

P is called a *potential function*.

Monderer and Shapley (1996) generalize the result of Rosenthal (1973) and prove the existence of pure Nash equilibria for finite potential games.

In this section we first focus on *infinite potential games* as defined in Sandholm (2001). Then we show that infinite potential games are limits of sequences of finite potential games.

3.4.1 Infinite potential games

3.19. Definition. A population game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$ is a *potential game* if there exists a C^1 function $P : \Pi_{i \in I} \Delta_i \to \mathbb{R}$ such that for every probability profile $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_I) \in \Pi_{i \in I} \Delta_i$, $i \in I$ and $a \in A_i$,

$$\frac{\partial}{\partial x_i^a} P(\bar{x}) = U_i^a(\bar{x}).$$

P is called a *potential function* of the population game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$.

If a population game has a potential function, it is unique up to an additive constant.

To my knowledge, such a function appeared for the first time in a game theoretical context in the framework of routing games, in order to establish existence of equilibria (see Section 4.1). The link between finite potential games and potential function for nonatomic games was noticed by Sandholm (2001) and we shall present this result in the next section.

A potential game is a population game, hence has an equilibrium state. However existence of equilibrium state may be established directly through a simple maximization problem. This constitutes the next theorem.

3.20 Theorem. *Any potential game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$ has an equilibrium state. Moreover if the potential function is strictly concave, the equilibrium state is unique.*

Proof: Consider the maximization problem

$$\max_{x \in \Pi_{i \in I} \Delta_i} P(x). \tag{3.7}$$

The Lagrangian for this maximization problem is

$$L(x, p, q) = P(x) + \sum_{i \in I} p_i \left(1 - \sum_{a=1}^{m_i} x_i^a\right) + \sum_{i \in I} \sum_{a=1}^{m_i} q_i^a x_i^a.$$

We will show that any point satisfying the Kuhn-Tucker first order necessary conditions is an equilibrium state of $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ and reciprocally. Hence the existence of a solution for the problem (3.7) will imply the existence of an equilibrium state.

The Kuhn-Tucker first order necessary conditions are for all $i \in I$ and $a \in A_i$

$$(i)\ \tfrac{\partial}{\partial x_i^a} P(\tilde{x}) = p_i - q_i^a, \qquad (ii)\ q_i^a \tilde{x}_i^a = 0,$$

$$(iii)\ q_i^a \geq 0 \qquad \text{and} \quad (iv)\ \textstyle\sum_{a\in A_i} \tilde{x}_i^a = 1.$$

(i), (ii), (iii) and (iv) implies:

$$\tilde{x}_i^a > 0 \Longrightarrow U_i^a(\tilde{x}) = p_i \geq p_i - q_i^{b_i} = U_i^b(\tilde{x}), \quad \forall b \in A_i.$$

Hence $\tilde{x}$ is an equilibrium state.

If $\tilde{x}$ is an equilibrium state, since

$$\frac{\partial}{\partial x_i^a} P(\tilde{x}) = U_i^a(\tilde{x}),$$

(i), (ii), (iii) and (iv) are satisfied by

$$\tilde{x},\ p_i := \max_{a=1}^{m_i} U_i^a(\tilde{x}) \text{ and } q_i^a := p_i - U_i^a(\tilde{x}).$$

Since all constraints are linear, an equilibrium state exists. Moreover if P is strictly concave, then there is a unique solution to the problem (3.7), thus equivalently, a unique equilibrium state. ∎

3.4.2 Potential games as limits of finite player games

In this section, we present a result of Sandholm (2001) on the approximation of infinite potential (IP) games by finite potential (FP) games and establish results for the sequence of corresponding equilibrium states. For notational convenience we restrict attention to infinite potential games with a unique population: $(\underline{T}, A, U)$ with potential function P.

3.21. Remark. *On finite potential games.*
(A) (Voorneveld et al., 1999, Theorem 2.1) shows that FP games may be characterized as the class of games which admit a representation

$$u_i(a_i, a_{-i}) = P(a_i, a_{-i}) + v_i(a_{-i}),$$

for all $a_i \in A_i$ and $a_{-i} \in A_{-i}$, and where v_i maps A_{-i} to $\mathbb{R}$.
(B) (Monderer and Shapley, 1996, Corollary 2.2) shows that FP games have the property to posses pure Nash equilibria: clearly the profile $(a_1, \ldots, a_n)$ which maximizes P on $\Pi_{i\in I} A_i$ is a Nash equilibrium.

To approach IP games with a unique population, we work with FP games in which players are identical and anonymous: An *e-anonymous finite potential (AFP) game*, is a FP game where all players have same action set A and same evaluation function $\{U^a\}_{a\in A}$ depending only on the aggregate choice of the population. Consider the following sets:

$$\begin{aligned}
\Delta(n) &:= \{x \in \mathbb{R}^m : x^a \geq 0,\ \sum_{a\in A} x^a = 1 \text{ and } nx^a \in \mathbb{N},\ \forall a \in A\}\\
\Delta^a(n) &:= \{x \in \Delta(n) : x^a > 0,\ \forall a \in A\}\\
\Delta'(n) &:= \{x \in \mathbb{R}^m : x^a \geq 0,\ \sum_{a\in A} x^a = 1 - 1/n \text{ and } nx^a \in \mathbb{N},\ \forall a \in A\}.
\end{aligned}$$

$\Delta(n)$ represents the set of all possible distributions of strategies across the set of n players. With the previous definitions and Remark 3.21(A), we are able to define formally AFP games.

3.22. Definition. An *e-anonymous n-player potential game*, is a FP game where the n players have same action set A and same vector-valued evaluation function U_n of the form

$$U_n^a(x) = P_n(x) + v_n(x - \frac{\delta_a}{n}), \tag{3.8}$$

where P_n is the potential function and x belongs to $\Delta^a(n)$. The domains of U_n, P_n and v_n are respectively $\Delta^a(n)$, $\Delta(n)$ and $\Delta'(n)$.

Extend the domain of definition of P_n from $\Delta(n)$ to $\Delta(n) \cup \Delta'(n)$ by defining $P_n(y) = -v_n(y)$ for all $y \in \Delta'(n)$, then the payoff becomes

$$U_n^a(x) = P_n(x) - P_n(x - \frac{\delta_a}{n}).$$

Given a set of actions A, a n-player AFP game is characterized by its evaluation and potential functions and is denoted by (U_n, P_n).

3.23. Definition. A sequence of AFP games $\{(U_n, P_n)\}_{n=n_0}^{\infty}$ *converges* if there exists a $\mathcal{C}^1$ function $P:\ \Delta \to \mathbb{R}$ and a vanishing sequence of real numbers $\{K_n\}_{n=n_0}^{\infty}$ such that

$$\left|\left(\frac{1}{n}P_n(x) - P(x)\right) - \left(\frac{1}{n}P_n(y) - P(y)\right)\right| \leq K_n\|x - y\|$$

for all n and x and y in $\Delta(n) \cup \Delta'(n)$.

To understand this definition note that defining P_k on $\mathbb{R}^m$, it follows that

$$\lim_{n\to\infty} \frac{P_k(x) - P_k(x - \frac{\delta_a}{n})}{n} = \frac{\partial P_k}{\partial x^a}(x).$$

3.24 Theorem. *Sandholm (2001)*
Let $\{(U_n, P_n)\}_{n=n_0}^{\infty}$ be a convergent sequence of AFP games with limit potential function P. Then

(i) The sequences of evaluation functions are uniformly convergent: for each $a \in A$, there exists a function $U^a : \Delta \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup_{x\in\Delta^a(n)} |U_n^a(x) - U^a(x)| = 0.$$

(ii) The limit evaluation function and potential function define an infinite player potential game. That is

$$U^a(x) = \frac{\partial}{\partial x^a} P(x) \quad \forall x \in \Delta,\ \forall a \in A.$$

Proof: Fix $a \in A$ and $x \in \Delta$. By the mean value theorem, we have

$$P(x) - P\left(x - \frac{\delta_a}{n}\right) = \frac{1}{n}\frac{\partial}{\partial x^a} P\left(z_n^a(x)\right),$$

for some $z_n^a(x)$ in the segment $\{z = \alpha x + (1-\alpha)\left(x - \frac{\delta_a}{n}\right); 0 \le \alpha \le 1\}$.

The definition of convergence presented previously implies that

$$\begin{aligned}
\left|U_n^a(x) - \frac{\partial}{\partial x^a}P(x)\right| &= \left|\left(P_n(x) - P_n(x - \frac{\delta_a}{n})\right) - \frac{\partial}{\partial x^a}P(x)\right| \\
&\le \left|\left(P_n(x) - P_n(x - \frac{\delta_a}{n})\right) - n\left(P(x) - P(x - \frac{\delta_a}{n})\right)\right| \\
&\qquad + \left|\frac{\partial}{\partial x^a}P\left(z_n^a(x)\right) - \frac{\partial}{\partial x^a}P(x)\right| \\
&\le n\left|\left(\frac{1}{n}P_n(x) - P(x)\right) - \left(\frac{1}{n}P_n(x - \frac{\delta_a}{n}) - P(x - \frac{\delta_a}{n})\right)\right| \\
&\qquad + \left|\frac{\partial}{\partial x^a}P\left(z_n^a(x)\right) - \frac{\partial}{\partial x^a}P(x)\right| \\
&\le K_n + \left|\frac{\partial}{\partial x^a}P\left(z_n^a(x)\right) - \frac{\partial}{\partial x^a}P(x)\right|.
\end{aligned}$$

Since K_n vanishes and P is C^1 on the compact set Δ, we conclude that

$$\lim_{n\to\infty} \sup_{x\in\Delta^a(n)} \left|U_n^a(x) - \frac{\partial}{\partial x^a}P(x)\right| = 0.$$

∎

Dealing with p-anonymous n-player potential games rather than e-anonymous ones does not allow to obtain the relation (3.8).

We now establish results concerning relations between equilibria of IP and AFP games. We state first a definition.

3.25. Definition. Given a real-valued function P on X, a set $B \subset X$ is a *local maximizer set* of P if (i) B is connected; (ii) P is constant on B; and (iii) there exists a neighborhood B_ε of B such that $P(y) < P(x)$ for all $y \in B_\varepsilon \setminus B$ and $x \in B$.

3.26 Proposition. *Let $\{(U_n, P_n)\}$ be a convergent sequence of AFP games with limit potential function P and vanishing sequence $\{K_n\}$. Let B be a local maximizer set of P. If $B \cap \Delta(n_1) \neq \emptyset$, for some n_1, then there exists a sequence of integers $\{kn\}_{k\in\mathbb{N}}$, for some n, such that $x \in \Delta(kn)$ is a $\sqrt{2}K_{kn}$-equilibrium of (U_{kn}, P_{kn}), for all k in $\mathbb{N}$.*

Proof: Let $x \in B \cap \Delta^a(n_1)$, for some n_1, denote $x_n^{a\to b} \in \Delta(n)$ the population state x_n where a player deviates from a to b, for all $n = kn_1$, k integer. Choose $N_1 > N$ such that for all $n = kn_1 \geq N_1$,

$$x_n^{a\to b} \in B_\varepsilon. \tag{3.9}$$

Suppose that $P_n(x_n^{a\to b}) > P_n(x)$, for some $n = kn_1 \geq N_1$. Then it follows from (3.9), the definition of local maximizer, this of convergence and the equality $\|x - x_n^{a\to b}\| = \sqrt{2}/n$ that

$$|P_n(x) - P_n(x_n^{a\to b})| < \sqrt{2}K_n.$$

Remark that $x - \frac{\delta_a}{n} = x_n^{a\to b} - \frac{\delta_b}{n}$, to assert by definition of the potential that

$$\begin{aligned} P_n(x) - P_n(x_n^{a\to b}) &= P_n(x) + P_n(x - \frac{\delta_a}{n}) - \left(P_n(x_n^{a\to b}) + P_n(x_n^{a\to b} - \frac{\delta_b}{n})\right) \\ &= U_n(x) - U_n(x_n^{a\to b}). \end{aligned}$$

Hence x is a $\sqrt{2}K_n$-equilibrium of (U_n, P_n). ∎

3.27 Corollary. *Let $\{(U_n, P_n)\}$ be a convergent sequence of AFP games with limit potential function P. Let B be a local maximizer set of P and $\{x_n\}$, $x_n \in \Delta(n)$, be a sequence converging to B. There exists a vanishing sequence of real numbers $\{\varepsilon_n\}$ such that x_n is an ε_n-equilibrium of (U_n, P_n).*

Proof: Follows from Proposition 3.26 and the continuity of the potential function P. ∎

3.28 Proposition. *Let $\{(U_n, P_n)\}$ be a convergent sequence of AFP games with limit potential function P. Let $\{x_n\}$, $x_n \in \Delta(n)$, be a sequence of equilibrium converging to x and such that $\|x - x_n\| \leq 2/n$. Then x is a local maximizer of P.*

Proof: Since x_n is an equilibrium of (U_n, P_n), it is a local maximizer of P_n (Monderer and Shapley, 1996, Lemma 2.1):

$$\forall y \in \Delta(n) \cap B_{2\sqrt{2}/n}(x_n), \quad P_n(x_n) \geq P_n(y), \tag{3.10}$$

where $B_\varepsilon(x)$ denotes the open ball of radius ε and center x. This equation and the assumption $\|x - x_n\| \leq 2/n$ imply that for some $\varepsilon > 0$, $\liminf_{n\to\infty} P_n(x_n) - P_n(y) \geq 0$, for all $y \in \Delta(n) \cap B_\varepsilon(x)$.

By contradiction, assume now that x is not a local maximizer of P. Then for all $\varepsilon > 0$, there exists $y \in \Delta \cap B_\varepsilon(x)$ such that $P(y) > P(x)$. It follows that there exists N such that for all $n \geq N$, $P(y_n) > P(x_n)$ for some $y_n \in \Delta(n) \cap B_\varepsilon(x)$.

Consider the limit in the equation defining convergence, that is

$$\lim_{n\to\infty} \left| \left(\frac{1}{n} P_n(x_n) - P(x_n) \right) - \left(\frac{1}{n} P_n(y) - P(y) \right) \right| \leq \lim_{n\to\infty} K_n \|x_n - y\|.$$

Equation (3.10) and the previous inequality yield

$$\lim_{n\to\infty} |P(x_n) - P(y_n)| \leq \lim_{n\to\infty} K_n \|x_n - y_n\| = 0, \tag{3.11}$$

a contradiction. ∎

We conclude this section on potential games with an example showing that neither weak convergence nor λ-convergence implies convergence of FP games.

3.29. Example. *Failure of convergence of FP games when the evaluation functions are common to every game.*
Consider a population of players who are randomly matched to play in pure strategies the coordination game

$1,1$	$0,0$
$0,0$	$1,1$

The payoffs of this game are given by $U^1(x^1, x^2) = x^1$ and $U^2(x^1, x^2) = x^2$. Clearly the sequence of finite-player games λ-converges (resp. weakly converges) to the $\mathcal{S}$-game (resp. $\mathcal{M}$-game).

These games are potential games: For the nonatomic game the potential function is

$$P(x) = (1/2)((x^1)^2 + (x^2)^2).$$

For finite-player games, the potential function is

$$P^n(\frac{k^1}{n}, \frac{k^2}{n}) = \frac{k^1(k^1+1)}{n} + \frac{k^2(k^2+1)}{n}.$$

To extend P^n to the set $\Delta'(n)$, note that, if $k^1 \geq 1$,

$$\frac{k^1}{n} = U^1(\frac{k^1}{n}, \frac{k^2}{n}) = P^n(\frac{k^1}{n}, \frac{k^2}{n}) - P^n(\frac{k^1-1}{n}, \frac{k^2}{n}).$$

It follows that

$$P^n(\frac{k^1-1}{n}, \frac{k^2}{n}) = -\left(\frac{k^1(k^1-1) + (k^2)^2}{n} + 1 \right).$$

We now prove the following result:
The sequence of FP games $\{(U, P^n)\}$ does not converge to the IP game (U, P).
Assume that there exists a sequence of real numbers $\{K_n\}$ such that

$$\left|\left(\frac{1}{n}P^n(x) - P(x)\right) - \left(\frac{1}{n}P^n(y) - P(y)\right)\right| \leq K_n\|x - y\|.$$

Let $x = (1, 0)$ and $y = ((n-1)/n, 1/n)$. It comes

$$\left(\frac{1}{n}P^n(x) - P(x)\right) - \left(\frac{1}{n}P^n(y) - P(y)\right) = 2 + \frac{1}{n}.$$

So $2 + \frac{1}{n} \leq \frac{K_n}{n}$ and $K_n > 2n$. Thus the sequence $\{K_n\}$ does not vanish and convergence of a sequence of evaluation functions does not imply convergence of the corresponding sequence of FP games.

Chapter 4

Applications

This chapter is devoted to three applications of population games which expanded independently ones of the others.

The first one, *routing games*, takes place in the context of road traffic (or telecommunication, Internet networks) and modelizes how selfish routing is done in networks. The concept of equilibrium of these games goes back to the fifties and is called *Wardop equilibrium*, it is equivalent to the equilibrium state of Section 3.3. A result of approximation of Wardrop equilibria by Nash equilibria of Haurie and Marcotte (1985) is presented.

The second one, *crowding games*, is an extension of congestion games of Rosenthal (1973) (see Section 3.4) where players tend to avoid the crowd. They were introduced in (Milchtaich, 2000).

The last one, *evolutionary games*, is an application of Game Theory to Evolutionary Biology.

4.1 Routing games

First road traffic networks are introduced and selfish routing in these networks is modelized as population game. Then the concept of equilibrium of routing games, called *Wardrop equilibrium* is defined; it corresponds to equilibrium state of population games, hence existence of Wardrop equilibrium and assumptions for its uniqueness will follow. Finally, Section 4.1.2 deals with approximation of Wardrop equilibria by Nash equilibria, a result of Haurie and Marcotte (1985).

4.1.1 Nonatomic routing games

Suppose that a large number of vehicles have to go from a location o to another one d which are connected by different routes. Each vehicle has to choose one route and wants to minimize its journey time which is a function of (i) the route it chooses and (ii) the flow of vehicles on this road. The number of vehicles being large, the influence of a single vehicle on the road traffic is negligeable.

A *flow configuration* is the measure on the set of routes induced by the choice of the vehicles. Wardrop (1952, p. 345) gives a minimal requirement for a flow configuration to be stable:

(W) The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

A flow configuration which does not respect this condition, is not stable: if some vehicles go through a route which has a delay strictly greater that another one, anyone of these vehicles would have interest in deviating of this route in order to decrease its journey time.

A flow configuration satisfying (W) is called a *Wardrop equilibrium*.

The negligeability of "individual influence" entails the modelisation of such situations by nonatomic games, more precisely by population games. Thus, from now on a vehicle is called a player or an individual. To modelize road traffic with game theory need first some "network's" definitions.

4.1. Definition. Consider a graph where (i) each vertex represents a location and (ii) an edge $l = (u, v)$ represents a road connecting u and v. Call a directed edge l a *link*, the vertex $o(l)$ is the *origin* of l. and the vertex $d(l)$ its *destination*. Let $\mathcal{L}$ denote the set of links. A *path* p is a finite sequence of links $(l_1, l_2, \ldots, l_n)$ such that for all $k \neq 1, n$, if l_k goes from a to b, then l_{k-1} ends at a and l_{k+1} starts from b $(d(l_k) = o(l_{k+1}))$.

Assume that different players may have different origins and different destinations, the number of pairs (origin, destination) being finite. Hence introduce a population set I where any member of population $i \in I$ uses paths having o_i for origin and d_i for destination. Denote the finite set of paths of population i, by $\mathcal{P}_i = \{1, \ldots, m_i\}$. Hence a path p in $\mathcal{P}_i$ is a sequence of links $(l_1, \ldots, l_n)$, such that $o(l_1) = o_i$ and $d(l_n) = d_i$.

4.2. Definition. A *flow configuration* is a vector $x = (x_1, \ldots, x_I)$, where x_i is a probability on $\mathcal{P}_i$. Let $\mathcal{P} =: \Pi_{i\in I}\mathcal{P}_i$. The set of flow configurations is denoted $\Sigma_{\mathcal{P}}$.

Each individual has a vector-valued cost function depending on the flow configuration (the p^{th} component representing the cost of path p). Assume that any two members of population i have same cost function. Thus the cost function of i's members is denoted by $F_i : \Sigma_{\mathcal{P}} \to \mathbb{R}^{m_i}$.

4.3. Definition. $(I, (\mathcal{P}_i)_{i\in I}, (F_i)_{i\in I})$ is a *routing game*.

Obviously, if the F_i's are continuous functions, then $(I, (\mathcal{P}_i)_{i\in I}, (-F_i)_{i\in I})$ is a population game and any flow configuration is a populations state.

4.4. Definition. Let $(I, (\mathcal{P}_i)_{i\in I}, (F_i)_{i\in I})$ be a routing game. The flow configuration $x \in \Sigma_{\mathcal{P}}$ is a *Wardrop equilibrium* if it satisfies the following condition:

$$\forall i \in I,\ \forall \bar{p}, p \in \mathcal{P}_i, \quad x_i^{\bar{p}} > 0 \Longrightarrow F_i^{\bar{p}}(x) \leq F_i^p(x). \tag{4.1}$$

Consider the population game $(I, (\mathcal{P}_i)_{i\in I}, (-F_i)_{i\in I})$, then the definition of Wardrop equilibrium is equivalent to the equilibrium condition (3.3). Thus, if the F_i's are continuous, the existence of Wardrop equilibrium follows from Theorem 3.14, and if the vector function $F = (F_i)_{i\in I}$ is strictly monotone (Definition 3.16) its uniqueness follows from Theorem 3.17.

Usually in routing games, the cost of a path is the sum of the costs of the links belonging to this path. This kind of cost functions leads to the existence of a potential function for routing game and hence allows to obtain existence of Wardrop equilibrium by Theorem 3.20.

The specification of the cost functions requires to endow the set I with a measure ν, ν_i being the weight (or "size") of population i.

For $l \in \mathcal{L}$, $i \in I$ and $p \in \mathcal{P}_i$, let δ^{lp} be the incident indicator:

$$\delta^{lp} = \begin{cases} 1 & \text{if } l \in p, \\ 0 & \text{otherwise.} \end{cases}$$

Any flow configuration x induces a measure $x_{\mathcal{L}}$ over the set of links,

$$x^l_{\mathcal{L}} := \sum_{i\in I} \sum_{p\in\mathcal{P}_i} \delta^{lp} \nu_i \, x^p_i,$$

$x^l_{\mathcal{L}}$ is the flow going through link l when the flow configuration is x.

The cost function of link l for is denoted by f^l, it depends only on the flow $x^l_{\mathcal{L}}$ which goes through this link ($f^l : \mathbb{R} \to \mathbb{R}$).

Member of i's cost function F_i is a function from $\Sigma_{\mathcal{P}}$ to $\mathbb{R}^{m_i}$ defined by

$$F^p_i(x) = \sum_{l\in\mathcal{L}} \delta^{lp} f^l\left(x^l_{\mathcal{L}}\right), \quad \forall p \in \mathcal{P}_i.$$

4.5 Proposition. *Let $(I, (\mathcal{P}_i)_{i\in I}, (F_i)_{i\in I})$ be a routing game where F_i is defined by*

$$\forall i \in I,\ \forall p \in \mathcal{P}_i,\ \forall x \in \Sigma_{\mathcal{P}}, \quad F^p_i(x) = \sum_{l\in\mathcal{L}} \delta^{lp} f^l\left(x^l_{\mathcal{L}}\right).$$

If for all $l \in \mathcal{L}$, f^l is continuous, then $(I, (\mathcal{P}_i)_{i\in I}, (F_i)_{i\in I})$ is a potential game.

Proof: Let $\tilde{f}$ be the function from $\Sigma_{\mathcal{P}}$ to $\mathbb{R}$ defined by

$$\tilde{f}(x) = \sum_{l\in\mathcal{L}} \int_0^{x^l_{\mathcal{L}}} f^l(x)\, dx.$$

$\tilde{f}$ is a C^1 function.

For all $i \in I$, $p \in \mathcal{P}_i$, we have

$$\frac{\partial}{\partial x^p_i} \tilde{f}(x) = \sum_{l\in\mathcal{L}} \delta^{lp} f^l\left(x^l_{\mathcal{L}}\right) = F^p_i(x).$$

Hence $\tilde{f}$ is a potential function. ∎

Now, the existence of Wardrop equilibrium follows from Theorem 3.20. Moreover if the f_l's are strictly increasing then $F = (F_i)_{i\in I}$ is strictly monotone and therefore Wardrop equilibrium is unique (Theorem 3.17).

4.1.2 Approximation of Wardrop equilibria by Nash equilibria

Haurie and Marcotte (1985) establish that given a sequence of replica of finite-player routing games possessing unique Nash equilibrium, a subsequence of these Nash equilibria converges to a Wardrop equilibrium of the "limit" (nonatomic) routing game. This result is the topic of this section. For notational convenience, it is established in the case where all players have same origin and same destination, it can be extended to multiple coules (origin-destination).

First finite routing games are defined and a characterization of Nash equilibria for these games is given. Then a sequence of replica is constructed and finally the approximation result is proved.

Consider a (finite) set of vertices $\mathcal{V}$, a (finite) set of links $\mathcal{L}$ and a finite set of players I. Each player i has (1) an origin o_i and a destination d_i, (2) an amount of flow ν_i to route from o_i to d_i. (Compared with routing games, population i's members do not choose their route: there is a single decision maker for the all population, consequently ν_i corresponds to the weight of population i.)

Let $x_i^l := x_i^{uv}$ be the flow send by player i through link $l = (u, v)$. For any vertex $v \in \mathcal{V}$, denote the set of its in-going (resp. out-going) links by $In(v)$ (resp. $Out(v)$).

A strategy of player i is a link-flow vector $x_i = (x_i^1, \ldots, x_i^L) \in \mathbb{R}^L$ such that

$$\forall l \in \mathcal{L},\ x_i^l \geq 0 \qquad (\textit{positivity constraints}), \tag{4.2}$$

$$\forall v \in \mathcal{V},\ \sum_{l \in Out(v)} x_i^l = \sum_{l \in In(v)} x_i^l + \nu_i^v \qquad (\textit{conservation constraints}), \tag{4.3}$$

where $\nu_i^{o_i} = \nu_i$, $\nu_i^{d_i} = -\nu_i$ and $\nu_i^v = 0$ for all $v \neq o_i, d_i$.

Denote by X_i the set of strategies of player i and by $X := \Pi_{i \in I} X_i$ the set of strategy profiles. Each player i has a real-valued cost function f_i defined on X.

4.6. Definition. The tuple $(I, (X_i), (f_i))$ is a *finite-player routing game.* A strategy profile $x \in X$ is a *Nash equilibrium* of $(I, (X_i), (f_i))$ if it satisfies

$$\forall i \in I, \quad f_i(x_i, x_{-i}) = \min_{y_i \in X_i} f_i(y_i, x_{-i}). \tag{4.4}$$

The next lemma gives a usual characterization of the set of Nash equilibria of the finite-player routing game $(I, (X_i), (f_i))$ under the assumption:
(A) f_i is continuously differentiable on X and convex on X_i.

4.7 Lemma. *Let $(I, (X_i), (f_i))$ be a finite-player routing game satisfing assumtion* **(A)**. *Then the strategy profile $x \in X$ is a Nash equilibrium if and only if*

$$\forall i \in I,\ \nabla f_i(x_i, x_{-i}) \cdot (y_i - x_i) \geq 0, \quad \forall y_i \in X_i. \tag{4.5}$$

Proof: Since $f_i(\cdot, x_{-i})$ is a C^1, convex function and X_i is a closed convex set (in $\mathbb{R}^L$), the equivalence between equations (4.4) and (4.5) follows from (Kinderlehrer and Stampacchia, 2000, I, Proposition 5.1 and Proposition 5.2). ∎

Let $\nabla f := (\nabla f_1, \dots, \nabla f_I)$, then (4.5) is equivalent to

$$\nabla f(x) \cdot (y - x) \geq 0, \quad \forall y \in X. \tag{4.6}$$

Since ∇f is continuous on a compact and convex set, it follows from (Kinderlehrer and Stampacchia, 2000, I, Theorem 3.1) that (4.6) has a solution, in other words $(I, (X_i), (f_i))$ has a Nash equilibrium.

We are in position to construct a *sequence of replica* of finite-player routing games converging to a "nonatomic" routing game.

Let $\Gamma := (\mathcal{P}, F)$ be a (nonatomic) routing game with a single population: all individuals have same origin o, same destination d and same vector-valued cost function $F : \Delta(\mathcal{P}) \to \mathbb{R}^m$.

Given a flow configuration $x \in \Delta(\mathcal{P})$, denote by $x_\mathcal{L} \in \mathbb{R}^{\#\mathcal{L}}$ the link-flow vector that x induces. The cost of path p is $F^p(x) = \sum_{l \in p} f^l(x^l_\mathcal{L})$.

To Γ associate a one-player routing game $\Gamma(1) := (\{1\}, X(1), f_1)$, where
(i) $X(1)$ is the set of flow vectors satisfying positivity and conservation constraints (4.2) and (4.3) with origin o, destination d and amount of flow $\nu_1 = 1$, and
(ii) $f_1 : X(1) \to \mathbb{R}$ is defined by

$$f_1(x) = \sum_{l \in L} x^l f^l(x^l).$$

Then replicate the player: construct new routing games, for all $n \in \mathbb{N}$, $\Gamma(n) := (\{1, \dots, n\}, X(n), f_n)$ where all players have
(i) same strategy set $X(n)$ which is the set of flow vectors satisfying positivity and conservation constraints (4.2) and (4.3) with origin o, destination d and amount of flow $\nu_n = 1/n$, and
(ii) same cost function $f_n : X(n) \times X(n)^{n-1} \to \mathbb{R}$ defined by

$$f_n(x_i, x_{-i}) = \sum_{l \in L} x^l_i f^l(x^l_\mathcal{L}),$$

where $x_\mathcal{L} \in X(1)$ is the link-flow vector induced by (x_i, x_{-i}).

The sequence $\{\Gamma(n)\}$ is a *sequence of replica* of finite-player routing games.

Assume that for all $l \in \mathcal{L}$, f^l is a C^1, increasing function. Then Γ has a Wardrop equilibrium, and for all $n \in \mathbb{N}$, $\Gamma(n)$ satisfies assumption **(A)** and thus has a Nash equilibrium $x(n)$. Moreover, since f_n is strictly convex with respect to its first argument, it follows from the next proposition that the Nash equilibrium is unique (see Section 4.1.1).

4.8 Proposition. *(Kinderlehrer and Stampacchia, 2000, I, Proposition 5.3)*
Let $g : E \to \mathbb{R}$ $(E \subset \mathbb{R}^n)$ be a C^1 and strictly convex function. Then ∇g is strictly monotone.

Let $x_{\mathcal{L}}(n)$ be the total flow vector induced by the unique Nash equilibrium $x(n)$ of $\Gamma(n)$. Equation (4.6) gives

$$\sum_{i=1}^{n}\sum_{l\in\mathcal{L}}\left(f^l\left(x^l_{\mathcal{L}}(n)\right)+x^l_i\nabla f^l\left(x^l_{\mathcal{L}}\right)\right)\left(y^l_i-x^l_i\right)\geq 0,\quad \forall y\in X. \tag{4.7}$$

It remains to state the approximation result of Haurie and Marcotte (1985).

4.9 Theorem. *(Haurie and Marcotte, 1985, Theorem 3.2)*
There exist a Wardrop equilibrium with link-flow vector $x_{\mathcal{L}}\in\mathbb{R}^L$ of Γ and a sequence $\{\Gamma(n_k)\}_{k\in\mathbb{N}}$ admitting unique Nash equilibria $x(n_k)$ such that the resulting flow vectors $x_{\mathcal{L}}(n_k)$ satisfy

$$\lim_{k\to\infty} x_{\mathcal{L}}(n_k)=x_{\mathcal{L}}.$$

Proof: From (Orda et al., 1993, Lemma 3.1), it follows that, for all n, the Nash equilibrium $x(n)$ is symmetric, that is for all players $i,j\in I$, we have $x_i(n)=x_j(n)\in\mathbb{R}^L$. This common Nash equilibrium strategy can be written $x_i(n)=(1/n)x_{\mathcal{L}}(n)$. The vector $x_{\mathcal{L}}(n)$ is uniformly bounded for all n, hence there exists a convergent subsequence $\{x_{\mathcal{L}}(n_k)\}_{k\in\mathbb{N}}$, let $x_{\mathcal{L}}$ be its limit. To simplify notations, assume that the sequence $\{x_{\mathcal{L}}(n)\}$ converges.

Replacing x_i by $(1/n)x_{\mathcal{L}}(n)$ in (4.7) gives

$$\sum_{i=1}^{n}\sum_{l\in\mathcal{L}}f^l(x^l_{\mathcal{L}}(n))\left(y^l_i-\frac{x^l_{\mathcal{L}}(n)}{n}\right)$$
$$+\frac{1}{n}\sum_{i=1}^{n}\sum_{l\in\mathcal{L}}x^l_{\mathcal{L}}(n)\nabla f^l(x^l_{\mathcal{L}}(n))\left(y^l_i-\frac{x^l_{\mathcal{L}}(n)}{n}\right)\geq 0,\qquad \forall y\in X.$$

Since X is compact,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{l\in\mathcal{L}}x^l_{\mathcal{L}}(n)\nabla f^l(x^l_{\mathcal{L}}(n))\left(y^l_i-\frac{x^l_{\mathcal{L}}(n)}{n}\right)=0.$$

Let x be any flow configuration that induces the link-flow vector $x_{\mathcal{L}}$. Then the continuity of the f_l's yields

$$F(x)\cdot(y-x)=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{l\in\mathcal{L}}f^l(x^l_{\mathcal{L}}(n))\left(y^l_i-\frac{x^l_{\mathcal{L}}(n)}{n}\right)\geq 0,\quad \forall y\in X.$$

Hence x is a Wardrop equilibrium of Γ and the result is established. ∎

4.2 Crowding games

Large crowding games are defined by Milchtaich (2000). They are specific routing games.

First the model is introduced and some properties of equilibria are stated. Then a result of approximation is given.

4.2.1 Large crowding games

Let I be a finite set of populations, the relative size of population i is denoted by ν_i, hence $\nu = (\nu_i)$ is a probability on I. The finite action set, common to every population, is by denoted $A = \{1, \ldots, m\}$.

A populations state $x = (x_i)$ induces the vector $\int_I x_i \, d\nu \in \Delta$, whose a^{th} coordinate, $\int_I x_i^a \, d\nu$, (henceforth $\int x^a$), is interpreted as the total proportion of the players choosing action a.

Given x, for all $a \in A$, the payoff of a member of i choosing action a depends only on $\int x^a$ (as for the class of games defined by Rosenthal (1973), see Section 3.4). The evaluation function U_i of a member of i is defined by

$$U_i(x) = \left(U_i^1 \left(\int x^1 \right), \ldots, U_i^m \left(\int x^m \right) \right),$$

where each function U_i^a is a continuous and strictly decreasing function from $[0, 1]$ to $\mathbb{R}$.

4.10. Definition. A *large crowding game* is a tuple $(I, \nu, A, (U_i))$. A populations state x is an *equilibrium* of a large crowding game $(I, \nu, A, (U_i))$ if, for every i and for every action a,

$$x_i^a > 0 \quad \Longrightarrow \quad U_i^a \left(\int x^a \right) = \max_{b \in A} U_i^b \left(\int x^b \right).$$

The existence of equilibria for large crowding games follows from Theorem 3.14.

The monotonicity of the U_i^a's implies:

4.11 Proposition. *(Milchtaich, 2000, Proposition 3.3)*
For any two equilibria, $\hat{x}$, $\bar{x}$ of a large crowding game

$$\int_I \hat{x}_i \, d\nu = \int_I \bar{x}_i \, d\nu.$$

It follows that payoffs are unique at equilibrium.

Proof: Since for all $i \in I$, $\hat{x}_i$, $\bar{x}_i$ belongs to Δ,

$$\sum_{a \in A} \hat{x}_i^a = \sum_{a \in A} \bar{x}_i^a = 1.$$

Then for every subset A' of A, and every measurable subset I' of I, we have:

$$\begin{aligned} \sum_{a \in A'} \left(\int \hat{x}^a - \int \bar{x}^a \right) &= \int_I \sum_{a \in A'} (\hat{x}_i^a - \bar{x}_i^a) \, d\nu - \int_{I'} \sum_{a \in A} (\hat{x}_i^a - \bar{x}_i^a) \, d\nu \\ &\leq \int_{I \setminus I'} \sum_{a \in A'} \hat{x}_i^a \, d\nu + \int_{I'} \sum_{a \in A \setminus A'} \bar{x}_i^a \, d\nu. \end{aligned} \tag{4.8}$$

Let $\hat{x}, \bar{x}$ be two equilibria of a large crowding game $\mathbf{u}$, $A' := \{a \in A : \int \hat{x}^a > \int \bar{x}^a\}$ and $I' := \{i \in I : max_{b\in A} U_i^b \left(\int \hat{x}^b\right) < max_{b\in A} U_i^b \left(\int \bar{x}^b\right)\}$.

If $i \in I \setminus I'$ and $a \in A'$, then by the monotonicity of the U_i^b's

$$max_{b\in A} U_i^b \left(\int \hat{x}^b\right) \geq max_{b\in A} U_i^b \left(\int \bar{x}^b\right) \geq U_i^a \left(\int \bar{x}^a\right) > U_i^a \left(\int \hat{x}^a\right).$$

Therefore from the equilibrium condition it follows that $\hat{x}_i^a = 0$ for almost every $i \in I \setminus I'$ and every $a \in A'$. Similarily if $i \in I'$ and $a \in A \setminus A'$, then

$$max_{b\in A} U_i^b \left(\int \bar{x}^b\right) > max_{b\in A} U_i^b \left(\int \hat{x}^b\right) \geq U_i^a \left(\int \hat{x}^a\right) \geq U_i^a \left(\int \bar{x}^a\right).$$

and $\bar{x}_i^a = 0$ for almost every $i \in I'$ and every $a \in A \setminus A'$.

From (4.8) A' is empty: $\int \hat{x}^a \leq \int \bar{x}^a$ for all $a \in A$. By symmetry, we conclude that $\int \hat{x}^a = \int \bar{x}^a$. ∎

Note that the proof presented does not require any continuity assumption on the payoff function.

4.2.2 Approximation

Milchtaich (2000) establishes that the sets of equilibria of converging sequences of finite crowding games converge on a dense G_δ to a singleton which is the unique equilibrium of the corresponding large crowing game.

Milchtaich (1996) defines n-person crowding games. These are extensions of the games introduced in (Rosenthal, 1973) (see Section 3.4) with player-specific payoff functions.

An n-person crowding game is a finite strategic game where the payoff of each player is a function of the number of players choosing the same action than him and this function is decreasing. A formal definition follows.

4.12. Definition. A *n-player crowding game* is a tuple $(I, \nu, A, (U_i))$, where I is a finite index set, ν is a probability measure on I such that $n\nu_i \in \mathbb{N}$, [1] A is a finite set of actions. U_i is the evaluation function of player of population i, the function the real-valued function U_i^a is a continuous and strictly decreasing function.

4.13. Definition. An *equilibrium* of an n-person crowding game $(I, \nu, A, (U_i))$ is a populations state $x \in \Delta^I$ such that for every player i and every action a, $x_i^a \nu_i n$ is an integer and

$$x_i^a > 0 \quad \Longrightarrow \quad U_i^a \left(\int x^a\right) \geq \max_{b\in A} U_i^b \left(\int x^b + \frac{1}{n}\right). \tag{4.9}$$

Equation (4.9) states that if action a is chosen by a player of population i at equilibrium, then the payoff of this action has to be greater than the payoff of another action when a player of population i goes from a to b.

[1] $n\nu_i$ is interpreted as the number of individuals of population i, all players have the same "weight".

Milchtaich (1996) establishes that n-person crowding games admit equilibria (in pure strategies).

We are now in position to define *sequence of replica* for crowding games.

4.14. Definition. Given a n-person crowding game $\Gamma(1) := (I, \nu, A, (U_i))$, define a mn-person crowding game $\Gamma(m) = (I, \nu, A, (U_i))$ similarly to $\Gamma(1)$ except the fact that the number of individuals of each population is multiplied by m, $\Gamma(m)$ is a *m-replica* of the n-person crowding game $\Gamma(1)$. The sequence $\{\Gamma(m)\}$ is a *sequence of replica*, the limit of this sequence is a large crowding game.

For every n-person crowding game $(I, \nu, A, (U_i))$, there is a corresponding large crowding game, which is defined by the same tuple $(I, \nu, A, (U_i))$. This large crowding game may be seen as the limit of the sequence of m-replicas of the n-person game.

4.15. Definition. Consider a large crowding game $\Gamma := (I, \nu, A, (U_i))$. For $m \in \mathbb{N}$, let $n_m \in \mathbb{N}$ and let $\Gamma(m) := (I, \nu, A, (U_i(m)))$ be an n_m-person crowding game where $U_i(m)$ is the evaluation function of members of i. If $U_i^a(m)(x) \to U_i^a(x)$ for every i, a, and x and if $n_m \to \infty$, then the sequence $\{\Gamma(m)\}$ *converges* to Γ.

In particular any sequence of replica $\{\Gamma(m)\}$ converges to Γ, where $\Gamma(m)$ and Γ are defined as in Definition 4.14.

4.16 Proposition. *(Milchtaich, 2000, Proposition 6.2)*
Let $\Gamma := (I, \nu, A, (U_i))$ be a large crowding game with a unique equilibrium. For $m \in \mathbb{N}$ let $\Gamma(m) := (I, \nu, A, (U_i(m)))$ be an nm-person crowding game that is defined over the same game structure as $\mathbf{u}$ and let $x(m)$ be an equilibrium of $\Gamma(m)$. If $\Gamma(m)$ converges to Γ, then a subsequence of $x(m)$ converges (pointwise) to the unique equilibrium of Γ.

Proof: By passing, if necessary, to a subsequence, it may be assumed that the sequence $\{x(m)\}_{m\in\mathbb{N}}$ is convergent. We show that the limit, x, is an equilibrium of Γ. For every i and every a, $x_i^a > 0$ only if $x_i^a(m) > 0$, and hence $U_i^a(m)\left(\int x^a(m)\right) \geq \max_{b\neq a} U_i^b(m)\left(\int x^b(m) + \frac{1}{n_m}\right)$, for all m large enough. Therefore, $x_i^a > 0$ implies $U_i^a\left(\int x^a\right) \geq \max_{b\in A} U_i^b\left(\int x^b\right)$. ∎

The assumption of the last proposition is frequently satisfied as asserted by the following result.

4.17 Theorem. *(Milchtaich, 2000, Theorem 4.3)*
The set of large crowding games with a unique equilibrium is a dense G_δ in the set of large crowding games.

4.3 Evolutionary games

First Nash equilibrium for n-player matrix game is interpreted in terms of population-statistics in Section 4.3.1. This interpretation was given by Nash himself under the name *mass-action* interpretation. This leads to consider the evolutionary game theoretical framework. In Section 4.3.2, two kind of population games with a single population are descrived: (i) *random matching games* where members are randomly selected from the population to play a symmetric two-player matrix game and (ii) *field games* where each member faces the entire population. Then a notion of stability of population state, called *evolutionarily stable strategy (ESS)*, is defined for these games. Finally, Section 4.3.2 considers population games and extends the notion of ESS, first for random matching game with n populations, then for population games.

4.3.1 From Nash to Maynard Smith: different interpretations of Nash equilibrium

Recall some definitions given in Definition 3.9. A *population* $i \in I$ is a collection of individuals, called also *members*, who have the same action set A_i and the same evaluation function U_i. A *population state* x_i is a probability on A_i, x_i^a represents the average frequency of action a in population i. A *populations state* x is a family of population state indexed by I, $x = (x_i)_{i \in I}$.

In his unpublished Ph.D. dissertation, John Nash provided two interpretations of his equilibrium concept for n-player games. One in terms of rationality and the other one in terms of population statistics.

The first one is the classical interpretation where n rational individuals play only once the n-player game. In this context, each player, given the strategy of others, will play a strategy maximizing his payoff. Suppose that a referee proposes a strategy profile to the players, then if this strategy profile is not a Nash equilibrium, it won't be played. So Nash equilibrium appears to be a necessary condition for a strategy profile to be played.

In the second interpretation, that he calls mass-action interpretation, Nash considers that there are n populations (here a population is not assumed to be necessarily uncountable even not infinite), one for each position in the game, and that the game is played over and over again by individuals of the populations. At each stage, a member of each population is selected at random to play the n-player game. This interpretation consider an element x_i in Δ_i not as a mixed strategy but as a probability where the a^{th} component of x_i represents the average frequency of individuals in i choosing action a (what we defined as a population state). I quote the relevant passage in (Nash, 1950, p. 21-22).

> It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.
>
> To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the "average playing" of the game involves n participants selected at random from the n populations, and that there is a

> stable average frequency with which each pure strategy is employed by the "average member" of the appropriate population.
>
> Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular n-tuple of pure strategies will be employed in a playing of the game should be the product of the probabilities indicating the chance of each of the n pure strategies to be employed in a random playing.

Now the assumption that individuals accumulate empirical information induces that, given a populations state, a member of population i will know the payoff given by each of his possible actions and thus will select only optimal pure strategies. Henceforth a populations state $x = (x_1, \ldots, x_n)$ will be "stable" if, the corresponding strategy profiles is a Nash equilibrium of the n-player game.

4.18. Remark. *On the size of populations.*
The population i has to be enough large to support the population state x_i. Explicitely, if N_i is the number of members of population i, then for all a, it has to exist an integer k such that $x_i^a = k/N_i$. To consider any point in the simplex Δ as a population state, one assumes that each population is infinitely large and therefore modelized by a nonatomic measure space.

The mass-action interpretation of the Nash equilibrium was rediscovered a few years ago and "the view that games are played over and over again by individuals who are randomly drawn from large populations was later indepently taken up by evolutionary biologists" (Weibull in Khun et al., 1996, p. 13), hence the view that a probability on an action set is a distribution of actions across a population. Nevertheless mass-action and evolutionary interpretations differ; namely in the first one individuals choose their action to be optimal, whereas in the second one, as we will see in the next section, individuals are given an action and the frequency of this action in the population will increase only if it is optimal.

4.3.2 Games with a single population

One of the major concerns in evolutionary biology is the stability of average frequency of behaviors within a species. Maynard Smith and Price (1973) modelized evolution with the help of Game Theory and introduced the notion of *evolutionarily stable strategy*. A new domain of game theory was born: Evolutionary Game Theory.

Consider a large population of individuals of the same species. Each individual has a given behavior (action in game terminology) which is inherited and will be passed on to his individual's offspring. Individuals interact generation after generation in game situations of the same kind. The reproductive success of an individual (his *fitness*) is determined by the games' outcomes. One is interested in stable population states; stable in the sense that if a mutant behavior y appears in a small fraction of the population, the average frequency of individuals with behavior y cannot increase faster than the average frequency of individuals with the original behavior(s). Thus a "mutant behavior" cannot invade a stable population state, on the contrary it tends to disappear.

Two kind of interactions are considered within population: (A) pairwise interactions between members of the population or (B) global competitions among all members of the population. In case (A), called *random matching games*, individuals are repeatdly drawn at random to play a symmetric two-player game. In case (B), called *field games*, the fitness of an individual is determined by interactions with the whole of the population.

A Random matching game

Consider a symmetric two-player matrix game (A, u) where A is the (finite) action set [2] and $u(a, b)$ is the payoff given by action a against action b. Consider, now, a population of individuals who are repeatedly drawn at randow to play this matrix game. Suppose that the population state is x, then the payoff of a player choosing action a is given by the expectation

$$U^a(x) = \sum_{b \in A} x^b u(a, b).$$

The tuple (A, U) constitutes a *random matching game with a single population* or simply *random matching (RM) game.* A formal definition is now given.

4.19. Definition. Given a population of individuals, a *random matching (RM) game* is a pair (A, U) where A is a finite action set and $U : \Delta \to \mathbb{R}^m$ is a vector-valued linear function. U is the *evaluation function* of the RM game.

RM games have the following obvious property: To any RM game (A, U) is *associated* a unique symmetric two-player matrix game (A, u) (henceforth *associated two-player game*) such that

$$U^a(x) = \sum_{b \in A} x^b u(a, b), \quad \forall x \in \Delta,\ a, b \in A.$$

An *equilibrium* of the random matching game is a population state x such that for any two actions a and b in the support of x,

$$U^a(x) = U^b(x).$$

This condition is equivalent to

$$x \cdot U(x) \geq \bar{x} \cdot U(x), \quad \forall \bar{x} \in \Delta.$$

In other terms, considering x as a mixed action, (x, x) is a symmetric Nash equilibrium of the symmetric two-player game (A, u).

We illustrate the previous definitions with a famous example: the Hawk-Dove game (reproduced from Hofbauer and Sigmund, 1998).

4.20. Example. *Random matching game: the Hawk-Dove game.*
Consider a population with two possible behaviors hawk (H) or dove (D) when

[2]By symmetic two-player matrix game, we understand a game where the payoffs are given by a (finite) matrix M such that if M gives the payoff of player 1, then M^t gives the payoff of player 2.

individuals are in conflict: An individual with the hawk behavior, henceforth a hawk, fights until either injury or escape of his opponent. An individual with the dove behavior, henceforth a dove, pretends to fight but escapes if the opponent effectively fights.

The conflict takes place over (for example) a morsel of food. Obtaining the food cooresponds to a gain in fitness of G, whereas an injury corresponds to a loss in fitness of C. Assume that the cost of the injury is bigger than the prize of the fight, $C \geq G$.

Then
- If a hawk meets a hawk, both will fight until the injury of one of them. Assume that each hawk has the same chance to win, so that the expected change of fitness is $(G - C)/2$ for each one.
- If a hawk meets a dove, the hawk attacks and the dove escapes, so a hawk has a gain in fitness of G, and the dove's fitness remains unchanged.
- If a dove meets a dove, both will pretend to fight, until one eventually escapes. Assume that each dove escapes with the same chance, so that the expected change of fitness is $C/2$ for each one.

To modelize this situation, consider the two-player symmetric game with payoff matrix

	H	D
H	$(G-C)/2, (G-C)/2$	$G, 0$
D	$0, G$	$G/2, G/2$

Individuals are repeatedly drawn at random to play the previous game. Hence, denoting by x the frequency of hawks in the population, the expected payoff (or fitness) is $x(G - C)/2 + (1 - x)G$ for the hawks and $(1 - x)G/2$ for the doves, with equality if and only if $x = G/C$. If $x > G/C$, then the hawks do less than the doves, hence their frequency diminishes, if $x < G/C$ then the hawks do better than the doves and their frequency increases. Therefore the evolution will lead to a stable population state $(G/C, 1 - G/C)$. Obviously, $(G/C, 1 - G/C)$ is an equilibrium of the RM game considered.

4.21. Remark. *Different interpretations of population states.*
I quote a passage of (Hammerstein and Selten, 1994, p. 932-933) which explain clearly the different interpretations of a population state:

> A population state is *monomorphic* if every individual uses the same strategy and *polymorphic* if more than one strategy is present. A mixed strategy has a monomorphic and a polymorphic interpretation. On the one hand we may think of a monorphic population state in which every individual plays this mixed strategy. On the other hand a mixed strategy can also be interpreted as a description of a polymorphic population state in which only pure strategies occur; in this picture the probabilities of the mixed strategy describe the relative frequencies of the pure strategies.

Example 4.20 considers polymorphic population states.

Note that the mass-action interpretation of Nash is a polymorphic interpretation (any population state x_i is polymorphic). In particular, in this context, individuals act only in pure strategies.

Since populations are assumed to be "large", the polymorphic interpretation, via the law of large numbers, is valid when mixed strategies occur. Formally, let $\underline{T}$ be the population space and $f : T \to \Delta$ a strategy profile, then $\int_T f(t)\, d\lambda(t)$ is a polymorphic population state.

B Field game

In the previous paragraph, were considered situations where individuals of a population met in pairwise interactions as in Example 4.20. This paragraph focus on global interactions among all members of the population. Models of such situation are called play against the field.

First an example reproduced from (Hofbauer and Sigmund 1998, Chapter 6) is presented.

4.22. Example. *Field game: the "sex ratio" game.*
The sex ratio of an individual is the probability that his child be a male. We examine why the sex ratio 1/2 prevals in animal populations. Why are they so many males, whereas a female biaised sex ratio leads to a higher overall growth rate? We shall presently see that although the number of children is not affected by the sex ratio, the number of grandchildren is. Roughly speaking, if there were more females, then males would have good prospects. Since the same holds vice versa, this leads to a sex ratio of 1/2.

To check this, denote by p the sex ratio of a given individual, and by m the average sex ratio in the population. Let N_1 be the population number in the daughter generation F_1 (of which mN_1 will be male and $(1-m)N_1$ female) and N_2 the number in the following generation F_2. Each member of F_2 has one mother and one father: the probability that a given male in the F_1 generation is its father is $\frac{1}{mN_1}$, and the expected number of children produced by a male in the F_1 generation is therefore $\frac{N_2}{mN_1}$ (assuming random mating). Similarly, a female in the F_1 generation contributes an average of $\frac{N_2}{(1-m)N_1}$ children. The expected number of grandchildren of an individual with sex ratio p will be proportional to

$$p\frac{N_2}{mN_1} + (1-p)\frac{N_2}{(1-m)N_1},$$

i.e. its fitness is proportional to

$$w(p,m) = \frac{p}{m} + \frac{1-p}{1-m}.$$

(We may clearly exclude the cases $m = 0$ and $m = 1$ which lead to immediate extinction.) For given $m \in (0,1)$, the function $p \to w(p,m)$ is affine linear, inceasing for $m < 1/2$, decreasing for $m > 1/2$ and constant for $m = 1/2$.

Let us now consider a sex ratio q, and ask wether it is evolutionarily stable in the sense that no other sex ratio p can invade. If such a deviant sex ratio is introduced in a small proportion ε, the average sex ratio is $r = \varepsilon p + (1-\varepsilon)q$. The q-ratio fares better than the p-ratio if and only if

$$w(p, r) < w(q, r).$$

This is obviously the case for every p when $q = 1/2$ (it is enough to note that p and r are either both smaller, or both large then $1/2$). For $q < 1/2$, a sex ratio $p > q$ will do better, however, and consequently spread; similarly, any $q > 1/2$ can be invaded by a smaller p. Thus $1/2$ is the unique sex ratio which is evolutionarily stable. The strategy $(1/2, 1/2)$ has to be understood as the average of the sex ratio of all individuals present in the population, thus is a polymorphic state in which mixed strategies occur.

The sex-ratio game is called a *field game*.

4.23. Definition. Given a population of individuals, a *field game* is a tuple (A, U) where A is a finite action set and $U : \Delta \to \mathbb{R}^m$ is a vector valued function. U is the *evaluation function* of the field game.

A direct consequence of Definitions 4.19 and 4.23 is:
A RM game is a field game with a linear evaluation function.

We now present the key concept of Evolutionary Game Theory: the evolutionarily stable strategy (ESS) introduced by Maynard Smith and Price (1973).

C Evolutionarily stable strategy

4.24. Definition. Let (A, U) be a field game. A strategy $x \in \Delta$ is an *evolutionarily stable strategy* (ESS) of (A, U) if there exists an *invasion barrier* $\bar{\varepsilon} > 0$ such that for all $y \in \Delta$ and all $0 < \varepsilon < \bar{\varepsilon}$,

$$x \cdot U((1-\varepsilon)x + \varepsilon y) > y \cdot U((1-\varepsilon)x + \varepsilon y).$$

ESS was first defined in the context of RM-game and monomorphic population state. In this setting an ESS is a strategy played by all members of the population such that if a mutant strategy appears and is played by a small fraction of the population, the mutant strategy has the lower fitness in this situation.

Now if x is interpreted as a polymorphic population state, then the average fitness of the population state x has to be better than the fitness of any mutant population y. Note that mutations are assumed to be rare and spaced in time, therefore the strategy y is always interpreted as a monomorphic population state.

The population state $(G/C, 1-G/C)$ in Example 4.20 and the sex ratio $(1/2, 1/2)$ in Example 4.22 are ESS.

An equivalent condition for x to be an ESS of (A, U) with U continuous is

$$x \cdot U(x) \geq y \cdot U(x), \quad \forall y \in \Delta \tag{4.10}$$

$$x \cdot U(x) = y \cdot U(x),\ y \neq x \Longrightarrow x \cdot U(y) > y \cdot U(y) \tag{4.11}$$

(see, e.g., (Hofbauer and Sigmund, 1998, Section 6.4)).

Equation (4.10) is the equilibrium condition for the field game (A, U). Hence an ESS of (A, U) is an equilibrium.

4.25. Definition. Let (A, U) be a field game. A population state x is a *strict equilibrium* if for any strategy $y \neq x \in \Delta$,

$$x \cdot U(x) > y \cdot U(x).$$

If x is a strict equilibrium, it lies necessarily in A, in other words x is a pure strategy.

From the definitions, it follows that for a field game (U, A), a strict equilibrium is an ESS.

4.26. Remark. *On the ESS.*
In the original definition of the ESS (Maynard Smith and Price, 1973), the invasion barrier may depend on the mutant strategy y. However the two definitions are equivalent see, e.g., (Weibull, 1995, Proposition 2.5).

4.3.3 Stability in games with n-populations

Settings with several populations are now considered. First different definitions of evolutionary stability in n-population random matching games are given and it is shown that all these definitions are equivalent since any of them rejects all nonstrict Nash equilibria. Then a notion of stability for population games is defined which coincides with the ESS in the case of field games.

A n-population random matching games

Let $(I, (A_i), (u_i))$ be a n-player matrix game where $I = \{1, \ldots, n\}$ is the set of players, A_i is player i's action set and $u_i(a_1, \ldots, a_n)$ his payoff when the action profile is $(a_1, \ldots, a_n)$.

Consider I as a set of populations. Given a populations state $x = (x_i)$, a member of i selecting action $a \in A_i$ will have the payoff

$$U_i^a(x_{-i}) = \sum_{a_{-i} \in A_{-i}} \Pi_{j \neq i} x_j^{a_j} u_i(a, a_{-i}).$$

Note that the payoff of an individual depends on the average frequency of each action in each population, excepted the one that he belongs to.

The tuple $(I, (A_i), (U_i))$ is called a *n-population random matching game.* A formal definition is now given.

4.27. Definition. A *n-population random matching (RM) game* is a tuple $(I, (A_i), (U_i))$ where A_i is the finite action set of any member of i and $U_i^a : \Pi_{j \neq i} \Delta_j \to \mathbb{R}$ is a real-valued multilinear function which represents action a's payoff function for members of i (Δ_j is the simplex in $\mathbb{R}_{m_j}$, where m_j is the cardinality of A_j). The vector-valued function U_i is the *evaluation function* of members of i.

As for *RM* game, to any n-population *RM* game $(I, (A_i), (U_i))$ is *associated* a unique n-player matrix game $(I, (A_i), (u_i))$ (henceforth *associated n-player game*) such that

$$U_i^a(x_{-i}) = \sum_{a_{-i} \in A_{-i}} \Pi_{j \neq i} x_j^{a_j} u_i(a, a_{-i}).$$

A populations state $x = (x_i)$ is an equilibrium of $(I, (A_i), (U_i))$ if for any two actions a, b in the support of x_i, $U_i^a(x_{-i}) = U_i^b(x_{-i})$.

A n-population random matching game corresponds to the mass-action interpretation of Nash where each population is a nonatomic measure space.

4.28. Remark. *Differences between random matching game and n-population random matching game.*
(i) In a n-population random matching game $(I, (A_i), (U_i))$, two members of the same population never play together the n-player matrix game $(I, (A_i), (u_i))$, whereas in a random matching game, each play opposes two members of the same population.
(ii) Any Nash equilibrium of n-player matrix game $(I, (A_i), (u_i))$ corresponds to an equilibrium of the n-population *RM* game $(I, (A_i), (U_i))$. This is not true for symmetric two player matrix game and random matching game. Indeed in this context the previous assertion becomes: Any symmetric Nash equilibrium (x, x) of (A, u) corresponds to an equilibrium x of the *RM* game (A, U).

4.29. Definition. Let $(I, (A_i), (U_i))$ be a n-population *RM* game. A populations state $x = (x_i)$ is a *strict equilibrium* if for any j and any strategy $\tilde{x}_j \neq x_j$,

$$x_j \cdot U_j(x_{-j}) > \tilde{x}_j \cdot U_j(x_{-j}).$$

Consider a n-population random matching game $(I, (A_i), (U_i))$. It follows from Remark (i) above that for any populations state $x = (x_i)$, a mutation of a fraction of population j has no impact on the payoff of an individual of the original population state x_j. As we shall see, this implies that the evolutionary stable populations states are the strict equilibria.

If we assume that, being rare and spaced in time, two mutations in two distinct populations do not happen in a same time, a first definition of evolutionary stability is the following.

4.30. Definition. The populations state $x = (x_i)$ of the n-population random matching game $(I, (A_i), (U_i))$ is *evolutionary stable* if for any $j \in I$, there exists $\bar{\varepsilon}_j > 0$ such that for every strategy $\tilde{x}_j \neq x_j$, and $0 < \varepsilon < \bar{\varepsilon}_j$,

$$x_j \cdot U_j((1-\varepsilon)x_{-j} + \varepsilon\tilde{x}_{-j}) > \tilde{x}_j \cdot U_j((1-\varepsilon)x_{-j} + \varepsilon\tilde{x}_{-j}),$$

where $\tilde{x}$ is the populations state $(\tilde{x}_j, x_{-j})$.

Since $\tilde{x}_{-j} = x_{-j}$, this definition is exactly the definition of a strict equilibrium. So we state a weaker condition of evolutionary stability (Weibull, 1995, Definition 5.1). This definition considers familly of mutations $(\tilde{x}_i)_{i \in I}$.

4.31. Definition. The populations state $x = (x_i)$ of the n-population random matching game $(I, (A_i), (U_i))$ is *evolutionary stable* if there exists $\bar{\varepsilon} > 0$ such that for every strategy profile $\tilde{x} \neq x$ and $0 < \varepsilon < \bar{\varepsilon}$, and with $\bar{x} = (1-\varepsilon)x + \varepsilon\tilde{x}$,

$$x_j \cdot U_j(\bar{x}_{-j}) > \tilde{x}_j \cdot U_j(\bar{x}_{-j}), \quad \text{for some } j \in I.$$

In other terms, a populations state x is evolutionary stable if for any family of mutations $\tilde{x}$, present in a small fraction of the populations, at least one population state x_i has a better increase in fitness than the corresponding mutant strategy $\tilde{x}_i$ in the new situation.

Under this definition, any evolutionary stable populations state is again a strict Nash equilibrium and vice-versa (Weibull, 1995, Proposition 5.1). The first result between equivalence of ESS and strict Nash equilibria is given by Selten (1980) in random matching games where individuals condition their action according to their position in the game (the *RM* game is no more defined with a symmetric two-player matrix game).

B Population games

Recall that a population game is a tuple $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$ where I is a set of populations, A_i is the finite action set of i's members ($\#A_i = m_i$) and $U_i : \Pi_{j \in I} \Delta_j \to \mathbb{R}^{m_i}$ their evaluation function (recall that $\Delta_j = \Delta_{m_j}$).

Obviously a population game with a single population is a field game and the notion of stability given below corresponds for field games to the ESS.

4.32. Definition. Consider the population game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$. A populations state x is *stable* if there exists a positive number $\bar{\varepsilon}$ such that for all ε with $0 < \varepsilon < \bar{\varepsilon}$, for all $\tilde{x} \neq x \in \Pi_{i \in I} \Delta_i$,

$$\sum_{i \in I} x_i \cdot U_i((1-\varepsilon)x + \varepsilon\tilde{x}) > \sum_{i \in I} \tilde{x}_i \cdot U_i((1-\varepsilon)x + \varepsilon\tilde{x}).$$

Chapter 5

Equilibrium's refinement: stability

In Section 4.3 a notion of stability was defined for population games. It is now extended to $\mathcal{S}$-games. Stability in $\mathcal{S}$-games yields an equilibrium selection. A characterization of the set of stable strategy profiles is given. Then in the framework of potential games, relationships between stable states and maximizers of the potential function are studied. Finally a recursive process is proposed to select equilibrium strategy profiles and it is shown that this process lets stable strategy profiles unchanged.

5.1 $\mathcal{S}$-games

The action set A is finite of cardinality m and is identified with the standard basis of $\mathbb{R}^m$.

5.1. Definition. Given a subset T_0 of T with positive λ-measure, denote by $(f; g, T_0)$ the strategy profile defined by

$$(f; g, T_0)(t) = \begin{cases} f(t) & \text{if } t \in T \setminus T_0, \\ g(t) & \text{if } t \in T_0. \end{cases}$$

The strategy profile $(f; g, T_0)$, with $0 < \lambda(T_0) = \varepsilon$ is called an *ε-perturbation* of f. Given an ε-perturbation h of f, the set on which h and f differ is denoted by T_h.

5.2. Definition. Consider an $\mathcal{S}$-game $\mathbf{u} : T \to C(A \times \mathcal{F}_A)$. The strategy profile f is *stable* in $\mathbf{u}$ if there exists a positive number $\bar{\varepsilon}$ such that for any ε-perturbation h of f, with $0 < \varepsilon < \bar{\varepsilon}$, the following holds

$$\int_{T_h} u_t(f(t), h)\, d\lambda(t) > \int_{T_h} u_t(h(t), h)\, d\lambda(t).$$

The real $\bar{\varepsilon}$ is called the *ray of stability* of f.

Explicitely, a strategy profile f is stable if, given any strategy profile h which differs from f on a "small" set of players, the average payoff of the group of deviators (t such that $f(t) \neq h(t)$) is strictly less than the average payoff of deviators in which any deviator would have not "individually" deviated.

Assumption: We assume in this section that any $\mathcal{S}$-game $\mathbf{u}$ is such that the family $\{u_t\}$ is uniformly bounded.

We first show that a stable strategy profile is an equilibrium.

5.3 Lemma. *A stable strategy profile of an $\mathcal{S}$-game is an equilibrium.*

Proof: Consider the $\mathcal{S}$-game $\mathbf{u}$. Let f be a stable strategy profile of $\mathbf{u}$. Assume by contradiction that f is not an equilibrium. There exists a nonnull subset T_0 of T such that the players in T_0 are dissatisfied:

$$\exists a_t \in A,\ u_t(f(t), f) < u_t(a_t, f),\ \ \lambda\text{-a.e. on } T_0,$$

where a_t is a best reply for player t against f.

Let g be the strategy profile equals to f on the set $T \setminus T_0$ and which is a best reply to f on T_0, $g(t) = f(t)$ on $T \setminus T_0$ and $g(t) = a_t$ on T_0. Then

$$\int_{T_0} u_t(g(t), f)\, d\lambda(t) > \int_{T_0} u_t(f(t), f)\, d\lambda(t).$$

Now, since, for all t, u_t is continuous in its second argument and the u_t's are uniformly bounded, it follows from the Lebesque dominated convergence theorem (Aliprantis and Border, 1999, Theorem 11.20) that there exists a nonnull subset $T_1 \subset T_0$, such that

$$\int_{T_1} u_t(g(t), (f; g, T_1))\, d\lambda(t) > \int_{T_1} u_t(f(t), (f; g, T_1))\, d\lambda(t).$$

This last equation contradicts the stability of f. ∎

We now prove that stability for $\mathcal{S}$-games corresponds to evolutionary stability for field games.

For this purpose, we introduce a definition.

5.4. Definition. Let (A, U) be a field game. Define the function $u : A \times \Delta \to \mathbb{R}$ by $u(a, x) = U^a(x)$, hence $u \in C(A \times \mathcal{F}_\Delta)$. Consider the $\mathcal{S}$-game $\mathbf{u} : t \in T \to u$. Then the $\mathcal{S}$-game $\mathbf{u}$ is the *representation* of the field game (A, U).

5.5 Proposition. *Let (A, U) be a field game and the $\mathcal{S}$-game $\mathbf{u}$ be its representation. Then*
(1) If the strategy profile f is stable in $\mathbf{u}$, with ray of stability $\bar{\varepsilon}$, the image measure x of λ under f is an ESS of the field game (A, U), with invasion barrier $\bar{\varepsilon}$.
(2) If the population state x is an ESS of the field game (A, U), with invasion barrier $\bar{\varepsilon}$, there exists a strategy profile f that satisfies $x = f[\lambda]$ and is stable in $\mathbf{u}$, with ray of stability $\bar{\varepsilon}$.

Proof: (1) Assume that f is a stable strategy profile of $\mathbf{u}$, with ray of stability $\bar{\varepsilon}$ and let x be the image measure of λ under f. First, note that by definition of pure strategy profiles, $x^a = \int f^a$.

Consider a mixed strategy y, the mutant strategy. Given ε $(0 < \varepsilon < \bar{\varepsilon})$, we shall construct a perturbation g of f such that the population state $(1-\varepsilon)x + \varepsilon y$, for some $y \in \Delta$, be the image measure under g. Then we shall conclude that

$$x \cdot U((1-\varepsilon)x + \varepsilon y) > y \cdot U((1-\varepsilon)x + \varepsilon y)$$

and the first part of the proof will be completed.

So fix ε in $(0, \bar{\varepsilon})$. Due to the nonatomicity of λ, for any $a \in A$, there exists a measurable partition $\{T_f^a, \tilde{T}_f^a\}$ of the set $f^{-1}(a)$ in proportion $(1-\varepsilon), \varepsilon$ $(\lambda(T_f^a) = (1-\varepsilon)\lambda(f^{-1}(a)))$. The equality

$$\int_{\cup_{a\in A}\tilde{T}_f^a} f(t)\, d\lambda(t) = \varepsilon x$$

follows.

Invoke again the nonatomicity of λ to obtain a measurable partition $\{T_g^a\}_{a\in A}$ of the set $\cup_{a\in A}\tilde{T}_f^a$ such that $\lambda(T_g^a) = \varepsilon y^a$.

The family $\{T_f^a, T_g^a\}_{a\in A}$ is a measurable partition of T, hence the strategy profile g with

$$g(t) = \begin{cases} f(t) & \text{if } t \in \cup_{a\in A} T_f^a, \\ a & \text{if } t \in T_g^a, \end{cases}$$

is well defined.

It is easily checked that $\int g^a = (1-\varepsilon)x^a + \varepsilon y^a$, for all $a \in A$, in other words $(1-\varepsilon)x + \varepsilon y$ is the image measure of λ under g. Moreover g is an ε-perturbation of f, indeed g differs from f on the set $\cup_{a\in A}\tilde{T}_f^a$ of λ-measure ε.

Finally, it remains to show that the equation

$$\int_{T_g} u_t(f(t), g)\, d\lambda(t) > \int_{T_g} u_t(g(t), g)\, d\lambda(t)$$

implies

$$x \cdot U((1-\varepsilon)x + \varepsilon y) > y \cdot U((1-\varepsilon)x + \varepsilon y).$$

Considering the equalities

$$\int_{T_g} u_t(f(t), g)\, d\lambda(t) = \sum_{a\in A} \lambda(\tilde{T}_f^a) U^a \left(\int g\right) = \varepsilon x \cdot U((1-\varepsilon)x + \varepsilon y)$$

and

$$\int_{T_g} u_t(g(t), g)\, d\lambda(t) = \sum_{a\in A} \lambda(T_g^a) U^a \left(\int g\right) = \varepsilon y \cdot U((1-\varepsilon)x + \varepsilon y),$$

this becomes straightforward.

(2) Let x be an ESS of U, with invasion barrier $\bar{\varepsilon}$. Suppose per absurdum that there exists a strategy profile f such that (i) $x = f[\lambda]$ and (ii) there exists an ε-perturbation g of f $(0 < \varepsilon < \bar{\varepsilon})$ that satisfies

$$\int_{T_g} u_t(f(t), g)\, d\lambda(t) \leq \int_{T_g} u_t(g(t), g)\, d\lambda(t). \tag{5.1}$$

Then equation (5.1) is equivalent to

$$\begin{aligned}
& \sum_{a\in A} \lambda(f^{-1}(a)\cap T_g) U^a\left(\int g\right) \le \sum_{a\in A} \lambda(g^{-1}(a)\cap T_g) U^a\left(\int g\right) \\
\iff & \sum_{a\in A} \left[\lambda(f^{-1}(a)\cap T_g) - \lambda(g^{-1}(a)\cap T_g)\right] U^a\left(\int g\right) \le 0 \\
\iff & \sum_{a\in A} \left[f[\lambda](a) - g[\lambda](a)\right] U^a\left(\int g\right) \le 0.
\end{aligned}$$

The last equivalence is due to the coincidence of f and g on the set $T \setminus T_g$.

Note that
(i) $f[\lambda](a) = x^a$,
(ii) $g[\lambda](a) = \int_{T\setminus T_g} f^a + \int_{T_g} g^a$ and
(iii) there exists $0 \le \varepsilon^a \le \varepsilon$ such that $\int_{T\setminus T_g} f^a = (1-\varepsilon^a)\int f^a = (1-\varepsilon^a)x^a$.

Denote by v^a the value $\int_{T_g} g^a$, it lies in $[0,\varepsilon]$. Since $\sum_{a\in A}\varepsilon^a x^a = \sum_{a\in A} v^a = \varepsilon$, the vector y in $\mathbb{R}^m$ defined by

$$y^a = \frac{1}{\varepsilon}\left[(\varepsilon - \varepsilon^a)x^a + v^a\right]$$

is a population state. With this notation, (i)-(iii) yield

$$f[\lambda](a) - g[\lambda](a) = \varepsilon x^a - \varepsilon y^a.$$

Therefore (5.1) is equivalent to

$$x\cdot U((1-\varepsilon)x + \varepsilon y) \le y \cdot U((1-\varepsilon)x+\varepsilon y).$$

This last equation contradicts the assumption that x is an ESS. ∎

We now characterize stable strategy profiles of an $\mathcal{S}$-game $\mathbf{u}: T \to C(A\times\mathcal{F}_A)$. Considering a vanishing sequence of games in adapted form $\{\mathbf{u}^n\}$ which λ-converges to $\mathbf{u}$, we prove that stable strategy profiles of $\mathbf{u}$ are limits of sequences of strict Nash equilibria of the games $\mathbf{u}^n$.

5.6 Theorem. *Let $\{\mathbf{u}^n\}$ be a sequence of finite-player games λ-converging to an $\mathcal{S}$-game $\mathbf{u}$; let f be a pure stable strategy profile of $\mathbf{u}$. If the sequence $\{f^n\}$, $f^n \in \mathcal{F}_A^n$, converges λ-almost everywhere to f, then there exists N such that, for all $n \ge N$, f^n is a strict Nash equilibrium of $\mathbf{u}^n$.*

Proof: Let f be a pure stable strategy profile of $\mathbf{u}$ with a ray of stability greater or equal than $\bar{\varepsilon}$ and let $\{f^n\}$ be a sequence converging λ-almost everywhere to f.

First, invoke the finiteness of the action set A to assert that if $\{f^n\}$ converges λ-almost everywhere to f, there exists N such that for all $n \ge N$, $f^n \sim f$. Thus for all $n \ge N$, all $i \le n$ and $a \in A$, the strategy profile $f^{n(i\to a)}$ is a $\lambda(T_i^n)$-perturbation of f (which obviously differs from f on the set T_i^n).

Second, the stability of f and the Lebesque dominated convergence theorem (Aliprantis and Border, 1999, Theorem 11.20) (for all t, u_t is continuous in its second argument and the u_t's are uniformly bounded) imply that for any ε-perturbation g of f ($\varepsilon < \bar{\varepsilon}$), there exists a neighborhood U_g of g such that for all h, $k \in U_g$,

$$\int_{T_g} u_t(f(t), h)\, d\lambda(t) > \int_{T_g} u_t(g(t), k)\, d\lambda(t).$$

Third, the λ-almost everywhere convergence of $\{f^n\}$ induces that for some N_1 and all $n \geq N_1$, f^n belongs to $U_{f^{n(i \to a)}}$, for any $i \leq n$ and $a \in A$.

Combining these three facts leads to

$$\int_{T_i^n} u_t(f^n(t), f^n)\, d\lambda(t) > \int_{T_i^n} u_t(a, f^{n(i \to a)})\, d\lambda(t),$$

for all $n \geq N_1$.

It remains to apply the λ-convergence of $\{\mathbf{u}^n\}$ to conclude that there exists some $N_2 \geq N_1$ such that for all $n_2 \geq N_2$ and all $n \geq N_1$,

$$\int_{T_i^n} u_t^{n_2}(f^n(t), f^n)\, d\lambda(t) > \int_{T_i^n} u_t^{n_2}(a, f^{n(i \to a)})\, d\lambda(t).$$

Thus, for all $n \geq N_2$,

$$u_i^n(f_i^n, f^n) > u_i^n(a, f^{n(i \to a)}),$$

namely the strategy profile f^n is a strict Nash equilibrium of $\mathbf{u}^n$. ∎

5.2 Potential games

A populations state x of the population game $(I, (A_i)_{i \in I}, (U_i)_{i \in I})$ is *stable* if there exists a positive number $\bar{\varepsilon}$ such that for all ε with $0 < \varepsilon < \bar{\varepsilon}$, for all $\tilde{x} \in \Pi_{i \in I} \Delta_i$ and all $i \in I$,

$$\sum_{i \in I} x_i \cdot U_i((1-\varepsilon)x + \varepsilon \tilde{x}) > \sum_{i \in I} \tilde{x}_i \cdot U_i((1-\varepsilon)x + \varepsilon \tilde{x}). \tag{5.2}$$

First note that, even in potential games, a stable populations state does not necessarily exist. Indeed, consider a population game which admits as potential function, a concave, but not strictly concave, function P such that the set of maximizers of P, $argmax\{P\}$, be included in $\Pi_{i \in I} \Delta_i$. Then, if x is a maximizer of P and if a small part of players changes their action in a way that the populations state still lies in $argmax\{P\}$, then any deviator will have the same payoff that he would have had if had not deviate. This is done in the next example.

5.7. Example. *A potential game with no stable population state.*
Consider the population game with a unique population whose action set is $A = \{1,2\}$ and whose evaluation function is defined by

$$U^1(x) = \begin{cases} -2(x-\frac{1}{4}) & \text{if } x \le \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} \le x \le \frac{3}{4}, \\ -2(x-\frac{3}{4}) & \text{if } \frac{3}{4} \le x, \end{cases} \quad \text{and } U^2 = U^1.$$

U^1 (resp. U^2) takes as argument the fraction of the population who chooses action 1 (resp. 2).

This population game admits the potential function:

$$P(x) = \begin{cases} -(x-\frac{1}{4})^2 & \text{if } x \le \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} \le x \le \frac{3}{4}, \\ (x-\frac{3}{4})^2 & \text{if } \frac{3}{4} \le x, \end{cases}$$

where $x \in [0,1]$ is the fraction of the population who chooses action 1.

Let $x = 1/2$, x is an equilibrium and consider the new state $x_\varepsilon = 1/2 - \varepsilon$ where 2ε of the players who had chosen 1 in x chooses now 2. For any $\varepsilon \le 1/4$, the payoff of any deviator is the same that the one he would have had if had not deviate. Hence x is not stable.

Remark that, given a tuple $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$, we can define either a population game where each population has measure $1/\#I$ or an anonymous $n(\#I)$-player game where there are n players of each population. So, call $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ a *population structure*. Given a population structure $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$, T denotes the population game and $n(\#I)$ the anonymous $n(\#I)$-player game defined over this population structure.

The converse statement of Theorem 5.6 is not true as shown by the following examples.

5.8. Example. *Stable populations states and sequences of (strict) Nash equilibria.*
(A) *Anonymous games: the weak limit of a sequence of Nash equilibria which is not stable.*
Consider the random matching game:

$0,0$	$2,2$	$0,1$
$2,2$	$0,0$	$0,1$
$1,0$	$1,0$	$0,0$

If the set of players is nonatomic, then the set of equilibrum states is $\{(x,y,z) \in \mathbb{R}^3_+ : x = y,\ x+y+z = 1\}$. Moreover, if the set of players is finite, the population state $(0,0,1)$ is a Nash equilibrium. Thus $(0,0,1)$ is the limit of the sequence of Nash equilibria $(0,0,1)$, but it is easy to see that it is not stable.

(B) *E-anonymous game: stable populations states and sequences of strict Nash equilibria.*
Consider a population structure with three actions denoted a, b and c and a single population: all players have same payoff functions, denoted U^a, U^b and U^c and

defined by

$$U^a(x) = (x^a)^2 + x^b, \quad U^b(x) = x^a + (x^b)^2, \quad U^c(x) = x^c.$$

(i) *The weak limit of a sequence of strict Nash equilibria which is not stable.* The population state $(1, 0, 0)$ is a strict Nash equilibrium for all games with a finite set of players. If the set of players is nonatomic, it is also an equilibrium (not strict: actions a and b lead to the same payoff) but it is not stable: In fact, if a group of players of size $\varepsilon > 0$ deviates from a to b, then the payoff of action a is $(1-\varepsilon)^2+\varepsilon$ which is equal to $1-\varepsilon+\varepsilon^2$, the payoff of action b.
(ii) *A stable population state limit of a sequence of strict Nash equililibria.* The population state $(0, 0, 1)$ is stable for the nonatomic game and is a strict Nash equilibrium for any finite-player game.
(iii) *An unstable equilibrium state which is not the limit of a sequence of Nash equililibria.* The population state $(1/2, 1/2, 0)$ is an equilibrium of the nonatomic game, actions a and b lead to a payoff of $3/4$, this state is not stable, as previously, if a group of players of size $\varepsilon > 0$ deviates from a to b, the payoff of action a will be the same than the payoff of action b. Assume that there is a even number of players, N, then the population state $(1/2, 1/2, 0)$ is not a Nash equilibrium: In fact, if a player deviates from a to b his payoff becomes $1/2 - 1/N + (1/2 + 1/N)^2 = 3/4 + 1/N^2$ strictly greater than $3/4$, the previous payoff.

If we consider Example (B) for p-anonymous finite-player games, the population state
(i) $(1, 0, 0)$ is not a strict Nash equilibrium;
(ii) $(0, 0, 1)$ is a strict Nash equilibrium;
(iii) $(1/2, 1/2, 0)$ is a Nash equilibrium if the number of players n is even.

We link stable populations states of potential games and local maximizers of their potential function. Let $m = \sum_{i\in I} m_i$.

5.9 Proposition. *If the population game $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ admits a potential function $P : \mathbb{R}^m \to \mathbb{R}$, then any stable populations state $x \in \Pi_{i\in I}\Delta_i$ of $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ is a strict local maximizer of P.*

Proof: Let x be a stable populations state of $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$. There exists $\bar{\varepsilon} > 0$ such that

$$\forall 0 < \varepsilon < \bar{\varepsilon}, \quad \nabla P((1-\varepsilon)x + \varepsilon y) \cdot \varepsilon(x - y) > 0. \tag{5.3}$$

The Taylor expansion of the potential function P at the point $(1-\varepsilon)x+\varepsilon y$ gives

$$P(x) = P((1-\varepsilon)x + \varepsilon y) + [\nabla P((1-\varepsilon)x + \varepsilon y) \cdot \varepsilon(x - y)] + \circ(\varepsilon).$$

Choose $0 < \underline{\varepsilon} < \bar{\varepsilon}$ sufficiently small in order that $\circ(\underline{\varepsilon})$ be negligeable relative to $[\nabla P((1-\underline{\varepsilon})x + \underline{\varepsilon}y) \cdot \underline{\varepsilon}(x - y)]$. Due to condition (5.3), such an $\underline{\varepsilon}$ exists.

Invoke again condition (5.3) to conclude that for all $0 < \varepsilon < \underline{\varepsilon}$

$$P(x) > P((1-\varepsilon)x + \varepsilon y).$$

Hence x is a strict local maximizer of P. ∎

5.10 Proposition. *Let $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ be a population game with potential function $P : \mathbb{R}^m \to \mathbb{R}$. If (i) the populations state x is a local maximizer of P and (ii) P is stricly concave on a neighborhood of x, then x is a stable populations state of $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$.*

Proof: Let x be a local maximizer of P and W be a neighborhood on which P is strictly concave. Consider the set $V := W \cap \Pi_{i\in I}\Delta_i$. The strict concavity of P implies

$$P(\underline{x}) < P(\overline{x}) + [\nabla P(\overline{x}) \cdot (\underline{x} - \overline{x})], \quad \forall \underline{x}, \overline{x} \in V.$$

Let $\bar{\varepsilon} > 0$ such that for all $0 < \varepsilon < \bar{\varepsilon}$, the populations state $(1-\varepsilon)x + \varepsilon y$ belongs to V for any $y \in \Pi_{i\in I}\Delta_i$. Substitute in the previous equation x for $\underline{x}$ and $(1-\varepsilon)x+\varepsilon y$ for $\overline{x}$, it follows

$$P(x) < P((1-\varepsilon)x + \varepsilon y) + [\nabla P((1-\varepsilon)x + \varepsilon y) \cdot \varepsilon(x - y)]. \tag{5.4}$$

Since x maximizes P on V, we have $P(x) \geq P((1-\varepsilon)x + \varepsilon y)$, for all $0 < \varepsilon < \bar{\varepsilon}$. Thus (5.4) yields $\nabla P((1-\varepsilon)x + \varepsilon y) \cdot \varepsilon(x - y) > 0$. Or equivalently

$$\forall 0 < \varepsilon < \bar{\varepsilon}, \quad \sum_{i\in I} x_i \cdot U_i((1-\varepsilon)x + \varepsilon y) > \sum_{i\in I} y_i \cdot U_i((1-\varepsilon)x + \varepsilon y).$$

This proves that x is stable. ∎

5.11 Corollary. *If the population game $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ admits a strictly concave potential function, then the unique equilibrium state of $(I, (A_i)_{i\in I}, (U_i)_{i\in I})$ is stable.*

5.3 Stability based on a recursive process

Given a strategy profile f and a pertubation g_0 of f, we construct a sequence of perturbations $(g_1, g_2, \ldots)$ such that for all $p \geq 1$,
(1) g_p is a perturbation of f which takes the values of the initial perturbation g_0 on T_p, and
(2) $T_{p+1} \subset T_p$.
This constrution allows to define a new notion of stability: f is *stable against* g_0 if $\lambda(\cap_{p\geq 1} T_p) = 0$. We show that this notion is strictly weaker than stability.

Let f be a strategy profile, and g_0 be an ε-perturbation of f, let $T_0 = T_{g_0}$. Let T_1 be the set of players in T_0 which are satisfied by their deviation:

$$T_1 := \{t \in T_0; u_t(g_0(t), g_0) \geq u_t(f(t), g_0)\}.$$

Consider the new perturbation g_1 of f where the deviators are in T_1 defined by

$$g_1(t) = \begin{cases} f(t) & \text{if } t \in T \setminus T_1, \\ g_0(t) & \text{if } t \in T_1. \end{cases}$$

Do the same procedure again to obtain a set

$$T_2 := \{t \in T_1; u_t(g_0(t), g_1) \geq u_t(f(t), g_1)\}$$

and a new perturbation g_2 of f defined by

$$g_2(t) = \begin{cases} f(t) & \text{if } t \in T \setminus T_2, \\ g_0(t) & \text{if } t \in T_2. \end{cases}$$

Repeat the procedure until the rank p if $T_p = T_{p-1}$ or $\lambda(T_p) = 0$, othewise the procedure never stops. The strategy profile f is said *stable against* g if either $T_p = 0$ for some p or $\lambda(\cap_{p\leq 1} T_1) = 0$.

5.12 Lemma. *Let* $\mathbf{u}$ *be an* $\mathcal{S}$*-game. A stable strategy profile* f *of the* $\mathbf{u}$*, with ray of stability* $\bar{\varepsilon}$ *is stable against any* ε*-perturbation* g *of* f $(0 < \varepsilon < \bar{\varepsilon})$*.*

The proof is straighforward and therefore omitted.

The converse statement of this lemma is not true as established by the following example.

5.13. Example. *Stability against any perturbation does not imply stability.* Consider the $\mathcal{S}$-game $\mathbf{u}$ defined as follows. The action set is made of two actions, $A = \{a, b\}$ and the payoff functions of player t are

$$U_t^a(f) = \begin{cases} 0 & \text{if } t = 0, \\ 2\int_{B_t(t)} f^a(s)\, d\lambda(s) & \text{if } t \in (0, 1/2], \\ 2\int_{B_{1-t}(t)} f^a(s)\, d\lambda(s) & \text{if } t \in [1/2, 1), \\ 0 & \text{if } t = 1. \end{cases}$$

and

$$U_t^b(f) = \begin{cases} 0 & \text{if } t = 0, \\ \int_{B_t(t)} f^b(s)\, d\lambda(s) & \text{if } t \in (0, 1/2], \\ \int_{B_{1-t}(t)} f^b(s)\, d\lambda(s) & \text{if } t \in [1/2, 1), \\ 0 & \text{if } t = 1. \end{cases}$$

So in $\mathbf{u}$, the players prefer to act as their neighbours, and the more a player is far from the center of the population ($1/2 \in [0, 1]$), the less his neighborhood is large.

Consider the constant strategy profile $f \equiv a$ which is an equilibrium of $\mathbf{u}$, since for any player t ($t \neq 0, 1$) all his neighbours act as him. We shall show that (i) f is stable against any ε-perturbation with $0 < \varepsilon < 1/2$ and (ii) f is not stable.

(i) For any ε, $0 < \varepsilon < 1/2$, consider the ε-perturbation g_ε of f defined by

$$g_\varepsilon(t) = \begin{cases} f(t) & \text{if } t \in [0, 1-\varepsilon), \\ b & \text{if } t \in [1-\varepsilon, 1] =: T_\varepsilon. \end{cases}$$

Then the players in $T_{(3/4)\varepsilon} := [1-(3/4)\varepsilon, 1]$ are satisfied by their deviations, whereas the others are not. Indeed the player $t_0 = 1-(3/4)\varepsilon$, has for neighborhood the open interval $(1-(3/2)\varepsilon, 1)$ where $1/3$ of the players play a and $2/3$ play b, then $U_{t_0}^a(g_\varepsilon) = U_{t_0}^b(g_\varepsilon)$. A player t at the right of t_0 ($t > 1-(3/4)\varepsilon$) will have a neighborhood where more than 2/3 of the players select action b, then $U_t^a(g_\varepsilon) < U_t^b(g_\varepsilon)$. Conversely a player t at the left of t_0 will have a neighborhood where more than 1/3 of the players select action a, then $U_t^a(g_\varepsilon) > U_t^b(g_\varepsilon)$. It follows that f is stable against any perturbation g_ε with $0 < \varepsilon < 1/2$.

Now, observe that any ε-perturbation g of f cannot do better than g_ε (better in the sense that there is more deviator players satisfied in g than in g_ε). Thus f is stable against any ε-perturbation ($0 < \varepsilon < 1/2$).

(ii) For any ε, $0 < \varepsilon < 1/2$, consider the perturbation g_ε defined above. We shall show that the equation defining stability is not satisfied by the perturbation g_ε, that is

$$\int_{T_\varepsilon} u_t(f(t), g_\varepsilon)\, d\lambda(t) \leq \int_{T_\varepsilon} u_t(g_\varepsilon(t), g_\varepsilon)\, d\lambda(t), \tag{5.5}$$

where T_ε is defined as previously.

Recall that the players in $[0, 1-\varepsilon)$ select action a and those in $[1-\varepsilon, 1]$ action b. Thus, for a player $t \geq 1 - \varepsilon/2$, all his neighbours select action b and for a player $t < 1 - \varepsilon/2$, his neighbours in $(2t-1, 1-\varepsilon)$ select action a and those in $[1-\varepsilon, 1]$ action b. This yields

$$\begin{aligned}
\int_{T_\varepsilon} u_t(f(t), g_\varepsilon)\, d\lambda(t) &= \int_{T_\varepsilon} 2 \int_{B_{1-t}(t)} g_\varepsilon^a(s)\, d\lambda(s)\, d\lambda(t) \\
&= 2 \int_{[1-\varepsilon, 1-\frac{\varepsilon}{2}]} 2 - \varepsilon - 2t\, d\lambda(t) \\
&= 2\left((2-\varepsilon)\frac{\varepsilon}{2} - (1-\frac{\varepsilon}{2})^2 + (1-\varepsilon)^2\right) \\
&= -\frac{\varepsilon^2}{4} < 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{T_\varepsilon} u_t(g_\varepsilon(t), g_\varepsilon)\, d\lambda(t) &= \int_{T_\varepsilon} \int_{B_{1-t}(t)} g_\varepsilon^b(s)\, d\lambda(s)\, d\lambda(t) \\
&= \int_{[1-\varepsilon, 1-\frac{\varepsilon}{2}]} \varepsilon\, d\lambda(t) + \int_{[1-\frac{\varepsilon}{2}, 1]} 2 - 2t\, d\lambda(t) \\
&= \frac{\varepsilon^2}{2} + 2\frac{\varepsilon}{2} - \left(1 - (1-\frac{\varepsilon}{2})^2\right) \\
&= \frac{3}{4}\varepsilon^2 > 0.
\end{aligned}$$

Therefore equation (5.5) is established and f is not stable.

Appendix

A Referenced theorems

As usual, T is the interval $[0, 1]$, endowed with Lebesgue measure λ.

We first present Fan-Glicksberg fixed point theorem.

A.1 Theorem. *Fan-Glicksberg fixed-point theorem*
Let C be a nonempty convex and compact set in a locally convex topological vector space; if Γ is an upper semicontinuous correspondence from C to C and if, for all $x \in C$, Γx is convex and nonempty, there exists $x^* \in C$ such that

$$x^* \in \Gamma x^*.$$

Proof: e.g. (Berge, 1965, Ch 9.5). ∎

The following propositions are used to establish the weak compactness of the set of strategy profiles $\mathcal{F}$.

A.2 Theorem. *Dunford-Pettis*
A subset K of $L_1(\lambda)$ has a weakly compact closure if and only if

(i) $\sup_{f \in K} \int_T \mid f(t) \mid \, d\lambda < \infty$, *and*

(ii) given $\epsilon > 0$ there is a $\delta > 0$ such that if $\lambda(A) \leq \delta$, then $\int_A \mid f \mid \, d\lambda \leq \epsilon$ for all $f \in K$.

Proof: e.g. (Diestel, 1984, p. 93). ∎

A.3 Theorem. *Mazur's theorem*
If K is a convex subset of the normed vector space X, then the closure of K in the norm topology coincides with the weak closure of K.

Proof: (Diestel, 1984, Theorem 1, p. 11). ∎

We now present a characterization of upper semicontinuous correspondence in terms of closedness. Recall first that a correspondence is closed if its graph is closed.

A.4 Proposition. *Let X and Y be topological spaces. If Y is compact, then the correspondence $\Gamma : X \to Y$ is upper semicontinuous if an only if it is closed.*

Proof:(Berge, 1965, Ch VI). ∎

The following theorems deal with weak topology, the first assert equivalence between weak compactness and weak sequential compactness, the second allows to extract a convergent sequence from any converging net.

A.5 Theorem. *Eberlein-Smulien*
Let A be a subset of a Banach space X. Then the following statements are equivalent:
(i) A is relatively weakly sequentially compact;
(ii) every countably infinite subset of A has a weak limit point in X;
(iii) A is relatively weakly compact

Proof: (Dunford and Schwart, 1988, V.6.1). ∎

A.6 Theorem. *Suppose that E is a vector space with a vector topology which is metrizable and that A is a subset of E with the property that every sequence of points of A has a weak closure point in E. Then any point of the weak closure of A is the weak limit of a sequence of points of A.*

Proof: (Kelley and Namioka, 1963, Problem 17L, p. 165). ∎

We give different theorems established by Aumann on convexity and integrals. We begin with some definitions.

A.7. Definition. A *limit point* of a sequence of sets $X_1, X_2, \ldots$ is a point x such that there exists a subsequence of points $x_{k_n} \in X_{k_n}$ with $\lim_n x_{k_n} = x$. The set of all limit points is denoted $\limsup_n X_n$.

A.8. Definition. A correspondence F from T to a normed space is bounded by a real-valued function h if for all measurable selection f of F ($f(t) \in F(t)$), we have $\|f(t)\| \leq h$.

A.9 Theorem. *(Aumann, 1965, Proposition 4.1)*
If F_1, F_2, $\ldots$ is a sequence of correspondences from $T = [0,1]$ to an euclidean n-space that are all bounded by the same integrable real-valued function h, then

$$\int \limsup F_k \supset \limsup \int F_k.$$

The following theorem use Lyapounov's theorem:

A.10 Theorem. *(Aumann, 1965, Theorem 1)*
Let F be a correspondence from T to an euclidean n-space. Then $\int_T F \, d\lambda$ is convex.

A.11 Theorem. *(Aumann, 1965, Theorem 3)*
Let F be correspondence from T to an euclidean n-space E^n. If F is nonnegative and if its graph $\{(t,x) : x \in F(t)\}$ is a Borel subset of $T \times E^n$, then $\int_T F(t)\, dt = \int_T conv(F(t))\, dt$.

A.12 Theorem. *(Aumann, 1965, Proposition 6.1)*
Let A be a convex compact subset of E^n and B the set of its extreme points. Then the set of extreme points of $\mathcal{F}_A$ is $\mathcal{F}_B$.

The following theorem of Aumann proves that under appropriate boundedness conditions, the upper semicontinuity of a correspondence $F_p(t)$ in a parameter p is preserved by integration.

A.13 Theorem. *Let F be a correspondence from T to E^n. Let $\{x_k\}$ be a sequence of measurable functions from T to E^n, all of which are bounded by a same integrable real-valued function. Assume that for each t, each limit point of $\{x_k(t)\}$ belongs to $F(t)$. Then each limit point of $\int x_k$ belongs to $\int F$.*

Proof: (Aumann, 1976). ∎

A representation theorem frequently used, when we deal with measures is

A.14 Theorem. *Skorohod's theorem*
Let M be a complete separable metric space and (μ_n) a weakly converging sequence of measures on M with limit μ. Then there exist measurable mappings f and f_n $(n = 1, \ldots)$ of T into M such that $\mu = f[\lambda]$, $\mu_n = f_n[\lambda]$ and $\lim_n f_n = f$, λ-a.e.

Proof: e.g. (Billingsley, 1986, Theorem 29.6). ∎

We finally state two theorems needed in Section 2.

A.15 Theorem. *Egoroff's theorem*
Let (Ω, Σ, μ) be a measure space, and let f and $f_1, f_2, \ldots$ be Σ-measurable functions with values on a Banach space X. If μ is finite and if $\{f_n\}$ converges to f μ-almost everywhere, then for each positive number ϵ there is a subset B of Ω that belongs to Σ, satisfies $\mu(\Omega \backslash B) < \epsilon$, and is such that $\{f_n\}$ converges to f uniformly on B.

Proof: (Dunford and Schwartz, 1988, III.6.12). ∎

To state the last theorem, we introduce a definition.

A.16. Definition. A topological space $(X, \mathcal{X})$ is normal if for every two closed nonempty disjoint subsets can be separated by open sets.

A.17 Theorem. *Arzelà-Ascoli*
If S is a normal compact linear space, then a set in $C(S)$ is relatively compact if and only if it is bounded and equicontinuous.

Proof: (Dunford and Schwartz, 1988, IV.6.7). ∎

B Measurability conditions

We prove in this appendix that the measurability condition of Schmeidler is strictly less restrictive that the measurability of the payoff map.

B.1. Definition. Let $C[0,1]$ be the set of real-valued continuous functions on $[0,1]$. For points $x_1, \ldots, x_n$ in $[0,1]$, let $\pi_{x_1,\ldots,x_n}$ be the mapping that carries the point f of $C[0,1]$ to the point $(f(x_1), \ldots, f(x_n))$ of $\mathbb{R}^n$. $\pi_{x_1,\ldots,x_n}$ is the *projection function* on $x_1, \ldots, x_n$.

We first present a result on equivalent measurability conditions.

B.2 Proposition. *(Billingsley, 1968, p. 57)*
Let $(I, \mathcal{I}, \nu)$ be a measure space and $\mathcal{C}$ be the set of Borel subsets of $C[0,1]$ (endowed with the supremum norm). A function $h : I \to C[0,1]$ is measurable if and only if, for every $0 \leq x \leq 1$, the real-valued function $h(x) : i \mapsto h(i)(x) =: h(i,x)$ is measurable.

Proof: Suppose that h is measurable, that is $h^{-1}\mathcal{C} \subset \mathcal{I}$. For all $x \in [0,1]$ and for all $a \in \mathbb{R}$, the set $\{f \in C[0,1] : f(x) \leq a\} = \pi_x^{-1}\{f(x) \leq a\}$ is measurable (the projection from $C[0,1]$ to $\mathbb{R}$ is continuous). Since

$$\{i \in I : h(i,x) \leq a\} := h^{-1}(\{f \in C[0,1] : f(x) \leq a\}),$$

the measurability of $\{i \in I : h(i,x) \leq a\}$ follows from the measurability of h. Hence for all $x \in [0,1]$, the function $h(x)$ is measurable with respect to $\mathcal{I}$ and $\mathcal{B}(\mathbb{R})$.

Suppose now that for all $x \in [0,1]$, the function $h(x)$ is measurable with respect to $\mathcal{I}$ and $\mathcal{B}(\mathbb{R})$. Let B be the closed sphere in $C[0,1]$ with center f and radius δ, then

$$h^{-1}(B) = \{i \in I : h(i) \in B\} = \cap_{r \in \mathbb{Q}: 0 \leq r \leq 1}\{i \in I : f(r) - \delta \leq h(i,r) \leq f(r) + \delta\},$$

hence $h^{-1}(B)$ is $\mathcal{I}$-measurable. But $C[0,1]$ is separable and each open set of a separable space is a countable union of open spheres and hence of closed spheres (Dunford and Schwartz, 1988, Proof of I.6.12). Thus $h^{-1}\mathcal{C} \subset \mathcal{I}$ ∎

The previous lemma holds substituting X for $[0,1]$, for any separable metric space X.

Hence given an $\mathcal{S}$-game $(\underline{T}, \mathcal{F}_\Delta, (u_t)_{t \in T})$ such that for all $t \in T$, for all $a \in A$, $u_t(a, \cdot) : \Delta \to \mathbb{R}$ [1], the measurability of the payoff map $\mathbf{u}$ which associates to a player its payoff function is equivalent to the measurability of the functions $t \mapsto u_t(a,x)$ for all $a \in A$ and $x \in \Delta$.

If the measurability of $t \mapsto u_t(a,x)$ for all $a \in A$ and $x \in \Delta$ implies condition (b) in Section 1.2, the converse is not true a shown by the following lemma.

[1] The payoff function of any player depends on the strategy profile $f \in \mathcal{F}_\Delta$ only through $\int_T f(t)\, d\lambda(t)$.

B.3 Lemma. *The condition*
(b) for all $x \in \Delta$ and $a, b \in A$, the set $\{t \in T; u_t(a,x) > u_t(b,x)\}$ is measurable does not imply that
for all $a \in A$, $x \in \Delta$, the function $t \mapsto u_t(a,x)$ is measurable.

Proof: Define a relation $\sim$ on $\mathbb{R}$ by letting $x \sim y$ hold if and only if $x-y$ is a rational number. $\sim$ is an equivalence relation. Each equivalence class under $\sim$ has the form $\mathbb{Q}+x$ for some x, and so is dense in $\mathbb{R}$. Since these equivalence classes are disjoint, and since each intersects the interval $(0,1)$, we can use the axiom of choice to form a subset E of $(0,1)$ that contains exactly one element from each equivalence class. E is not Lebesgue measurable (see, for example, (Cohn, 1980, Theorem 1.4.7)).

Now select the u_t's such that
(i) condition (b) is satisfied.
(ii) for $x \in \Delta$, there exists an action $a \in A$ which is strictly dominated: for all $b \in A$, $u_t(a,x) < u_t(b,x)$, and
(iii) for all $t \in E$, $_t(a,x) < c$ and for $t' \in T \setminus E$, $u_{t'}(a,x) \geq c$.

(ii) and (iii) do not violate (i), however, the set $\{t \in T : u_t(a,x) < c\} = E$ is not measurable. Hence the function which maps $t \in T$ to $u_t(a,x)$ is not measurable. ∎

C Proof of Proposition 2.24

Due to assertion (2) of Proposition 2.22, f is a strategy profile of the $\mathcal{S}$-game $\mathbf{u}$. We shall prove by contradiction that f is an equilibrium of $\mathbf{u}$.

Denote by ν the image measure of λ under f. Assume that f is not in equilibrium, then there exists a subset T_0 of T with positive measure such that

$$\forall t \in T_0, \exists s_t \in S,\ u_t(f(t),\nu) < u_t(s_t,\nu) - \delta,\ \text{for some } \delta > 0. \tag{C.1}$$

We first claim that there exists a subset of T with positive λ-measure for which s_t and δ_t in the previous equation are independent of t. To see this, note that S is separable (being a compact metric space) and that C_S is also separable. Since u_t is continuous, it follows that there exist countable subsets $\mathcal{S} \subset S$ and $\mathcal{C} \subset C_S$ such that for all $t \in T_0$, there exist $\bar{s}_t \in \mathcal{S}$ and $\bar{u}_t \in \mathcal{C}$ that satisfy

$$\exists \bar{\delta}_t > 0, \quad \bar{u}_t(s_t,\nu) - \bar{u}_t(\bar{s}_t,\nu) > \bar{\delta}_t. \tag{C.2}$$

Now, the nonatomicity of T_0 and the countability of $\mathcal{S}$ and $\mathcal{C}$ imply the existence of a couple $(\bar{s},\bar{u})$ for which (C.2) holds for some $t = t_0$ and such that for all $\delta, \varepsilon > 0$, the set

$$T_0(\bar{s},\delta,\bar{u},\varepsilon) = \{t \in T_0; \|f(t) - \bar{s}\| < \delta,\ \|u_t - \bar{u}\|_\infty < \varepsilon\}$$

has positive λ-measure.

Denote by s_1 (resp. $\bar{x}$) a strategy belonging to $argmax\{u_{t_1}(\cdot,\nu)\}$ (resp. $argmax\{\bar{u}(\cdot,\nu)\}$) which exists by continuity of u_{t_1} ($\forall t_1 \in T$) (resp. $\bar{u}$), and consider for all $t_1 \in$

$T_0(\bar{s}, \delta, \bar{u}, \varepsilon)$ the inequality

$$\begin{aligned} u_{t_1}(\bar{x}, \nu) - u_{t_1}(f(t_1), \nu) \geq\ & \bar{u}(\bar{x}, \nu) - \bar{u}(\bar{s}, \nu) \\ & -|\bar{u}(\bar{x}, \nu) - u_{t_1}(\bar{x}, \nu)| \\ & -|\bar{u}(\bar{s}, \nu) - \bar{u}(f(t_1), \nu)| \\ & -|\bar{u}(f(t_1), \nu) - u_{t_1}(f(t_1), \nu)|. \end{aligned} \tag{C.3}$$

By continuity of $\bar{u}$, for any $\varepsilon > 0$, we can choose $\delta > 0$ such that for all $t_1 \in T_0(\bar{s}, \delta, \bar{u}, \varepsilon)$, $|\bar{u}(\bar{s}, \nu) - \bar{u}(f(t_1), \nu)| < \varepsilon$. Moreover

$$|u_{t_1}(s_1, \nu) - \bar{u}(\bar{x}, \nu)| \leq \|u_{t_1} - \bar{u}\|_\infty < 2\varepsilon.$$

and there exists $\bar{\delta} > 0$ such that

$$\bar{u}(\bar{x}, \nu) - \bar{u}(\bar{s}, \nu) > \bar{\delta}.$$

Then equation (C.3) yields

$$u_{t_1}(\bar{x}, \nu) - u_{t_1}(f(t_1), \nu) > \bar{\delta} - 5\varepsilon.$$

Therefore the claim is established: there exists a subset $T_0(\bar{s}, \delta, \bar{u}, \varepsilon)$ of T with positive λ-measure such that

$$\exists \bar{\delta} > 0, \bar{x} \in S,\ \forall t \in T_0(\bar{s}, \delta, \bar{u}, \varepsilon), \quad u_t(\bar{x}, \nu) - u_t(f(t), \nu) > \bar{\delta}. \tag{C.4}$$

So without loss of generality assume that $T_0 = T_0(\bar{s}, \delta, \bar{u}, \varepsilon)$.

For all $t \in T_0$, by continuity of u_t, there exist open sets $V_t \in S$ and $W_t \in \mathcal{M}(S)$, such that for all $s' \in V_t$ and $m \in W_t$,

$$u_t(s, m) - u_t(s', m) > \delta, \text{ for some } \delta > 0.$$

Let $\Gamma = \cup_{t \in T_0} V_t$. Since S is separable (being a compact metric space), it follows from Lindelöf's theorem (Dunford and Schwartz, 1988, I.4.14) that there is a countable subset $Y \subset T$ such that $\{V_t; t \in Y\}$ is a cover of Γ. Since $\lambda(T_0) > 0$ and $T_0 \subset \{t; f(t) \in V_t\}$ and since Y is countable, there is some $\bar{t} \in Y$ such that $\lambda(\{t; f(t) \in V_{\bar{t}}\}) > 0$.

Define Q, $\tilde{G}^n = G^n \circ \alpha^n$ and $\tilde{f}^n = s^n \circ \alpha^n$ as in Proposition 2.22. For $n \in Q \subset \mathbb{N}$, the image measure of λ under $\tilde{f}^n$ is denoted ν^n and the one of the strategy profile $\tilde{f}^n$ where all players in the set $(\alpha^n)^{-1}(\alpha^n(t))$ deviate from $\tilde{f}^n$ to a is denoted ν^n_t.

By assumption $\{\nu^n\}_{n \in Q}$ weakly converges to ν so as $\{\nu^n_t\}_{n \in Q}$ (by definition of α^n) and $\{\tilde{f}^n\}_{n \in Q}$ converges λ-almost everywhere to f. Thus, for n enough large, ν^n and ν^n_t belong to $W_{\bar{t}}$ and λ-almost everywhere in $\{t; f(t) \in V_{\bar{t}}\}$ the strategy profile $\tilde{f}^n(t)$ belongs to $V_{\bar{t}}$. Hence there exists $N_1 \in Q$ such that for all $n \in Q$, $n \geq N_1$ and λ-almost everywhere in $\{t; f(t) \in V_{\bar{t}}\}$,

$$u_t(s, \nu^n_t) - u_t(\tilde{f}^n(t), \nu^n) > \delta.$$

Note that if $\tilde{f}^n(t) \in V_{\bar{t}}$, then $(\alpha^n)^{-1}(\alpha^n(t)) \subset \{t; f(t) \in V_{\bar{t}}\}$.

Now $\{\tilde{G}^n\}_{n\in Q}$ converges λ-almost everywhere to $\mathbf{u}$, thus there exists $N_2 \geq N_1$ such that for all $n \in Q$, $n \geq N_2$ and λ-almost everywhere in $\{t; f(t) \in V_{\bar{t}}\}$

$$\tilde{G}^n_t(s, \nu^n_t) - \tilde{G}^n_t(\tilde{f}^n(t), \nu^n) > \delta, \text{ for some } \delta > 0.$$

Finally, for all $n \in Q$, we have the equalities $\tilde{G}^n = G^n \circ \alpha^n$ and $\tilde{f}^n = s^n \circ \alpha^n$, hence for all $n \in Q$, $n \geq N_2$, there exists $t^n \in T^n$ such that ν^n_t is the image measure of λ^n under s^n where player t^n deviates from $s^n(t)$ to a and

$$G^n_{t^n}(s, \nu^n_t) - G^n_{t^n}(s^n(t^n), \nu^n) > \delta,$$

which contradicts the fact that s^n is a Nash equilibrium of G^n. ∎

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Glossary of symbols

A, A_i action sets

$\mathcal{B}(X)$, set of Borel sets of the metric space X
$B_\varepsilon(x)$, open ball of radius ε and center x
BR, best reply correspondence

$C(X)$, space of real-valued continuous functions on the compact metric space X, endowed with the supremum norm
$C_A = C(A \times \mathcal{M}(A))$

Δ_m, Δ, simplex in $\mathbb{R}^m$
Δ_i, simplex in $\mathbb{R}^{m_i}$
δ_x, Dirac probability on x

$f^{(i \to a)}$, function adapted to some $\mathcal{T}^n$ that coincides with f except on T_i^n where $f(t) = a$, 39
f^n, function from T to S adapted to $\mathcal{T}^n$, 38
$\mathcal{F}_S = \{\text{equivalence class of } f : t \to S, \text{ measurable}\}$, 20
$\mathcal{F}_S^n = \{\text{equivalence class of } f : T \to S, \text{ adapted to } \mathcal{T}^n\}, 38$

G^n, quasi-normal form of n-player large game, 45

λ, Lebesque measure

m, cardinality of the finite action set A
m_i, cardinality of the finite action set A_i
$\mathcal{M}(X)$, X metric space, set of probabilities on X
μ, probability on C_S
μ^n, probability on C_S, probability form of n-player large game, 47

$p(\cdot, \cdot)$, Prohorov metric, 49

S, S_i, strategy sets
σ, probability on $C_S \times S$

$T = [0, 1]$, set of players
$\underline{T} = (T, \mathcal{B}(T), \lambda)$, nonatomic measure space of players, 20

$\mathcal{T}^n$, uniform partition of T with n elements, 38

u, u_i, u_t, payoff function (real-valued)
$\mathbf{u}$ (resp. $\mathbf{u}^n$) measurable function from T (resp. adapted to $\mathcal{T}^n$) to a set of utility functions
U, U_i, evaluation function (vector-valued)

$\|\cdot\|$, euclidean norm in an euclidean space
$\|\cdot\|_1$, L_1-norm
$\|\cdot\|_\infty$, supremum norm

List of definitions

Part II

Network applications

Chapter 6

Mixed equilibrium for multiclass routing games [1]

Abstract **We consider a network shared by non-cooperative two types of users, group users and individual users, each user of the first type has a significant impact on the load of the network, whereas a user of the second type doesn't. Both group users as well as individual users choose their routes so as to minimize their costs. We further consider the case that the users may have side constraints. We study the concept of mixed equilibrium (mixing of Nash equilibrium and Wardrop equilibrium). We establish its existence and some conditions for its uniqueness. Then we apply the mixed equilibrium to a parallel links network and to a case of load balancing.**

6.1 Introduction

We consider in this paper the problem of optimal routing in networks. The entity that is routed is called a *job*. There are infinitely many jobs to ship from a source to a destination (sources, so as destinations, may be different according to the jobs). The decision maker is called a user, there exist two types of users, group users and individual users. A group user has a large amount of jobs to ship, while an individual user has only one job to ship. Each user has its own source(s) and destination(s), its own link costs functions, and its own optimization criterion. We further consider allow for side constraints, and even more generally, we consider a setting in which the space of decisions of users is not orthogonal (see [20] for a similar setting in the case of group equilibrium only).

We group the individual users into classes, and we call also a group user a class. Then there are several classes of jobs. Each class corresponds to a large number of single jobs. In each class the routes to be taken by the jobs of that class are determined either by a decision maker that centralizes all decisions for that class, or they are done individually by each individual user. We call the first type of per-class decision making a class-centralized optimization, and the second approach a

[1]This work was done in collaboration with E. Altman, H. Kameda and O. Pourtallier and published in *IEEE Trans. on Automatic Control*, vol 47 no 6, pp. 903-916, June 2002.

class-individual optimization.

When all class use a class-individual optimization approach then the natural optimization concept is the Wardrop equilibrium [21]. This concept was very much studied (e.g. [3, 4, 5, 8, 19] and references therein). Most of the work with this optimization approach has been done in the framework of road traffic. However, this concept has been also useful in the area of distributed computing [12, 13], and in telecommunication networks [6]. In the context of road traffic, an individual user (a "job" in our terminology) may correspond to a single driver, and the class may correspond to all the drivers of a given type of vehicle that have a given source and destination. In the context of distributed computing, a user may correspond to a single job that is sent to be processed at some computer in a computer-network. Finally, in the context of telecommunications, a single user may correspond to a single packet in networks in which the delay of each packet is minimized [6]. A generalized version of the Wardrop equilibrium which involves side constraints has been studied in [17] and references therein.

When all class use a class-centralized optimization approach then the optimization concept is the Nash equilibrium. There has been much recent interest in this framework in recent years [2, 1, 3, 10, 15, 16, 18]. In the context of road traffic, a class, or a group user, may correspond to a transportation company, or to a bus company; in both examples we may assume that the route of each vehicle is indeed determined by the company and not by the individual driver.

The concept of mixed-equilibrium (M.E.) has been introduced by Harker [7] (and further applied in [22] to a dynamic equilibrium and in [11] to a specific load balancing problem with a completely symmetrical network). Harker has established the existence of the M.E., characterized it through variational inequalities, and gave conditions for its uniqueness.

The first part of the paper consists of the mathematical model and the definition of mixed-equilibria (Sec. 6.2), then Sec. 6.3-6.4 establish the existence of equilibria under different approaches and assumptions, and Sec. 6.5, 6.6 derive uniqueness conditions under conditions related to strict monotonicity. This part of our paper extends Harker's model [7] in several directions: (i) A general cost function is considered, rather than the separable cost function given as the sum of link costs in [7]. This allows one to model routing games in which the performance measures are rejection probabilities of calls or loss probabilities of packets. Our general cost allows in particular different users to have different costs for the same links or the same paths, which allows to model priorities. In some cases we explicitly introduce the term of per-user "service-rate" for this purpose. (ii) We obtain existence and some uniqueness results for the case where the decisions of group users are constrained. This allows one to model side constraints, and to consider multiobjective problems faced by the users. For example, a group user might wish to find a strategy that minimizes its delay, and at the same time constraining its average loss probability to be below some bound.

In the second part of the paper we obtain new sets of conditions for the uniqueness of the mixed equilibrium for the case where conditions of the type of strict monotonicity (such as those that are used in [7]) do not apply. Some of the new conditions are obtained by making further assumptions on the structure of possible equilibria (Sec. 6.7) and others are obtained for specific topologies (Sec. 6.8): the parallel link topology, and load balancing models.

6.2 Mixed equilibrium (M.E.): model and assumptions

We consider a general network. We denote $\mathcal{M}$ the set of nodes, and $\mathcal{L} \subset \mathcal{M} \times \mathcal{M}$ the set of unidirectional links [2]. The unit entity that is routed through the network is called a *job*.

Each job j has an origin-destination pair (*O-D pair*) as well as a service rate vector, μ^j. We denote the origin, or the source by $s(j)$ and the destination by $d(j)$; $\mu^j = (\mu_l^j, l = 1, \ldots, L)$ is an L-vector (L is the number of elements of the set $\mathcal{L}$, *i.e.* $L = \#\mathcal{L}$). The interpretation of μ_l^j can be the speed at which job j is processed in link l and we assume that $\mu_l^i > 0$, $\forall i \in \mathcal{I}$, $\forall l \in \mathcal{L}$.

Each user u has a certain amount of jobs to route from a source s to a destination d, we call this amount the flow demand of the user u for the O-D pair (s,d) and we denote it by $\phi^u_{(s,d)}$.

The network is used by two types of users.

The first type of users, referred to as *group users* have to route a large amount of jobs. The choices made by each of these users have a significant impact on the load of the network, and then on the delays that any other user can expect. We denote by $\mathcal{N}$ the set of group users. Each user $i \in \mathcal{N}$ is characterized by

- one service rate vector $\mu^i = (\mu_l^i)_{l \in \mathcal{L}}$,
- a *set* of O-D pairs D^i and
- a vector of demands $\phi^i = \left(\phi^i_{(s,d)}\right)_{(s,d) \in D^i}$, $\phi^i_{(s,d)}$ denotes the rate of jobs of this class that have to be shipped from s to d.

(Note that having several sources and destinations allows in particular to handle multicast applications, in which several destinations are associated with a single source).

The second type of users, referred as *individual users* have a single job to route through the network from a given source to a given destination, with a given service rate. There are infinitely many individual users and the routing choice of a single individual user has a negligible impact on the load of the system. Individual users can be classified according to the pair source-destination and the service rate associated to their jobs. We denote $\mathcal{W}$ the set of classes of individual users. Each class i of the second type is characterized by

- one O-D pair (s^i, d^i),

[2] A bidirectional link may be transformed into a network of unidirectional ones where some are of null cost (Appendix B), then the results presented in this paper are also valid in networks with both unidirectional and bidirectional links unless the assumptions impose that the links' cost functions are strictly increasing.

- one service rate vector $\mu^i = (\mu_l^i)_{l \in \mathcal{L}}$ and
- one flow demand ϕ^i (the "number" of users belonging to class i).

Note that since all jobs of class i have the same service rate vector, we shall use μ^i to denote the service rate vector of any one of the jobs that belong to class i.

Note also that the elements of the set $\mathcal{N}$ or of the set $\mathcal{W}$ can be considered as a class of jobs characterized by a set of pair(s) of source and destination and a service rate. Nevertheless the routing decision of all the jobs of $i \in \mathcal{N}$ is taken by a single decision maker, while the routing decision of any single job of $i \in \mathcal{W}$ is taken by the individual user who is paired with it.

We denote by $\mathcal{I}$ the set of all possible classes of jobs, $\mathcal{I} = \mathcal{N} \cup \mathcal{W}$, and assume that $\mathcal{I}$ is finite.

A path p from $s \in \mathcal{M}$ to $d \in \mathcal{M}$ is a sequence of directed links that goes from s to d. For $i \in \mathcal{I}$ we denote by $\mathcal{P}^i$ the set of possible paths for class i, by $\mathcal{P}^i_{(u,v)}$ the set of possible paths for class i which go from u to v, $\mathcal{P}^i = \cup_{(u,v)\in D^i} \mathcal{P}^i_{(u,v)}$ and by $\mathcal{P}$ the set of all possible paths, $\mathcal{P} = \cup_{i \in \mathcal{I}} \mathcal{P}^i$.

In this paper we try to work as much as possible on paths (*i.e.*, the decision is what fraction of traffic of each class has to be routed over each path; this is in contrast to the more restrictive models such as [18] in which the routing decisions are how much jobs to route to each outgoing link of each node; this second type of models implicitly assumes that all sequences of directed links that lead from a source to a destination are admissible paths). Nevertheless it will be necessary, sometimes, to work on link models, *i.e.*, at each node we shall allow each class to route all the flow that it sends through that node to any of the out-going links of that node. So we introduce two notations for the flows, one in term of paths and one in term of links.

Each decision maker (a class within $\mathcal{N}$ or an individual user belonging to some class in $\mathcal{W}$) has to choose a (set of) path(s) to route its job(s). For $i \in \mathcal{I}$ and $p \in \mathcal{P}^i$ (*resp.* $l \in \mathcal{L}$), we denote by $x^i_{(p)}$(*resp.* x^i_l) the amount of jobs sent through path p (*resp.* link l) by class i. Note again that the meaning of $x^i_{(p)}$ is slightly different according to whether i belongs to the set $\mathcal{N}$ or $\mathcal{W}$. If $i \in \mathcal{N}$, then $x^i_{(p)}$ is the amount of jobs of user $i \in \mathcal{N}$ sent through p, if $i \in \mathcal{W}$, $x^i_{(p)}$ represents the amount of individual users of class $i \in \mathcal{W}$ that choose path p to ship their unique job.

Depending on the context, we will denote by x^i, the strategy of class i, either the vector $(x^i_{(1)}, \ldots, x^i_{(P^i)})^T$ of path flows, or the vector $(x^i_1, \ldots, x^i_L)^T$ of link flows, where P^i (*resp.* L) is the number of paths (*resp.* links) in the set $\mathcal{P}^i$ (*resp.* $\mathcal{L}$).

Let $\mathbf{x}$ be the flow configuration, *i.e.*, $\mathbf{x}^T$ is the vector $(x^1, \ldots, x^I)$, where $I = \#\mathcal{I}$, and $\mathcal{X}$ be the set of possible $\mathbf{x}$ (the "total" strategy set).

It will sometimes be necessary to distinguish in a routing profile $\mathbf{x}$ of $\mathcal{X}$ the part due to the group users, and the part due to the classes of individual users. We will then write $\mathbf{x} = \begin{pmatrix} \mathbf{x}^{\mathcal{N}} \\ \mathbf{x}^{\mathcal{W}} \end{pmatrix}$, where $(\mathbf{x}^{\mathcal{N}})^T = (x^1, \ldots, x^N) \in \mathcal{X}^{\mathcal{N}}$ corresponds to the choice of the group users, and
$(\mathbf{x}^{\mathcal{W}})^T = (x^1, \ldots, x^W) \in \mathcal{X}^{\mathcal{W}}$ corresponds to the choice of the classes of individual users, where $N = \#\mathcal{N}$ and $W = \#\mathcal{W}$. We assume that $\mathcal{X} = \mathcal{X}^{\mathcal{N}} \times \mathcal{X}^{\mathcal{W}}$, and that

both $\mathcal{X}^{\mathcal{N}}$ and $\mathcal{X}^{\mathcal{W}}$ are convex and compact. Note that, as in [20], we do not assume that $\mathcal{X}^{\mathcal{N}}$ has a product form, and thus the policy used by some classes may restrict the policies used by other class. This is a general way of introducing constraints over the policies.

Notations:

ρ_l is the total load on link l, $\rho_l = \sum_{i\in\mathcal{I}} \rho_l^i$, where

$\rho_l^i = \frac{1}{\mu_l^i}\sum_{p\in\mathcal{P}^i} \delta_{lp} x_{(p)}^i = \frac{x_l^i}{\mu_l^i}$, where $\delta_{lp} = \begin{cases} 1 & \text{if } l \in p, \\ 0 & \text{otherwise.}\end{cases}$

$\boldsymbol{\rho}$ is the utilization vector which is induced by $\mathbf{x}$, $\boldsymbol{\rho}^T = (\rho_1, \ldots, \rho_L)$,

Φ is the total vector of flow demand, $\Phi = (\phi^i)_{i\in\mathcal{I}}$.

Let "$\cdot$" be the inner product and $\nabla_i := \frac{\partial}{\partial x^i}$.

Cost functions:

- $J^i : \mathcal{X} \to [0, \infty)$ (or $[0, \infty]$, depending on the context) is the cost function of class $i \in \mathcal{N}$.
- $F_{(p)}^i : \mathcal{X} \to [0, \infty)$ (or $[0, \infty]$, depending on the context) is the cost function of path p for each individual user of class $i \in \mathcal{W}$.

The aim of each user is to minimize its cost (according to the constraint set), i.e. for $i \in \mathcal{N}$, $\min_{x^i\in\mathcal{X}^i} J^i(\mathbf{x})$, and for $i \in \mathcal{W}$, $\min_{p\in\mathcal{P}^i} F_{(p)}^i(\mathbf{x})$.

Let $\mathcal{P}^{i*}(\mathbf{x})$ be the set of paths for class i which have a flow strictly positive when the strategy of class i is x^i ($\forall p \in \mathcal{P}^{i*}(\mathbf{x}),\ x_{(p)}^i > 0$) and let $(\mathbf{x}^{-i}, y^i)$ be the flow configuration where class j ($j \neq i$) uses strategy x^j and class i uses strategy y^i.

Definition: $\mathbf{x} \in \mathcal{X}$ is a *Mixed Equilibrium* (M.E.) if

$\forall i \in \mathcal{N},\ \forall y^i\ s.t.\ (\mathbf{x}^{-i}, y^i) \in \mathcal{X},\ J^i(\mathbf{x}) \leq J^i(\mathbf{x}^{-i}, y^i)$ (*Nash equilibrium condition*), and

$\forall i \in \mathcal{W},\ \forall p \in \mathcal{P}^i,\ \forall p^* \in \mathcal{P}^{i*}(\mathbf{x}),\ F_{(p^*)}^i(\mathbf{x}) \leq F_{(p)}^i(\mathbf{x})$ (*Wardrop equilibrium condition*).

6.2.1. Remark. Wardrop equilibrium condition is equivalent to

$$F_{(p)}^i(\mathbf{x}) - A^i \geq 0; \quad (F_{(p)}^i(\mathbf{x}) - A^i)x_{(p)}^i = 0 \tag{6.2.1}$$

$\forall i \in \mathcal{W}$ and $\forall p \in \mathcal{P}^i$, where $A^i = A^i(\mathbf{x}) := \min_{p\in\mathcal{P}^i} F_{(p)}^i(\mathbf{x})$.

6.3 Existence of M.E. through variational inequalities

In this section, we present a simple variational inequality method to establish the existence of M.E. in the case of no extra constraints under general conditions on the cost functions for both types of classes. (An introduction to variational inequality methods may be found in [14].) More precisely, Let $n = \sum_{i\in\mathcal{N}} \#\mathcal{P}^i$, $w = \sum_{i\in\mathcal{W}} \#\mathcal{P}^i$, $n + w = \sum_{i\in\mathcal{I}} \#\mathcal{P}^i$ and define the (total) strategy set $\mathcal{X}$ as follows

$$\mathcal{X} = \{\mathbf{x}\in\mathbb{R}^{(n+w)} \mid \forall i \in \mathcal{I}, \forall (u,v) \in D^i, \forall p \in \mathcal{P}^i,\ x_{(p)}^i \geq 0, \sum_{p\in\mathcal{P}_{(u,v)}^i} x_{(p)}^i = \phi_{(u,v)}^i\}$$

Assumptions:

- ($\mathcal{A}1$) $\forall i \in \mathcal{N},\ J^i(\mathbf{x}) : \mathcal{X} \to [0,\infty)$ is convex in x^i and continuously differentiable w.r.t. $x^i_{(p)},\ \forall p \in \mathcal{P}^i$
- ($\mathcal{A}2$) $\forall i \in \mathcal{W}, \forall p \in \mathcal{P}^i \quad F^i_{(p)}(\mathbf{x}) : \mathcal{X} \to [0,\infty)$ is continuous.

For every $i \in \mathcal{N}$ we denote the derivative of $J^i(\mathbf{x})$ with respect to $x^i_{(p)}$ $K^i_{(p)}(\mathbf{x})$, *i.e.*, $K^i_{(p)}(\mathbf{x}) = \frac{\partial}{\partial x^i_{(p)}} J^i(\mathbf{x})$

Let us now reformulate the mixed equilibrium conditions. $\mathbf{x} \in \mathcal{X}$ is a M.E. if and only if $\mathbf{x}$ satisfies

- $\exists \boldsymbol{\alpha} = \boldsymbol{\alpha}(x)$, $\boldsymbol{\alpha} = (\alpha^i_{(u,v)})_{i\in\mathcal{N},(u,v)\in D^i}$, such that $\forall i \in \mathcal{N},\ \forall (u,v) \in D^i$ and $\forall p \in \mathcal{P}^i_{(u,v)}$

$$K^i_{(p)}(\mathbf{x}) - \alpha^i_{(u,v)} \geq 0; \quad (K^i_{(p)}(\mathbf{x}) - \alpha^i_{(u,v)})x^i_{(p)} = 0 \quad \text{and}^3$$

- $\forall i \in \mathcal{W}$ and $\forall p \in \mathcal{P}^i$

$$F^i_{(p)}(\mathbf{x}) - A^i \geq 0; \quad (F^i_{(p)}(\mathbf{x}) - A^i)x^i_{(p)} = 0.$$

where $A^i = A^i(\mathbf{x}) := \min_{p\in\mathcal{P}^i} F^i_{(p)}(\mathbf{x})$.

Denote by $K(\mathbf{x})$ the n-dimensional vector $K(\mathbf{x}) = (K^i_{(p)}(\mathbf{x}))_{i\in\mathcal{N},p\in\mathcal{P}^i}$,
by $F(\mathbf{x})$ the w-dimensional vector $F(\mathbf{x}) = (F^i_{(p)}(\mathbf{x}))_{i\in\mathcal{W},p\in\mathcal{P}^i}$,
by $T(\mathbf{x})$ the $(n+w)$-dimensional vector $T(\mathbf{x}) = \begin{pmatrix} K(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$,
by $\mathcal{A}$ the $\left(\sum_{i\in\mathcal{N}} \#D^i + W\right)$-dimensional vector $\mathcal{A} = \begin{pmatrix} \boldsymbol{\alpha} \\ (A^i)_{i\in\mathcal{W}} \end{pmatrix}$,
and by Δ the incidence matrix (see Appendix A).

Then we have

6.3.1 Lemma. *Assume $\mathcal{A}1$-$\mathcal{A}2$. $\mathbf{x} \in \mathcal{X}$ is a M.E. if and only if $\mathbf{x}$ satisfies*

$$T(\mathbf{x}) - \Delta\mathcal{A} \geq 0, \qquad (T(\mathbf{x}) - \Delta\mathcal{A}) \cdot \mathbf{x} = 0,$$
$$\Delta^T \mathbf{x} = \Phi, \qquad \mathbf{x} \geq 0. \tag{6.3.1}$$

Proof: We have just to note that the conditions (6.3.1) are equivalent to $\mathbf{x} \in \mathcal{X}$. □

6.3.2 Lemma. *Assume $\mathcal{A}1$-$\mathcal{A}2$. $\mathbf{x} \in \mathcal{X}$ is a M.E. if and only if*

$$T(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0 \quad \forall \mathbf{y} \in \mathcal{X}. \tag{6.3.2}$$

Proof: Similar to the proof of [3, Lem. 3.2] ((6.3.2) holds if and only if $\mathbf{x}$ is solution of the linear program $\min_{\mathbf{y}} T(\mathbf{x}) \cdot \mathbf{y}$, s.t. $\Delta^T \mathbf{y} = \Phi,\ \mathbf{y} \geq 0$). □

6.3.3 Theorem. *Assume $\mathcal{A}1$-$\mathcal{A}2$. Then there exists a mixed equilibrium.*

Proof: $\mathcal{X} \subset \mathbb{R}^n$ is a nonempty, bounded, convex set, $T : \mathcal{X} \to \mathbb{R}^n$ is a continuous mapping on $\mathcal{X}$, then there exists a solution to (6.3.2) (see [14, Ch I, Thm 3.1]). □

6.4 Existence of M.E.: a fixed point approach

In this section we relax the assumptions on the cost functions of the group users but restrict the cost functions of the individual users. With these new assumptions we obtain the existence of the M.E. (in a setting that allows one to include extra constraints) using the following approach. It is well known that one may compute the Wardrop equilibrium by transforming the problem into an equivalent optimization problem (as if there were only one decision maker) by transforming the costs in the network, see [19] and references therein. In our setting of M.E. we shall thus (i) transform in a similar way the optimization problem faced by all individual users into an optimization problem of a new equivalent single group user by transforming the cost in a similar way as is done for the Wardrop equilibrium. (ii) Then we will be faced with a game problem of group users only, for which we shall use Rosen's existence theorem [20].

Let f_l be the cost function of the link l, $f_l : (0, \infty) \to (0, \infty]$, this function is used only for the individual users. For any individual user $i \in \mathcal{W}$, we defined the cost function of path $p \in \mathcal{P}^i$ as follows

$$F^i_{(p)}(\mathbf{x}) = \sum_{l \in \mathcal{L}} \frac{\delta_{lp}}{\mu^i_l} f_l(\rho_l) = \sum_{l \in p} \frac{1}{\mu^i_l} f_l(\rho_l)$$

Assumptions:

- $(\mathcal{A}'1)$ $\mathcal{X}$ is a nonempty, convex and compact subset of $\mathbb{R}^n$
- $(\mathcal{A}'2)$ $\mathcal{X}^{\mathcal{W}} = \{\mathbf{x}^{\mathcal{W}} \in \mathbb{R}^w \mid \forall i \in \mathcal{W}, \forall p \in \mathcal{P}^i,\ x^i_{(p)} \geq 0,\ \sum_{p \in \mathcal{P}^i} x^i_{(p)} = \phi^i\}$
- $(\mathcal{A}'3)$ $J^i(\mathbf{x}) : \mathcal{X} \to [0, \infty]$ is a continuous function of $\mathbf{x}$ and is convex in x^i,
- $(\mathcal{A}'4)$ f_l is continuous and increasing in ρ_l,
- $(\mathcal{A}'5)$ for every system flow configuration $\mathbf{x}$, if not all costs of group users are finite then at least one class, $i \in \mathcal{N}$ with infinite cost can change its own flow configuration to make its cost finite, and similarly an individual user has always a path of finite cost that it can use.

These assumptions will be imposed in the rest of the paper. They imply that the policies of the group users may be constrained, since we do not assume that $\mathcal{X}^{\mathcal{N}}$ is an orthogonal set. Thus the choice of policies by some group users may restrict the set of policies available to other group users.

Define $\tilde{f}(\mathbf{s}, \mathbf{t})$ where $\mathbf{s} \in \mathcal{X}^{\mathcal{N}}, \mathbf{t} \in \mathcal{X}^{\mathcal{W}}$ by

$$\tilde{f}(\mathbf{s}, \mathbf{t}) = \frac{1}{\Phi} \left[\sum_{l \in \mathcal{L}} \int_0^{\rho_l} f_l(\tau) d\tau \right],$$

where

$$\rho_l = \sum_{i=1}^{N} \frac{1}{\mu^i_l} \left(\sum_{p \in \mathcal{P}^i} \delta_{lp} s^i_{(p)} \right) + \sum_{i=N+1}^{N+W} \frac{1}{\mu^i_l} \left(\sum_{p \in \mathcal{P}^i} \delta_{lp} t^i_{(p)} \right),\ \delta_{lp} = \begin{cases} 1 & \text{if } l \in p, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\tilde{f}$ is convex in $\mathbf{t}$ (f_l is increasing) and continuous in $\mathbf{s}$ and $\mathbf{t}$.

Introduce the convex minimization program

$$\min \tilde{f}(\mathbf{s},\mathbf{t}) \text{ with respect to } \mathbf{t} \in \mathcal{X}^{\mathcal{W}}. \tag{6.4.1}$$

6.4.1 Lemma. *Either the convex program (6.4.1) has an optimal solution which satisfies condition (6.2.1) where* $\mathbf{x} = (\mathbf{s},\mathbf{t})^T$ *or* $\forall \mathbf{t} \in \mathcal{X}^{\mathcal{W}},\ \tilde{f}(\mathbf{s},\mathbf{t}) = +\infty$.

Proof: We have $F^i_{(p)}(\mathbf{s},\mathbf{t}) = \frac{\partial}{\partial t^i_{(p)}}(\Phi \tilde{f}(\mathbf{s},\mathbf{t}))$. Define the Lagrangian function

$$\Lambda(\mathbf{t},\alpha) = (\Lambda_1(\mathbf{t},\alpha^1),\ldots,\Lambda_W(\mathbf{t},\alpha^W))^T,$$

where $\forall i \in \mathcal{W} \quad \Lambda_i(\mathbf{t},\alpha^i) = \Phi\tilde{f}(\mathbf{s},\mathbf{t}) + \alpha^i(\phi^i - \sum_{p\in\mathcal{P}^i} t^i_{(p)})$.

Since we minimize a continuous and convex function on a convex, compact set, therefore either there exists an optimal solution or $\forall t \in \mathcal{X}^{\mathcal{W}} \quad \tilde{f}(\mathbf{s},\mathbf{t}) = +\infty$.

Next we show that an optimal solution satisfies (6.2.1). $\bar{\mathbf{t}}$ is an optimal solution , if and only if $\bar{\mathbf{t}}$ satisfies the following necessary and sufficient Kuhn-Tucker conditions: for any $i \in \mathcal{W} \quad \exists \bar{\alpha}^i \in \mathbb{R}$ (which depends on $\bar{\mathbf{t}}$) such that:

$$\nabla_i \Phi\tilde{f}(\mathbf{s},\bar{\mathbf{t}}) - \bar{\alpha}^i(1,\ldots,1)^T \geq 0,$$

$$(\nabla_i \Phi\tilde{f}(\mathbf{s},\bar{\mathbf{t}}) - \bar{\alpha}^i(1,\ldots,1)^T)^T \bar{t}^i = 0,$$

then $\bar{\alpha}^i = A^i$ and the result follows. □

Notation: In order to simplify the reading, let $f^i_l(\rho_l) = \frac{1}{\mu^i_l} f_l(\rho_l)$.

We now apply the existence theorem in [20, Thm. 1] to the convex game (in the sense of [20]) with $(N+1)$ players: the original N group users as well as the additional one who minimizes $\tilde{f}(\mathbf{x}^{\mathcal{N}},\mathbf{x}^{\mathcal{W}})$ with respect to $\mathbf{x}^{\mathcal{W}} \in \mathcal{X}^{\mathcal{W}}$.

Therefore under assumptions $\mathcal{A}'1 - \mathcal{A}'5$, there exists a mixed equilibrium.

6.4.2. Remark. If we wish to include constraints that involve also the individual users, such as add constraints on the links capacities (which then involves constraints on all users) then the condition of Wardrop equilibrium (all the paths used are of same cost) may not hold anymore. Nevertheless Larsson and Patriksson in [17] show that the program $\min \tilde{f}(x^{\mathcal{W}})$ with respect to the new strategy set leads to another kind of equilibrium (which they call *generalized Wardrop equilibrium*, in this case, we can also apply our *Lemma 6.4.1* and *Theorem 1* of [20] to obtain the existence of a "generalized mixed equilibrium".

6.5 Uniqueness of M.E.: Rosen's type condition

Definition: Let $T(\mathbf{x}) \in \mathbb{R}^n$ be a vector, then $\sigma(\mathbf{x},r) = \sum_{i=1}^n r_i T^i(\mathbf{x})$ is DSI (Diagonally Strictly Increasing) for $\mathbf{x} \in \mathcal{X}$ and for some $r \geq 0$ if for any $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}} \in \mathcal{X}$ ($\bar{\mathbf{x}} \neq \tilde{\mathbf{x}}$) we have

$$\sum_{i=1}^n r_i \left((\bar{x}^i - \tilde{x}^i)(T^i(\bar{\mathbf{x}}) - T^i(\tilde{\mathbf{x}}))\right) > 0,$$

$$\text{or equivalently, } (\bar{\mathbf{x}} - \tilde{\mathbf{x}}) \cdot \gamma(\tilde{\mathbf{x}}, r) + (\tilde{\mathbf{x}} - \bar{\mathbf{x}}) \cdot \gamma(\bar{\mathbf{x}}, r) < 0, \tag{6.5.1}$$
$$\text{where } \gamma^T(\mathbf{x}, r) = (r_1 T^1(\mathbf{x}), r_2 T^2(\mathbf{x}), \ldots, r_n T^n(\mathbf{x})).$$

The notion of DSI comes from the DSC (Diagonal Strict Convexity) of [20], in fact J.B. Rosen introduces the Diagonal Strict Concavity for a maximization problem, when we talk about a minimization problem we have to reverse the inequality in order to obtain convexity. The DSC is a condition on the derivatives $(\nabla_i T_i)$, that we cannot apply in our case to the cost functions of individual users, that's why we introduce the DSI. Note that $\sigma(\mathbf{x}, r) = \sum_{i=1}^{n} r_i \nabla_i T^i(\mathbf{x})$ DSI is equivalent to $s(\mathbf{x}, r) = \sum_{i=1}^{n} r_i T^i(\mathbf{x})$ DSC.

In the previous section we considered general convex, compact sets $\mathcal{X}^i$. In this section we need that $\mathcal{X}$ be orthogonal, then we restrict to sets that can be described as follows. Let $\mathcal{X} = \mathcal{X}^1 \times \mathcal{X}^2 \times \ldots \times \mathcal{X}^I$, where for any $i \in \mathcal{I}$, $\mathcal{X}^i$ is a bounded, closed and convex set defined by

- $\forall i \in \mathcal{N}$, $\mathcal{X}^i = \{x^i | g^i(x^i) \leq 0\}$, where $g^{i^T}(x^i) = (g_1^i(x^i), \ldots, g_{c_i}^i(x^i))$, $g_j^i(x^i)$, $j = 1, \ldots, c_i$, is a convex function of x^i, continuously differentiable and c_i (for $i \in \mathcal{N}$) is a constant.

- $\forall i \in \mathcal{W} \quad \mathcal{X}^i = \{x^i | -x^i_{(p)} \leq 0, \ \sum_{p \in \mathcal{P}^i} x^i_{(p)} = \phi^i\}$,

Then $\mathcal{X}$ is an orthogonal constraint set, which is convex.
6.5.1. Remark. g^i may represent (for $i \in \mathcal{N}$) the positivity constraints, the flow conservation constraints and some "extra" constraints. For $i \in \mathcal{W}$ positivity and demand constraints are explicitly described.

We introduce the following assumptions:
Assumptions:

- $(\mathcal{B}1)$ There exists an interior point in the set of constraints which are not linear.

- $(\mathcal{B}2)$ Wherever finite, J^i is continuously differentiable in x^i (which imposes that J^i is continuous in x^i).

- $(\mathcal{B}3)$ J^i depends on $\mathbf{x}$ only through x^i and $\boldsymbol{\rho}$.

Notations:
$(\nabla J)^T(\mathbf{x}) = (\nabla_1 J^1(\mathbf{x}), \nabla_2 J^2(\mathbf{x}), \ldots, \nabla_N J^N(\mathbf{x}))$,
$f^T(\mathbf{x}) = (f_1(\rho_1), f_2(\rho_2), \ldots, f_L(\rho_L))$,
$\tilde{T} = \begin{pmatrix} \nabla J \\ f \end{pmatrix}$.

Let $\mathbf{y}$ be the function from $\mathcal{X}$ to $\mathcal{X}^{\mathcal{N}} \times \mathbb{R}^L$, defined by $\mathbf{y}(\mathbf{x}) = \mathbf{y} = (y^1, \ldots, y^N, y^{\mathcal{W}})^T$, where $\forall i \in \mathcal{N}, y^i = x^i$ and $y^{\mathcal{W}} = (y_1^{\mathcal{W}}, \ldots, y_L^{\mathcal{W}})^T$, and where $\forall l \in \mathcal{L}, y_l^{\mathcal{W}} = \rho_l^{\mathcal{W}} = \sum_{i \in \mathcal{W}} \frac{x_l^i}{\mu_l^i}$.
With some abuse of notation, we shall write for $\mathbf{y} = \mathbf{y}(\mathbf{x})$

$$J^i(\mathbf{y}) = J^i(\mathbf{y}(\mathbf{x})) = J^i(\mathbf{x}) \text{ and } f(\mathbf{y}) = f(\mathbf{y}(\mathbf{x})) = f(\mathbf{x}).$$

That J^i depends on $\mathbf{x}$ only through $\mathbf{y}(\mathbf{x})$ follows from $(\mathcal{B}3)$ and that f depends on $\mathbf{x}$ only through $\mathbf{y}(\mathbf{x})$ follows from the fact that f depends on $\mathbf{x}$ only through ρ. Let

$$\sigma(\mathbf{y}, r) = \sum_{i \in \mathcal{N} \cup \mathcal{W}} r_i \tilde{T}^i(\mathbf{y}),$$

where $\forall i \in \mathcal{N}$ $\tilde{T}^i(\mathbf{y}) = J^i(\mathbf{y})$ and $\tilde{T}^{\mathcal{W}}(\mathbf{y}) = f(\mathbf{y})$, and let

$$\gamma^T(\mathbf{y}, r) = (r_1 \tilde{T}^1(\mathbf{y}), r_2 \tilde{T}^2(\mathbf{y}), \ldots, r_N \tilde{T}^N(\mathbf{y}), r_{\mathcal{W}} \tilde{T}^{\mathcal{W}}(\mathbf{y})).$$

Note again that for $i \in \{1, \ldots, N\} \cup \{\mathcal{W}\}$, $\tilde{T}^i(\mathbf{x}) = \tilde{T}^i(\mathbf{y})$.

6.5.2 Theorem. *If $\sigma(\mathbf{y}, r)$ is DSI for some $r > 0$, then all mixed equilibria $\mathbf{x}$ have the same utilization on links and moreover $\mathbf{x}^{\mathcal{N}}$ is unique.*

Proof: Let $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ be two M.E. Then we have for $\mathbf{x} = \bar{\mathbf{x}}$ and for $\mathbf{x} = \tilde{\mathbf{x}}$, $\forall i \in \mathcal{N}$, by the necessity of the Kuhn-Tucker conditions,

$$g^i(x^i) \leq 0 \text{ and } \exists \alpha^i \geq 0 \text{ such that} \tag{6.5.2}$$

$$\alpha^i \cdot g^i(x^i) = 0 \quad \left(\alpha^i = (\alpha^i_1, \ldots, \alpha^i_{c_i})^T\right), \tag{6.5.3}$$

$$\nabla_i J^i(\mathbf{x}) + \sum_{j=1}^{c_i} \alpha^i_j \nabla_i g^i_j(x^i) = 0 \tag{6.5.4}$$

and $\forall i \in \mathcal{W}$ according to our *Lemma 6.4.1*, we have that

$$\text{for } p \in \mathcal{P}^i, \quad (F^i_{(p)}(\mathbf{x}) - A^i) x^i_{(p)} = 0$$

which can be restated as

$$x^i \geq 0,\ g^i(x^i) \leq 0 \text{ and } \exists \beta^i \geq 0 \text{ such that} \tag{6.5.5}$$

$$\beta^i \cdot x^i = 0 \quad \left(\beta^i = (\beta^i_{(1)}, \ldots, \beta^i_{(P^i)})^T\right) \tag{6.5.6}$$

$$F^i(\mathbf{x}) - \beta^i - A^i(1, \ldots, 1)^T = 0, \tag{6.5.7}$$

where $F^{i^T}(\mathbf{x}) = (F^i_{(1)}(\mathbf{x}), \ldots, F^i_{(P^i)}(\mathbf{x}))$. Note that $\boldsymbol{\beta} = (\beta^i,\ i \in \mathcal{W})$ depends on $\mathbf{x}$.

We multiply (6.5.4) (*resp.* (6.5.7)) by $r_i(\tilde{x}^i - \bar{x}^i)^T$ (*resp.* $r_{\mathcal{W}}(\tilde{x}^i - \bar{x}^i)^T$) for $\bar{\mathbf{x}}$ and by $r_i(\bar{x}^i - \tilde{x}^i)^T$ (*resp.* $r_{\mathcal{W}}(\bar{x}^i - \tilde{x}^i)^T$) for $\tilde{\mathbf{x}}$, and sum on i. This gives

$$d_{\mathcal{N}} + d_{\mathcal{W}} + \delta_{\mathcal{N}} + \delta_{\mathcal{W}} = 0, \tag{6.5.8}$$

where

$$d_{\mathcal{N}} = \sum_{i \in \mathcal{N}} r_i \left((\bar{x}^i - \tilde{x}^i) \cdot \tilde{T}^i(\tilde{\mathbf{x}}) + (\tilde{x}^i - \bar{x}^i) \cdot \tilde{T}^i(\bar{\mathbf{x}})\right),$$

$$d_{\mathcal{W}} = r_{\mathcal{W}} \sum_{i \in \mathcal{W}} \sum_{p \in \mathcal{P}^i} (\tilde{x}^i_{(p)} - \bar{x}^i_{(p)})(F^i_{(p)}(\bar{\mathbf{x}}) - F^i_{(p)}(\tilde{\mathbf{x}})),$$

$$\delta_{\mathcal{N}} = \sum_{i\in\mathcal{N}} r_i B_i, \quad \delta_{\mathcal{W}} = \sum_{i\in\mathcal{W}} r_{\mathcal{W}} B_i' + \sum_{i\in\mathcal{W}} r_{\mathcal{W}} C_i,$$

where

$$B_i = \sum_{j=1}^{c_i} \left\{ \bar{\alpha}_j^i (\tilde{x}^i - \bar{x}^i) \cdot \nabla_i g_j^i(\bar{x}^i) + \tilde{\alpha}_j^i (\bar{x}^i - \tilde{x}^i) \cdot \nabla_i g_j^i(\tilde{x}^i) \right\},$$

$$B_i' = \tilde{\beta}^i \cdot (\tilde{x}^i - \bar{x}^i) - \bar{\beta}^i \cdot (\tilde{x}^i - \bar{x}^i)$$

and

$$C_i = (\bar{A}^i - \tilde{A}^i) \sum_{p\in\mathcal{P}^i} (\bar{x}^i_{(p)} - \tilde{x}^i_{(p)}).$$

Since $\forall i \in \mathcal{N}$, $\forall j \in \{1, \ldots, c_i\}$ $\quad g_j^i$ is convex in x^i, therefore

$$(\tilde{x}^i - \bar{x}^i) \cdot \nabla_i g_j^i(\bar{x}^i) \leq g_j^i(\tilde{x}^i) - g_j^i(\bar{x}^i)$$

and

$$(\bar{x}^i - \tilde{x}^i) \cdot \nabla_i g_j^i(\tilde{x}^i) \leq g_j^i(\bar{x}^i) - g_j^i(\tilde{x}^i).$$

Moreover $\forall i \in \mathcal{N}$, $\forall j \in \{1, \ldots, c_i\}$ $\quad \alpha_j^i \geq 0$. Hence

$$\begin{aligned} B_i &\leq \sum_{j=1}^{c_i} \left[\bar{\alpha}_j^i \left(g_j^i(\tilde{x}^i) - g_j^i(\bar{x}^i) \right) + \tilde{\alpha}_j^i \left(g_j^i(\bar{x}^i) - g_j^i(\tilde{x}^i) \right) \right] \\ &\leq \left(\bar{\alpha}^i \cdot g^i(\tilde{x}^i) + \tilde{\alpha}^i \cdot g^i(\bar{x}^i) \right). \end{aligned}$$

The last inequality is due to (6.5.3) and (6.5.6). By (6.5.2) we have $B_i \leq 0$. By (6.5.5) and (6.5.6) we have also that $B_i' \leq 0$. Since

$$\forall i \in \mathcal{W}, \sum_{p\in\mathcal{P}^i} \tilde{x}^i_{(p)} = \sum_{p\in\mathcal{P}^i} \bar{x}^i_{(p)} = \phi^i,$$

therefore $C_i = 0$. Further,

$$\begin{aligned} d_{\mathcal{W}} &= r_{\mathcal{W}} \sum_{i\in\mathcal{W}} \sum_{p\in\mathcal{P}^i} \sum_{l\in\mathcal{L}} (\tilde{x}^i_{(p)} - \bar{x}^i_{(p)}) \delta_{lp} \frac{1}{\mu_l^i} (f_l(\bar{\rho}_l) - f_l(\tilde{\rho}_l)) \\ &= r_{\mathcal{W}} \sum_{l\in\mathcal{L}} (f_l(\bar{\rho}_l) - f_l(\tilde{\rho}_l))(\tilde{y}_l^{\mathcal{W}} - \bar{y}_l^{\mathcal{W}}). \end{aligned}$$

Then

$$d_{\mathcal{N}} + d_{\mathcal{W}} = (\bar{\mathbf{y}} - \tilde{\mathbf{y}}) \cdot \gamma(\tilde{\mathbf{y}}, r) + (\tilde{\mathbf{y}} - \bar{\mathbf{y}}) \cdot \gamma(\bar{\mathbf{y}}, r)$$

Hence $\delta_{\mathcal{N}} \leq 0$, $\delta_{\mathcal{W}} \leq 0$ and it follows from (6.5.1) that $d_{\mathcal{N}} + d_{\mathcal{W}} < 0$ if $\bar{\mathbf{y}} \neq \tilde{\mathbf{y}}$. But this contradicts (6.5.8), so $\bar{\mathbf{y}} = \tilde{\mathbf{y}}$, *i.e.*, $\bar{\mathbf{x}}^{\mathcal{N}} = \tilde{\mathbf{x}}^{\mathcal{N}}$ and $\forall l \in \mathcal{L}$ $\quad \bar{\rho}_l = \tilde{\rho}_l$. $\square$

6.5.1 Sufficient condition for DSI

Let $\mathcal{Y}$ be the set of $\mathbf{y}$ which correspond to a $\mathbf{x} \in \mathcal{X}$. Let the $(N+1) \times (N+1)$ matrix $\Gamma(\mathbf{y}, r)$ be the Jacobian of $\gamma(\mathbf{y}, r))$ for fixed $r > 0$. That is the j*th* column of $\Gamma(\mathbf{y}, r)$ is $\frac{\partial \gamma(\mathbf{y},r)}{\partial y^j}$, $j \in \{1, \ldots, N\} \cup \{\mathcal{W}\}$, where $\gamma(\mathbf{y}, r)$ is defined by (6.5.1). Then the condition given in [20, Thm. 6] holds for our definition of DSI, *i.e.*

6.5.3 Theorem. *A sufficient condition that $\sigma(\mathbf{y}, r)$ be diagonally strictly increasing for $\mathbf{y} \in \mathcal{Y}$ and fixed $r > 0$ is that the symmetric matrix*
$[\Gamma(\mathbf{y}, r) + \Gamma^T(\mathbf{y}, r)]$ be positive definite for $\mathbf{y} \in \mathcal{Y}$.

The corollary of this theorem given in [18] (*Corollary 3.1*) holds as well.
Define

$$\Gamma_l(y_l, r) = \left\{ r_i \frac{\partial \tilde{T}_l^i}{\partial y_l^j} \right\}_{i,j\{1,\ldots,N\}\cup\{\mathcal{W}\}},$$

where for $i \in \mathcal{N}$, $\tilde{T}_l^i = \frac{\partial J_l^i}{\partial y_l^i}$ and $\tilde{T}_l^{\mathcal{W}} = f_l$.

6.5.4 Corollary. *Assume that for some positive $r \in \mathbb{R}^{N+1}$, the symmetric matrix $\left(\Gamma_l(y_l, r) + \Gamma_l^T(y_l, r)\right)$ is positive definite for every possible y_l and $l \in \mathcal{L}$. Then all mixed equilibria $\mathbf{x}$ have the same utilization on links and moreover $\mathbf{x}^{\mathcal{N}}$ is unique.*

Proof: It may easily be seen that, up to re-indexing of rows and columns, $\Gamma(\mathbf{y}, r)$ equals $\mathrm{diag}\{\Gamma_l(y_l, r), l \in \mathcal{L}\}$, and the required conclusion follows from Theorem 6.5.3 and 6.5.2. □

Example: Linear Costs. To illustrate Theorem 6.5.2, we consider the following cost structure, for which uniqueness has already been obtained in [7] using an alternative approach. Define the cost functions as follows

- $\forall l \in \mathcal{L} \quad f_l(x_l) = p_l x_l + q_l$, where $p_l > 0$ and $q_l \geq 0$.
- $\forall i \in \mathcal{N}$, $\forall l \in \mathcal{L} \quad J_l^i(x_l^i, x_l) = x_l^i . f_l(x_l)$.
- $\forall i \in \mathcal{W}$, $\forall p \in \mathcal{P}^i \quad F_{(p)}^i(\mathbf{x}) = \sum_{l \in p} f_l(x_l)$.

Indeed in such a case we have

$$\Gamma_l(y_l, 1) = (\underline{1} \cdot \underline{1}^T + Q) p_l,$$

where $\underline{1}$ denotes the $(N+1)$-dimensional vector with entries all 1 and Q is the diagonal matrix with 1 everywhere on its diagonal, except at position $(N+1, N+1)$ where it is a 0.
For any $l \in \mathcal{L}$, $\Gamma_l(y_l, 1)$ is a symmetric positive definite matrix.

Note that this network is a special case of equal service rates. One may easily find examples of networks with linear costs but different service rates where one cannot satisfy the hypothesis of *Corollary 6.5.4*. In the next section we deal with such cases and we obtain a result on uniqueness for the links utilization.

6.6 Uniqueness of M.E.: linear costs

We next obtain uniqueness of the utilization of some of the links in general networks with linear costs allowing for prioritizations through different service rates (thus extending the uniqueness results of [7]).
Assumptions:

- $(\mathcal{C}1)$ We define the cost function of user $i \in \mathcal{N}$ as follows:

$$J^i(\mathbf{x}) = \sum_{p\in\mathcal{P}} x^i_{(p)} \sum_{l\in\mathcal{L}} \delta_{lp} \frac{1}{\mu^i_l} f_l(\rho_l),$$

- $(\mathcal{C}2)$ f_l is linear and increasing.

We make the following assumption on the links:

- $(\mathcal{D})$ The set $\mathcal{L}$ is composed of two disjoint sets of links:
  (i) $\mathcal{L}_{\mathcal{I}}$, for which $f_l(\rho_l)$ are strictly increasing,
  (ii) $\mathcal{L}_{\mathcal{C}}$, for which $f_l(\rho_l) = f_l$ are constant (independent of ρ_l).

6.6.1. Remark. The f_l's are the same for all users (group and individual users).

We have

$$K^i_{(p)}(\mathbf{x}) := \frac{\partial J^i(\mathbf{x})}{\partial x^i_{(p)}} = \sum_{l\in L} \delta_{lp} \frac{1}{\mu^i_l} \left(f_l(\rho_l) + \frac{x^i_l}{\mu^i_l} \frac{\partial f_l(\rho_l)}{\partial \rho_l} \right) \text{ where } T(\mathbf{x}) = \begin{pmatrix} K(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}.$$

6.6.2 Lemma. *Assume $\mathcal{C}1$, $\mathcal{C}2$ and $\mathcal{D}$. For arbitrary $\mathbf{x}$ and $\tilde{\mathbf{x}}$ ($\mathbf{x} \neq \tilde{\mathbf{x}}$), if $T(\mathbf{x})$ are finite or $T(\tilde{\mathbf{x}})$ are finite then*

$$(\mathbf{x} - \tilde{\mathbf{x}}) \cdot [T(\mathbf{x}) - T(\tilde{\mathbf{x}})] > 0 \textit{ if } \exists l \in \mathcal{L}_{\mathcal{I}} \textit{ such that } \rho_l \neq \tilde{\rho}_l. \tag{6.6.1}$$

Proof: Assume that $\exists l \in \mathcal{L}_{\mathcal{I}}$ such that $\rho_l \neq \tilde{\rho}_l$. Then

$$\begin{aligned}(\mathbf{x} - \tilde{\mathbf{x}}) \cdot [T(\mathbf{x}) - T(\tilde{\mathbf{x}})] = &\left[\sum_{i\in\mathcal{N}} \sum_{p\in\mathcal{P}^i} \left(x^i_{(p)} - \tilde{x}^i_{(p)}\right) \left(K^i_{(p)}(\mathbf{x}) - K^i_{(p)}(\tilde{\mathbf{x}})\right) \right] \\ &+ \left[\sum_{i\in\mathcal{W}} \sum_{p\in\mathcal{P}^i} \left(x^i_{(p)} - \tilde{x}^i_{(p)}\right) \left(F^i_{(p)}(\mathbf{x}) - F^i_{(p)}(\tilde{\mathbf{x}})\right) \right],\end{aligned}$$

where $\forall i \in \mathcal{N}$,

$$\sum_{p\in\mathcal{P}^i} \left(x^i_{(p)} - \tilde{x}^i_{(p)}\right) \left(K^i_{(p)}(\mathbf{x}) - K^i_{(p)}(\tilde{\mathbf{x}})\right)$$

$$= \sum_{p\in\mathcal{P}^i} \sum_{l\in\mathcal{L}} \left(x^i_{(p)} - \tilde{x}^i_{(p)}\right) \times \delta_{lp} \frac{1}{\mu^i_l} \left[\left(f_l(\rho_l) + \frac{x^i_l}{\mu^i_l} \frac{\partial f_l(\rho_l)}{\partial \rho_l} \right) - \left(f_l(\tilde{\rho}_l) + \frac{\tilde{x}^i_l}{\mu^i_l} \frac{\partial f_l(\tilde{\rho}_l)}{\partial \tilde{\rho}_l} \right) \right]$$

$$= \sum_{l\in\mathcal{L}} (\rho_l^i - \tilde{\rho}_l^i)(f_l(\rho_l) - f_l(\tilde{\rho}_l)) + \sum_{l\in\mathcal{L}} (\rho_l^i - \tilde{\rho}_l^i)^2 \nabla f_l,$$

where $\nabla f_l := \frac{\partial f_l(\rho_l)}{\partial \rho_l} = \frac{\partial f_l(\tilde{\rho}_l)}{\partial \tilde{\rho}_l} \geq 0$, since f_l is linear and increasing (assumption ($\mathcal{C}2$)), and $\forall i \in \mathcal{W}$

$$\sum_{p\in\mathcal{P}^i} \left(x_{(p)}^i - \tilde{x}_{(p)}^i\right)\left(F_{(p)}^i(\mathbf{x}) - F_{(p)}^i(\tilde{\mathbf{x}})\right)$$
$$= \sum_{p\in\mathcal{P}^i} \sum_{l\in\mathcal{L}} \left(x_{(p)}^i - \tilde{x}_{(p)}^i\right) \times \frac{\delta_{lp}}{\mu_l^i}(f_l(\rho_l) - f_l(\tilde{\rho}_l))$$
$$= \sum_{l\in\mathcal{L}} (\rho_l^i - \tilde{\rho}_l^i)(f_l(\rho_l) - f_l(\tilde{\rho}_l)).$$

Moreover we know that $\forall l \in \mathcal{L}_\mathcal{C},\ f_l(\rho_l) - f_l(\tilde{\rho}_l) = 0$. Then

$$(\mathbf{x} - \tilde{\mathbf{x}}) \cdot [T(\mathbf{x}) - T(\tilde{\mathbf{x}})] = \sum_{l\in\mathcal{L}_\mathcal{I}} (f_l(\rho_l) - f_l(\tilde{\rho}_l))(\rho_l - \tilde{\rho}_l) + \sum_{i\in\mathcal{N}} \sum_{l\in\mathcal{L}} (\rho_l^i - \tilde{\rho}_l^i)^2 \nabla f_l > 0.$$

The last inequality is due to the existence of a link $l \in \mathcal{L}_\mathcal{I}$ such that $\rho_l \neq \tilde{\rho}_l$. Therefore we have the relation (6.6.1). □

6.6.3 Theorem. *Assume $\mathcal{C}1$-$\mathcal{C}3$ and $\mathcal{D}$. Then all mixed equilibria have the same utilization on links $l \in \mathcal{L}_\mathcal{I}$.*

Proof: Let $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$ be two mixed equilibria. Then according to *Lemma 6.3.2* we have

$$T(\bar{\mathbf{x}}) \cdot (\tilde{\mathbf{x}} - \bar{\mathbf{x}}) \geq 0, \text{ and } T(\tilde{\mathbf{x}}) \cdot (\bar{\mathbf{x}} - \tilde{\mathbf{x}}) \geq 0.$$

Then we obtain that

$$(\tilde{\mathbf{x}} - \bar{\mathbf{x}}) \cdot (T(\tilde{\mathbf{x}}) - T(\bar{\mathbf{x}})) \leq 0.$$

We can apply *Lemma 6.6.2*, since if the assumptions imposed are
- $\mathcal{A}1$-$\mathcal{A}2$, then $T(\mathbf{x})$ is finite for all $\mathbf{x} \in \mathcal{X}$,
- $\mathcal{A}'1$-$\mathcal{A}'5$, then $T(\mathbf{x})$ is finite if $\mathbf{x}$ is a mixed equilibrium, due to ($\mathcal{A}'5$). It implies that $\forall l \in \mathcal{L}_\mathcal{I} \quad \bar{\rho}_l = \tilde{\rho}_l$. □

6.7 Uniqueness of M.E.: positive flows

The first Theorem of this section shows under quite general conditions that if the global load on some links are the same under two equilibria, then also the flows of each user on these links are the same for the group users. Under more restrictive conditions, the second theorem in the section then, which extends [18, Thm. 3.3], establishes conditions for the uniqueness of the global load at equilibrium.

Assumptions:

- ($\mathcal{D}'$) The set $\mathcal{L}$ is composed of two disjoint sets of links:
  (i) $\mathcal{L}_\mathcal{I}$, for which $h_l(\rho_l)$ are strictly increasing,
  (ii) $\mathcal{L}_\mathcal{C}$, for which $h_l(\rho_l) = h_l$ are constant (independent of ρ_l).

- ($\mathcal{E}1$) All the individual users are grouped in a unique class, denoted $\mathcal{W}$, then we have $\mathcal{I} = \{1, \ldots, N\} \cup \{\mathcal{W}\}$.
- ($\mathcal{E}2$) The service rate μ_l^i can be represented as $a^i \mu_l$, and $0 < \mu_l^i$ is finite for all $i \in \mathcal{I}$, and $l \in \mathcal{L}$.
- ($\mathcal{E}3$) At each node, each class may re-route all the flow that it sends through that node to any of the out-going links of that node.
- ($\mathcal{E}4$) $J^i(\mathbf{x}) = \sum_{l \in \mathcal{L}} J_l^i(x_l^i, \rho_l)$, where
- ($\mathcal{E}5$) J_l^i the cost function on link l for user i satisfies

$$J_l^i(x_l^i, \rho_l) = \frac{x_l^i}{\mu_l^i} h_l(\rho_l) = \rho_l^i h_l(\rho_l), \text{ where}$$

- ($\mathcal{E}6$) h_l is continuous and increasing and J_l^i is continuously differentiable wherever finite.
- ($\mathcal{E}7$) ϕ_v^i is the amount of traffic of class i that enters the network at node $v \in \mathcal{M}$, if this quantity is negative this means that traffic of class i leaves node v at an amount of $|\phi_v^i|$. We assume that $\sum_{v \in \mathcal{M}} \phi_v^i = 0$.

Note that in this section the cost functions on links for the group users, h_l, may be different of those used by individual users, f_l; while in the previous section we required that both types of users have the same ones, *i.e.*, $h_l = f_l$.

For each node v, we denote by $In(v)$ the set of its in-going links, and by $Out(v)$ the set of its out-going links. For each node v we have the following demand-conservation constraint

$$\sum_{l \in Out(v)} x_l^i = \sum_{l \in In(v)} x_l^i + \phi_v^i.$$

In order to minimize cost functions, we introduce the Lagrangian function

$$\Lambda^i(\mathbf{x}, \alpha^i) = \sum_{l \in \mathcal{L}} \rho_l^i h_l(\rho_l) - \sum_{v \in \mathcal{M}} \alpha_v^i \left[\sum_{l \in Out(v)} x_l^i - \sum_{l \in In(v)} x_l^i - \phi_v^i \right],$$

where $\alpha^i = (\alpha_1^i, \alpha_2^i, \ldots, \alpha_M^i)^T$ is the vector of Lagrange multipliers for class i. Then for $\mathbf{x}$ to be a mixed equilibrium the following conditions on group users are necessary for any $i \in \mathcal{N}$ there exists $\alpha^i = \alpha^i(x)$ such that for any $l \in \mathcal{L}$

$$\frac{\partial \Lambda^i(\mathbf{x}, \alpha^i)}{\partial x_l^i} \geq 0, \quad \left(\frac{\partial \Lambda^i(\mathbf{x}, \alpha^i)}{\partial x_l^i} \right) x_l^i = 0, \tag{6.7.1}$$

$$x_l^i \geq 0, \sum_{l \in Out(v)} x_l^i = \sum_{l \in In(v)} x_l^i + \phi_v^i.$$

We have

$$K_l^i(x_l^i, \rho_l) := \frac{\partial \rho_l^i h_l(\rho_l)}{\partial x_l^i} = \frac{1}{\mu_l^i} \left(\rho_l^i \frac{\partial h_l(\rho_l)}{\partial \rho_l} + h_l(\rho_l) \right).$$

With $l = (u, v)$, conditions (6.7.1) can be rewritten as

$$K_l^i(x_l^i, \rho_l) - \alpha_u^i + \alpha_v^i \geq 0; \quad \left(K_l^i(x_l^i, \rho_l) - \alpha_u^i + \alpha_v^i\right) x_l^i = 0,$$

6.7.1. Remark. For $l \in \mathcal{L}_{\mathcal{I}},\ K_l^i(x_l^i, \rho_l)$ is strictly increasing in both of its arguments.

Definition: $\mathcal{L}_0(\mathbf{x})$ is the set of links l such that for any $i \in \mathcal{I},\ x_l^i > 0$.

6.7.2 Theorem. *Assume $\mathcal{D}'$ and $\mathcal{E}3$-$\mathcal{E}7$. If for all links $l \in \mathcal{L}_{\mathcal{I}}$ we have $\rho_l = \tilde{\rho}_l$ then $\forall l \in \mathcal{L}_{\mathcal{I}},\ \forall i \in \mathcal{N},\ x_l^i = \tilde{x}_l^i$.*

Proof: Suppose that there exists a link $\bar{l} \in \mathcal{L}_{\mathcal{I}}$ and a group user $i \in \mathcal{N}$ such that $\tilde{x}_{\bar{l}}^i > x_{\bar{l}}^i$. We construct a directed network $G'(\mathcal{M}', \mathcal{L}')$, whose set of nodes is identical to the one of our original network, *i.e.*, $\mathcal{M}' = \mathcal{M}$ and the set of links $\mathcal{L}'$ is constructed as follows

- for each link $l = (u, v) \in \mathcal{L}$ such that $\tilde{x}_l^i > x_l^i$ we have a link $l' = (u, v) \in \mathcal{L}'$ which we assign a flow value $x_{l'} = \tilde{x}_l^i - x_l^i > 0$,
- for each link $l = (u, v) \in \mathcal{L}$ such that $\tilde{x}_l^i < x_l^i$ we have a link $l' = (v, u) \in \mathcal{L}'$ which we assign a flow value $x_{l'} = x_l^i - \tilde{x}_l^i > 0$.

In other words, we redirect links according to the relation between $\tilde{x}_l^i$ and x_l^i. Remark that $\forall l' \in \mathcal{L}',\ x_{l'} > 0$ and $\mathcal{L}' \neq \emptyset$ since for the link $\bar{l},\ \tilde{x}_{\bar{l}}^i > x_{\bar{l}}^i$. Then obviously the values $x_{l'}$ constitute a positive, directed flow in the network. Since at each node $w \in \mathcal{M}'$ we must have

$$\sum_{l \in Out(w)} x_l^i - \sum_{l \in In(w)} x_l^i = \phi_w^i = \sum_{l \in Out(w)} \tilde{x}_l^i - \sum_{l \in In(w)} \tilde{x}_l^i,$$

this flow has no sources (it is a circulation). Then there exists a cycle C of links in G' such that $x_{l'} > 0$ for all $l' \in C$ and such that $\bar{l} \in C$.
Consider now a link $l' = (u, v) \in C$. Therefore either $\tilde{x}_{uv}^i > x_{uv}^i$ or $x_{vu}^i > \tilde{x}_{vu}^i$. Then in the first case we have

$$\tilde{\alpha}_u^i - \tilde{\alpha}_v^i = K_{uv}^i(\tilde{x}_{uv}^i, \tilde{\rho}_{uv}) > K_{uv}^i(x_{uv}^i, \rho_{uv}) \geq \alpha_u^i - \alpha_v^i, \tag{6.7.2}$$

if $l \in \mathcal{L}_{\mathcal{I}}$, where the first equality and the last inequality follow from Kuhn-Tucker conditions and the first inequality follows from the hypothesis $\tilde{x}_{uv}^i > x_{uv}^i$ and the fact that $\tilde{\rho}_{uv} = \rho_{uv}$. Furthermore, if $l \in \mathcal{L}_{\mathcal{C}}$ then

$$\tilde{\alpha}_u^i - \tilde{\alpha}_v^i = K_{uv}^i(\tilde{x}_{uv}^i, \tilde{\rho}_{uv}) = K_{uv}^i(x_{uv}^i, \rho_{uv}) \geq \alpha_u^i - \alpha_v^i. \tag{6.7.3}$$

In the second case we have

$$\alpha_v^i - \alpha_u^i = K_{vu}^i(x_{vu}^i, \rho_{vu}) > K_{vu}^i(\tilde{x}_{vu}^i, \tilde{\rho}_{vu}) \geq \tilde{\alpha}_v^i - \tilde{\alpha}_u^i, \text{ if } l \in \mathcal{L}_{\mathcal{I}}\ , \tag{6.7.4}$$

$$\alpha_v^i - \alpha_u^i = K_{vu}^i(x_{vu}^i, \rho_{vu}) = K_{vu}^i(\tilde{x}_{vu}^i, \tilde{\rho}_{vu}) \geq \tilde{\alpha}_v^i - \tilde{\alpha}_u^i, \text{ if } l \in \mathcal{L}_{\mathcal{C}}. \tag{6.7.5}$$

Note that the results of equations (6.7.2) and (6.7.4) are in fact identical as well as those of (6.7.3) and (6.7.5).

Denote $\delta\alpha_w = \tilde{\alpha}_w^i - \alpha_w^i$ (for all $w \in \mathcal{M}$). ¿From (6.7.2)-(6.7.5) we conclude that

for a link $l' = (u,v) \in \mathcal{L}'$ which comes from a link $l \in \mathcal{L}_{\mathcal{I}}$ in the original network

$$x_{l'} > 0 \text{ implies that } \delta\alpha_u > \delta\alpha_v.$$

and for a link $l' = (u,v) \in \mathcal{L}'$ which comes from a link $l \in \mathcal{L}_{\mathcal{C}}$ in the original network

$$x_{l'} > 0 \text{ implies that } \delta\alpha_u \geq \delta\alpha_v.$$

This means that along the cycle C we would have a monotonically increasing sequence of $\delta\alpha$'s where a step is with a strict increase (due to the existence of the link $\bar{l}$), then $\delta\alpha_u > \delta\alpha_u$, which is a contradiction.

We conclude that $\forall l \in \mathcal{L}_{\mathcal{I}},\ \forall i \in \mathcal{N},\ x_l^i = \tilde{x}_l^i$. □

6.7.3 Theorem. *Assume all users have the same source and destination. Assume $\mathcal{D}'$ and $\mathcal{E}1 - \mathcal{E}7$. Let $\mathbf{x}$ and $\tilde{\mathbf{x}}$ be two mixed equilibria. Assume that $\forall i \in \mathcal{I},\ \forall l \notin \mathcal{L}_0(\mathbf{x})\quad x_l^i = 0$ and $\forall i \in \mathcal{I},\ \forall l \notin \mathcal{L}_0(\tilde{\mathbf{x}}),\ \tilde{x}_l^i = 0$. Then $\forall l \in \mathcal{L}_{\mathcal{I}},\ \rho_l = \tilde{\rho}_l$ and moreover $\forall l \in \mathcal{L}_{\mathcal{I}},\ \forall i \in \mathcal{N},\ x_l^i = \tilde{x}_l^i$.*

Proof: Denote $\alpha_u = \sum_{i\in\mathcal{N}} a^i \alpha_u^i$ (where a^i is defined in ($\mathcal{E}2$)) and

$$S_l(\rho_l) = \rho_l \frac{\partial h_l(\rho_l)}{\partial \rho_l} + N h_l(\rho_l).$$

$S_l(\rho_l)$ are finite (due to our assumption ($\mathcal{A}'5$)).
Let $\boldsymbol{\alpha}$ (*resp.* $\tilde{\boldsymbol{\alpha}}$) be the vector of Lagrange multipliers associated to $\mathbf{x}$ (*resp.* $\tilde{\mathbf{x}}$).
Since $\rho_l^{\mathcal{W}} \geq 0$, we have

$$S_l(\rho_l) \geq \sum_{i\in\mathcal{N}} \left(\rho_l^i \frac{\partial h_l(\rho_l)}{\partial \rho_l} + h_l(\rho_l) \right).$$

(6.7.2) implies that

$$\frac{1}{\mu_{uv}} S_{uv}(\rho_{uv}) \geq \alpha_u - \alpha_v, \tag{6.7.6}$$

with equality for $(u,v) \in \mathcal{L}_0(\mathbf{x})$. A similar relation holds for $\tilde{\mathbf{x}}$.
We obtain for

$$\begin{aligned} 0 \ &\leq \sum_{(u,v)\in\mathcal{L}} (\rho_{uv} - \tilde{\rho}_{uv})(S_{uv}(\rho_{uv}) - S_{uv}(\tilde{\rho}_{uv})) \qquad (6.7.7) \\ &\leq \sum_{(u,v)\in\mathcal{L}} \mu_{uv}(\rho_{uv} - \tilde{\rho}_{uv})((\alpha_u - \tilde{\alpha}_u) - (\alpha_v - \tilde{\alpha}_v)) = 0. \end{aligned}$$

The first inequality follows from the monotonicity and the convexity of $h_l(\rho_l)$ for $l \in \mathcal{L}_{\mathcal{I}}$. The second inequality holds in fact for each pair u, v (and not just for the sum). Indeed, for $(u,v) \in \mathcal{L}_0(\mathbf{x}) \cap \mathcal{L}_0(\tilde{\mathbf{x}})$ this relation holds with equality due to (6.7.6). This is also the case for $(u,v) \notin \mathcal{L}_0(\mathbf{x}) \cup \mathcal{L}_0(\tilde{\mathbf{x}})$, since in that case

$\rho_{uv} = \tilde{\rho}_{uv} = 0$. Consider next the case $(u,v) \in \mathcal{L}_0(\mathbf{x})$, $(u,v) \notin \mathcal{L}_0(\tilde{\mathbf{x}})$. Then we have

$$\begin{aligned}(\rho_{uv} - \tilde{\rho}_{uv})(S_{uv}(\rho_{uv}) - S_{uv}(\tilde{\rho}_{uv})) &= \rho_{uv}(S_{uv}(\rho_{uv}) - S_{uv}(\tilde{\rho}_{uv})) \\ &\leq \mu_{uv}\rho_{uv}((\alpha_u - \tilde{\alpha}_u) - (\alpha_v - \tilde{\alpha}_v)).\end{aligned}$$

A symmetric argument establishes the case $(u,v) \notin \mathcal{L}_0(\mathbf{x})$, $(u,v) \in \mathcal{L}_0(\tilde{\mathbf{x}})$. We finally establish the last equality in (6.7.7).

$$\sum_{(u,v)\in\mathcal{L}} \mu_{uv}(\rho_{uv} - \tilde{\rho}_{uv})((\alpha_u - \tilde{\alpha}_u) - (\alpha_v - \tilde{\alpha}_v))$$

$$= \sum_{r\in\mathcal{M}}(\alpha_r - \tilde{\alpha}_r) \sum_{s\in\mathcal{M}(r,s)\in\mathcal{L}} (\rho_{rs} - \tilde{\rho}_{rs})\mu_{rs} - \sum_{r\in\mathcal{M}}(\alpha_r - \tilde{\alpha}_r) \sum_{s\in\mathcal{M},(s,r)\in\mathcal{L}} (\rho_{sr} - \tilde{\rho}_{sr})\mu_{sr}$$

$$\begin{aligned}&= \sum_{r\in\mathcal{M}}(\alpha_r - \tilde{\alpha}_r)\left(\sum_{l\in Out(r)} (\rho_l - \tilde{\rho}_l)\mu_l - \sum_{l\in In(r)} (\rho_l - \tilde{\rho}_l)\mu_l\right) \\ &= \sum_{i\in\mathcal{I}} \frac{1}{a^i}\left[\sum_{r\in\mathcal{M}}(\alpha_r - \tilde{\alpha}_r)\left(\sum_{l\in Out(r)} (x_l^i - \tilde{x}_l^i) - \sum_{l\in In(r)} (x_l^i - \tilde{x}_l^i)\right)\right] \\ &= \sum_{i\in\mathcal{I}} \frac{1}{a^i}\left[\sum_{r\in\mathcal{M}}(\alpha_r - \tilde{\alpha}_r)\left(\sum_{l\in Out(r)} x_l^i - \sum_{l\in In(r)} x_l^i - \left(\sum_{l\in Out(r)} \tilde{x}_l^i - \sum_{l\in In(r)} \tilde{x}_l^i\right)\right)\right] = 0.\end{aligned}$$

Since for all $l \in \mathcal{L}$ (resp. $\in \mathcal{L}_\mathcal{I}$), S_l is increasing (resp. strictly increasing), we conclude from (6.7.7) that $\rho_l = \tilde{\rho}_l$ for all links in $\mathcal{L}_\mathcal{I}$. The first part of the theorem is established.

From Theorem 6.7.2 we conclude that $\forall l \in \mathcal{L}$, $\forall i \in \mathcal{N}$ $x_l^i = \tilde{x}_l^i$. Thus the theorem is established. □

6.7.4. Remark. The above Theorem extends [18, Thm 3.3]. The latter first establishes, under a more restrictive setting, the uniqueness of global link flows. Then it proceeds to conclude the uniqueness of the actual flows by hinting at an argument different than Theorem 6.7.2, taken from the proof of [18, Thm 2.1], which deals with the case of parallel links. We have not been able to reconstruct that argument, as it uses the fact that the sum of link flows of each user between two nodes does not depend on the equilibrium; this indeed is trivially true in the case of a parallel link topology, but one still needs to show that this extends to general topology. Our Theorem 6.7.2 of course implies this.

6.8 Uniqueness of M.E. for specific topologies

We establish below the uniqueness of M.E. in networks with specific topologies: a network of parallel links, and two load balancing models from [10]. The uniqueness of the Nash equilibrium ($\mathcal{W} = \emptyset$) for these models in the equal service rate case has been established in [18, thm 2.1], [10, Thm 5.1] and [9]. For the load balancing networks, uniqueness and characterization of the M.E. has been derived in [11] for the case of a completely symmetric network. Introduce the following

Assumptions:

- ($\mathcal{F}1$) $J_l^i : [0,\infty)^2 \to [0,\infty]$, $J_l^i(x_l^i, \rho_l)$ is a continuous function, convex in x_l^i.
- ($\mathcal{F}2$) Wherever finite, J_l^i is continuously differentiable in x_l^i. We denote $K_l^i := \partial J_l^i / \partial x_l^i$.
- ($\mathcal{F}3$) K_l^i depends of two arguments x_l^i and ρ_l and is strictly increasing in both of them.

6.8.1 Parallel links

In a network with parallel links, all the users have the same origin and the same destination, and moreover each link is a path and vice-versa. Then we have $F_{(p)}^i(\mathbf{x}) = \sum_{l\in p} f_l^i(\rho_l) = f_l^i(\rho_l)$. Even for such a simple network, and even if we took equal service rates, the conditions in Harker [7] or the SDI condition are typically not satisfied; indeed, it is shown in [18] that these type of conditions do not hold in the special case of two links, two group users (with no individual users), and link costs that are of the type of an M/M/1 queue, except for very low traffic demands.

6.8.1 Lemma. *In a network of parallel links where the cost function of each user satisfies* ($\mathcal{A}'1$) $-$ ($\mathcal{A}'5$) *and* ($\mathcal{F}1$) $-$ ($\mathcal{F}3$)*, all mixed equilibria* $\mathbf{x}$ *have the same utilization on links and moreover* $\mathbf{x}^{\mathcal{N}}$ *is unique.*

Proof: We recall that $\rho_l^i = \frac{x_l^i}{\mu_l^i}$ and $\tilde{\rho}_l^i = \frac{\tilde{x}_l^i}{\mu_l^i}$.
Let $\mathbf{x}$ and $\tilde{\mathbf{x}} \in \mathcal{X}$ be two mixed equilibria. Then $\mathbf{x}$ and $\tilde{\mathbf{x}}$ satisfy the following conditions:
for $i \in \mathcal{W}$, $\forall l \in \mathcal{L}$

$$f_l^i(\rho_l) - A^i \geq 0; \quad (f_l^i(\rho_l) - A^i)x_l^i = 0, \tag{6.8.1}$$

$$f_l^i(\tilde{\rho}_l) - \tilde{A}^i \geq 0; \quad (f_l^i(\tilde{\rho}_l) - \tilde{A}^i)\tilde{x}_l^i = 0, \tag{6.8.2}$$

$\exists\, \boldsymbol{\alpha},\ \tilde{\boldsymbol{\alpha}}$ such that $\forall i \in \mathcal{N}$, $\forall l \in \mathcal{L}$

$$K_l^i(x_l^i, \rho_l) - \alpha^i \geq 0; \quad (K_l^i(x_l^i, \rho_l) - \alpha^i)x_l^i = 0, \tag{6.8.3}$$

$$K_l^i(\tilde{x}_l^i, \tilde{\rho}_l) - \tilde{\alpha}^i \geq 0; \quad (K_l^i(\tilde{x}_l^i, \tilde{\rho}_l) - \tilde{\alpha}^i)\tilde{x}_l^i = 0, \tag{6.8.4}$$

where $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^N)^T$ (*resp.* $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}^1, \ldots, \tilde{\alpha}^N)^T$) is the vector of Lagrange multipliers associated to $\mathbf{x}$ (*resp.* $\tilde{\mathbf{x}}$).

The first step is to establish that $\rho_l = \tilde{\rho}_l$, $\forall l \in \mathcal{L}$. To this end, we use the relations of the proof of [18, Thm. 2.1], *i.e* we prove that for each $l \in \mathcal{L}$ and $i \in \mathcal{N}$, the following relations hold

$$\{\tilde{\alpha}^i \leq \alpha^i,\ \tilde{\rho}_l \geq \rho_l\} \text{ implies that } \tilde{x}_l^i \leq x_l^i, \tag{6.8.5}$$

$$\{\tilde{\alpha}^i \geq \alpha^i,\ \tilde{\rho}_l \leq \rho_l\} \text{ implies that } \tilde{x}_l^i \geq x_l^i. \tag{6.8.6}$$

We shall only prove (6.8.5), since (6.8.6) is symmetric. Assume that $\tilde{\alpha}^i \leq \alpha^i$ and $\tilde{\rho}_l \geq \rho_l$ for some $l \in \mathcal{L}$ and some $i \in \mathcal{N}$. Note that (6.8.5) holds trivially if $\tilde{x}_l^i = 0$. Otherwise, if $\tilde{x}_l^i > 0$, then (6.8.3)-(6.8.4) together with our assumption imply that

$$K_l^i(\tilde{x}_l^i, \tilde{\rho}_l) = \tilde{\alpha}^i \leq \alpha^i \leq (K_l^i(x_l^i, \rho_l) \leq (K_l^i(x_l^i, \tilde{\rho}_l),$$

where the last inequality follows from the monotonicity of K_l^i in its second argument. Now, since K_l^i is non-decreasing in its first argument, this implies that $\tilde{x}_l^i \leq x_l^i$, and (6.8.5) is established.

Furthermore we have $l \in \mathcal{L}$ and $i \in \mathcal{W}$

$$\{\tilde{A}^i \leq A^i,\ \tilde{\rho}_l > \rho_l\} \text{ (or equivalently } \{\tilde{A}^i < A^i,\ \tilde{\rho}_l \geq \rho_l\}) \text{ implies that } \tilde{x}_l^i = 0, \tag{6.8.7}$$

$$\{\tilde{A}^i > A^i,\ \tilde{\rho}_l \leq \rho_l\} \text{ (or equivalently } \{\tilde{A}^i \geq A^i,\ \tilde{\rho}_l < \rho_l\}) \text{ implies that } x_l^i = 0. \tag{6.8.8}$$

Actually suppose $\tilde{x}_l^i > 0$, hence by (6.8.1)-(6.8.2) we obtain:

$$f_l(\tilde{\rho}_l) = \tilde{A}^i \leq A^i \leq f_l(\rho_l) \text{ (or } f_l(\tilde{\rho}_l) = \tilde{A}^i < A^i \leq f_l(\rho_l)),$$

which is a contradiction with our assumption on f_l. Therefore $\tilde{x}_l^i = 0$ and a fortiori $\tilde{x}_l^i \leq x_l^i$, (6.8.8) is symmetric.

Let $\mathcal{L}_1 = \{l :\ \tilde{\rho}_l > \rho_l\}$ and $\mathcal{I}_a = \{i :\ \tilde{\alpha}^i > \alpha^i \text{ or } \tilde{A}^i > A^i\}$,
$\mathcal{L}_2 = \mathcal{L} - \mathcal{L}_1 = \{l :\ \tilde{\rho}_l \leq \rho_l\}$. Assume that $\mathcal{L}_1$ is not empty. Since $\sum_l \tilde{x}_l^i = \sum_l x_l^i = \phi^i$, therefore (6.8.6) and (6.8.8) imply that for any $i \in \mathcal{I}_a$

$$\sum_{l \in \mathcal{L}_1} \tilde{x}_l^i = \phi^i - \sum_{l \in \mathcal{L}_2} \tilde{x}_l^i \leq \phi^i - \sum_{l \in \mathcal{L}_2} x_l^i = \sum_{l \in \mathcal{L}_1} x_l^i.$$

Since (6.8.5) and (6.8.7) imply that $\tilde{x}_l^i \leq x_l^i$ for $l \in \mathcal{L}_1$ and $i \notin \mathcal{I}_a$, therefore

$$\sum_{l \in \mathcal{L}_1} \tilde{\rho}_l = \sum_{l \in \mathcal{L}_1} \sum_{i \in \mathcal{I}} \frac{\tilde{x}_l^i}{\mu_l^i} \leq \sum_{l \in \mathcal{L}_1} \sum_{i \in \mathcal{I}} \frac{x_l^i}{\mu_l^i} = \sum_{l \in \mathcal{L}_1} \rho_l.$$

This inequality contradicts our non-emptiness assumption on $\mathcal{L}_1$, and then $\mathcal{L}_1 = \emptyset$. By symmetry it follows that the set $\{l :\ \tilde{\rho}_l < \rho_l\}$ is also empty. Thus, it has been established that

$$\tilde{\rho}_l = \rho_l,\ \forall l \in \mathcal{L}, \tag{6.8.9}$$

i.e., all mixed equilibria have the same utilization of links.

So it is now sufficient to establish that $\tilde{\alpha}^i = \alpha^i$ in order to prove the theorem. (6.8.5) may be strengthened as follows

$$\{\tilde{\alpha}^i < \alpha^i,\ \tilde{\rho}_l = \rho_l\} \text{ implies that either } \tilde{x}_l^i < x_l^i \text{ or } \tilde{x}_l^i = x_l^i = 0. \tag{6.8.10}$$

Indeed, if $\tilde{x}_l^i = 0$ then the implication is trivial. Otherwise, if $\tilde{x}_l^i > 0$, it follows that

$$K_l^i(\tilde{x}_l^i, \tilde{\rho}_l) = \tilde{\alpha}^i < \alpha^i \leq K_l^i(x_l^i, \rho_l),$$

so that $\tilde{x}_l^i < x_l^i$ as required.
Assume that $\tilde{\alpha}^i < \alpha^i$ for some $i \in \mathcal{N}$. Since $\sum_{l\in\mathcal{L}} \tilde{x}_l^i = \phi_i > 0$, then $\exists l \in \mathcal{L}$ such that $\tilde{x}_l^i > 0$ and (6.8.10) implies that

$$\sum_{l\in\mathcal{L}} x_l^i > \sum_{l\in\mathcal{L}} \tilde{x}_l^i = \phi_i,$$

which contradicts the demand constraint for user i. Hence $\tilde{\alpha}^i < \alpha^i$ does not hold for any user $i \in \mathcal{N}$, we obtain by a symmetric argument that $\tilde{\alpha}^i > \alpha^i$ does not hold as well.
Finally we have for $i \in \mathcal{N}$, $\tilde{\alpha}^i = \alpha^i$.
Combined with (6.8.9), this implies by (6.8.5) and (6.8.6) that $\tilde{x}_l^i = x_l^i$ for all $l \in \mathcal{L}$, $i \in \mathcal{N}$ and the lemma is proved. □

6.8.2 Load balancing with unidirectional links

We consider a model consisting of two processors, a and b, and a two-way communication lines, c and c', between them; the two processors can be seen as two links, a and b, between two different sources, u and v, and a unique destination.

This network can be reformulated in a network of parallel links ([9]), then we can apply *Lemma 6.8.1* and obtain the same uniqueness than previously for this model of load balancing. This result can be extended to a model of n processors with a two-way communication lines between each couple of processors (there are exactly $2(n-1)^2$ lines).

6.8.3 Load balancing with a communication bus

We now consider a model made up of two processors, a and b, and a communication bus, c, between them; the two processors can be seen as two links, a and b, between two different sources, u and v, and a unique destination, d.

Since a bidirectional link can be transformed in a network of unidirectional links (Appendix B) mixed equilibria in networks with unidirectional and bidirectional links exist.
Notation: We denote for $w = u, v$ by i_w the part of class i whose origin is w, *i.e.*, $s(i_w) = w$ and by ϕ_w^i the initial flow demand of user i at node w. In this model there exist four paths, $\mathcal{P} = \{(a),\ (ca),\ (b),\ (cb)\}$.

Each class $i \in \mathcal{N}$ is faced with the minimization program

$$\min_{x^i} J^i(\mathbf{x}) = \sum_{p\in\mathcal{P}^i} J_{(p)}^i(x_{(p)}^i, \boldsymbol{\rho}), \quad s.t.$$

$$x_{(a)}^i + x_{(cb)}^i = \phi_u^i, \quad x_{(a)}^i \geq 0,\ x_{(cb)}^i \geq 0,$$

$$x_{(b)}^i + x_{(ca)}^i = \phi_v^i, \quad x_{(b)}^i \geq 0,\ x_{(ca)}^i \geq 0.$$

Note that we allow $\phi_w^i = 0$, if the class $i \in \mathcal{N}$ has only one O-D pair, then $\forall i \in \mathcal{N}$, $\mathcal{P}^i = \mathcal{P}$.

For $l = a, b$, we have $J^i_{(l)}(x^i_{(l)}, \boldsymbol{\rho}) = J^i_l(x^i_{(l)}, \rho_l)$ and $J^i_{(cl)}(x^i_{(cl)}, \boldsymbol{\rho}) = J^i_l(x^i_{(cl)}, \rho_l) + J^i_c(x^i_{(cl)}, \rho_c)$.

We define for $p \in \mathcal{P}$

$$K^i_{(p)}(x^i_{(p)}, \boldsymbol{\rho}) := \frac{\partial J^i_{(p)}(x^i_{(p)}, \boldsymbol{\rho})}{\partial x^i_{(p)}}.$$

We have for $l = a, b$

$$K^i_{(l)}(x^i_{(l)}, \boldsymbol{\rho}) = K^i_l(x^i_{(l)}, \rho_l) \tag{6.8.11}$$

and

$$K^i_{(cl)}(x^i_{(cl)}, \boldsymbol{\rho}) = K^i_l(x^i_{(cl)}, \rho_l) + K^i_c(x^i_{(cl)}, \rho_c),$$

where for $p = l, cl$

$$K^i_l(x^i_{(p)}, \rho_l) = \frac{\partial J^i_l(x^i_{(p)}, \rho_l)}{\partial x^i_{(p)}},$$

and

$$K^i_c(x^i_{(cl)}, \rho_c) = \frac{\partial J^i_c(x^i_{(cl)}, \rho_c)}{\partial x^i_{(cl)}}.$$

Then for $\mathbf{x}$ to be a mixed equilibrium we have the following necessary Kuhn-Tucker conditions:
There exists $\boldsymbol{\alpha}$ such that $\forall i \in \mathcal{N}$

$$K^i_{(p)}(x^i_{(p)}, \boldsymbol{\rho}) - \alpha^i_w \geq 0; \quad (K^i_{(p)}(x^i_{(p)}, \boldsymbol{\rho}) - \alpha^i_w)x^i_{(p)} = 0,$$

where $p = a, ca$ if $w = u$ and $p = b, cb$ if $w = v$.

6.8.2 Lemma. *Assume that the service rate of each user does not depend on the links. Then in a network with two processors and a communication bus between them and where the cost function of each user satisfies* $(\mathcal{A}'1) - (\mathcal{A}'5)$ *and* $(\mathcal{F}1) - (\mathcal{F}3)$*, all mixed equilibria* $\mathbf{x}$ *have the same utilization on links.*

Proof: See Appendix C

6.9 Conclusion

We have focused in this paper the M.E. concept introduced in [7] and studied it under more general assumptions on the costs and for more general setting of optimization (which allows one to use constraints). M.E. involves groups that contain a continuum of users, where some of the groups have a single decision maker for the whole group and others have a decision maker per user. We further established uniqueness of the mixed equilibrium by either restricting to specific topologies or making some extra assumptions on the equilibrium flows.

A future research direction would be to add also extra constraints on the individual users (see [17]). We have not included these constraints here (except for a

Remark in the end of Section 6.4) since in their presence, the Wardrop principles need not hold anymore. For example, consider a network of two parallel links having both load independent costs, and in which there is a capacity constraint on a link with the lowest cost. If the latter link cannot accommodate all the flow then we would expect the outcome of individual optimization to yield the full utilization of that link and partial utilization of the other one. Hence the costs of different links (paths) that carry positive flow are not the same, thus violating Wardrop principle.

Appendix

A Constraints in general networks

In a general network the configuration flows which are feasible are the $\mathbf{x}$ which satisfy

$$\Delta^T \mathbf{x} = \Phi \ \& \ \mathbf{x} \geq 0, \text{ where } \mathbf{x} = \begin{pmatrix} \mathbf{x}^{\mathcal{N}} \\ \mathbf{x}^{\mathcal{W}} \end{pmatrix}.$$

In order to simplify the reading let $d^i = \# D^i$.

$$\mathcal{T} = \begin{pmatrix}
\tau_{11}^{11} & \tau_{12}^{11} & \cdots & \tau_{1d^1}^{11} & \tau_{11}^{12} & \cdots & \tau_{1d^N}^{1N} \\
\tau_{21}^{11} & \tau_{22}^{11} & \cdots & \tau_{2d^1}^{11} & \tau_{21}^{12} & \cdots & \tau_{2d^N}^{1N} \\
\vdots & \vdots & \ddots & \vdots & & & \\
\tau_{P^1 1}^{11} & \tau_{P^1 2}^{11} & \cdots & \tau_{P^1 d^1}^{11} & \tau_{P^1 1}^{12} & \cdots & \tau_{P^1 d^N}^{1N} \\
\tau_{11}^{21} & \tau_{12}^{21} & \cdots & \tau_{1d^1}^{21} & \tau_{11}^{22} & \cdots & \tau_{1d^N}^{2N} \\
\vdots & \vdots & \ddots & \vdots & & & \\
\tau_{(P^N-1)1}^{N1} & \tau_{(P^N-1)2}^{N1} & \cdots & \tau_{(P^N-1)d^1}^{N1} & \tau_{(P^N-1)1}^{N2} & \cdots & \tau_{(P^N-1)d^N}^{NN} \\
\tau_{P^N 1}^{N1} & \tau_{P^N 2}^{N1} & \cdots & \tau_{P^N d^1}^{N1} & \tau_{P^N 1}^{N2} & \cdots & \tau_{P^N d^N}^{NN}
\end{pmatrix}$$

whose element (q, r) is τ_{pd}^{ij}, where $i, j \in \mathcal{N}$, $d \in D^j$, $p \in \mathcal{P}^i$, $q = k + \sum_{k=1}^{i-1} P^k$ and $r = d + \sum_{s=1}^{j-1} d^s$ and
$\tau_{dk}^{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } k \in d, \\ 0 & \text{otherwise.} \end{cases}$

$$\Theta = \begin{pmatrix}
\theta_1^{11} & \theta_1^{12} & \cdots & \theta_1^{1W} \\
\theta_2^{11} & \theta_2^{12} & \cdots & \theta_2^{1W} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{P^1}^{11} & \theta_{P^1}^{12} & \cdots & \theta_{P^1}^{1W} \\
\theta_1^{21} & \theta_1^{22} & \cdots & \theta_1^{2W} \\
\theta_2^{21} & \theta_2^{22} & \cdots & \theta_2^{2W} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{P^W-1}^{W1} & \theta_{P^W-1}^{W2} & \cdots & \theta_{P^W-1}^{WW} \\
\theta_{P^W}^{W1} & \theta_{P^W}^{W2} & \cdots & \theta_{P^W}^{WW}
\end{pmatrix}$$

whose element (q, r) is θ_k^{ij}, where $i, j \in \mathcal{W}$, $k \in \mathcal{P}^i$, $q = k + \sum_{s=1}^{i-1} P^s$ and $r = j$ and
$\theta_k^{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

and

$$\Delta = \begin{pmatrix} \mathcal{T} & 0 \\ 0 & \Theta \end{pmatrix}$$

6.9.1. Remark. Here for all $i \quad x^i$ is expressed in term of paths, *i.e.*,
$\forall i \in \mathcal{N} \cup \mathcal{W} \quad x^i = (x^i_{(1)}, \ldots, x^i_{(P^i)})^T$.

B Relation between a bidirectional link and a network of unidirectional ones

A bidirectional link may always be expressed as an equivalent network of unidirectional ones. Indeed, consider a bidirectional link l between nodes u and v, where the cost function of this link is $f^i_l(x_l)$, and where x_l is the aggregate flow through link l. Then we can transform this link in the network of unidirectional links $(l1, l2, l3, l4, l5)$ in figure 5 with the cost functions: $f^i_{l1}(x_{l1}) = f^i_{l2}(x_{l2}) = f^i_{l4}(x_{l4}) = f^i_{l5}(x_{l5}) = 0$ and

$f^i_{l3}(x_{l3}) = f^i_l(x_{l3})$. These two subnetworks are not equivalent, since in the second one a user i can go from u to u (of from v to v) with the cost $f^i(x_{l3})$ which is not possible in the first one. Neverthelesss, they become equivalent if we add a constraint in the second network excluding cycles. This does not affect the equilibrium, since at equilibrium the paths $(l1, l3, l4)$ and $(l2, l3, l5)$ will not be used (as long as costs are nonnegative, of course).

C Proof of Lemma 6.8.2

Let $\mathbf{x}$ and $\tilde{\mathbf{x}} \in \mathcal{X}$ be two mixed equilibria.
Assume that $\tilde{\rho}_c \geq \rho_c$. Let $k \in \{a, b\}$ such that $\tilde{\rho}_k \geq \rho_k$ (which is equivalent to $\tilde{\rho}_l \leq \rho_l, \ l \neq k$). Let $\nu(l')$ be the source associated with l', *i.e.*, $l' = (\nu(l'), d)$, $l' \in \{l, k\}$, where d is the unique destination.
We recall that each class $i \in \mathcal{W}$ has only one O-D pair, then in this model, either $s(i) = u$ or $s(i) = v$; define $x^i_{(a)} = x^i_{(cb)} = 0$ if $s(i) = v$ and $x^i_{(b)} = x^i_{(ca)} = 0$ if $s(i) = u$.

First step: We first prove that for all $i \in \mathcal{W}$,

$$\tilde{x}^i_{(l)} \geq x^i_{(l)} \text{ or } \tilde{\rho}_l = \rho_l. \tag{6.9.1}$$

It is trivial if $s(i) = \nu(k)$, then we assume that $s(i) = \nu(l)$, we are faced with the two following cases

1) $\tilde{A}^i \geq A^i$
Then either $x^i_{(l)} = 0$ and $\tilde{x}^i_{(l)} \geq x^i_{(l)}$ or $x^i_{(l)} > 0$ and $\tilde{\rho}_l = \rho_l$.
The first implication is trivial, so we have only to check the second implication.
Let $x^i_{(l)} > 0$ then we have

$$F^i_{(l)}(\tilde{\mathbf{x}}) = f^i_l(\tilde{\rho}_l) \geq \tilde{A}^i \geq A^i = F^i_{(l)}(\mathbf{x}) = f^i_l(\rho_l).$$

Therefore, since f_l^i is strictly increasing in ρ_l, we have $\tilde{\rho}_l \geq \rho_l$ and finally $\tilde{\rho}_l = \rho_l$.

2) $\tilde{A}^i \leq A^i$

Then either $\tilde{x}^i_{(ck)} = 0$ and $\tilde{x}^i_{(l)} \geq x^i_{(l)}$ or $\tilde{x}^i_{(ck)} > 0$ and $\tilde{\rho}_l = \rho_l$.

As above we have only to check the second implication, the first one being trivial ($\tilde{x}^i_{(ck)} \leq x^i_{(ck)}$ is the same as $\tilde{x}^i_{(l)} \geq x^i_{(l)}$). Let $\tilde{x}^i_{(cl)} > 0$ then we have

$$F^i_{(ck)}(\tilde{\mathbf{x}}) = f^i_k(\tilde{\rho}_k) + f^i_c(\tilde{\rho}_c) = \tilde{A}^i \leq A^i \leq F^i_{(ck)}(\mathbf{x}) = f^i_k(\rho_k) + f^i_c(\rho_c),$$

which implies, since $\tilde{\rho}_k \geq \rho_k$, that $\rho_c \geq \tilde{\rho}_c$ which contradicts our assumption $\tilde{\rho}_c \geq \rho_c$, unless we have equality for c and for k. Then (6.9.1) is established.

Second step: We next prove a statement in the spirit of the first step, but for $i \in \mathcal{N}$:

$$\tilde{x}^i_{(l)} \geq x^i_{(l)} \text{ or equivalently } \tilde{x}^i_{(ck)} \leq x^i_{(ck)}. \tag{6.9.2}$$

First remark that the equivalence follows from the constraint on the sum of the flows. (6.9.2) holds trivially if $x^i_{(l)} = 0$, and so we have to check only the case $x^i_{(l)} > 0$. To do so, fix some $i \in \mathcal{N}$ and consider the following two subcases. Assume that

(a) $\tilde{\alpha}^i_{\nu(l)} \geq \alpha^i_{\nu(l)}$. Hence

$$K^i_{(l)}(\tilde{x}^i_{(l)}, \tilde{\boldsymbol{\rho}}) = K^i_l(\tilde{x}^i_{(l)}, \tilde{\rho}_l) \geq \tilde{\alpha}^i_{\nu(l)} \geq \alpha^i_{\nu(l)} = K^i_{(l)}(x^i_{(l)}, \boldsymbol{\rho}) = K^i_l(x^i_{(l)}, \rho_l) \geq K^i_l(x^i_{(l)}, \tilde{\rho}_l).$$

The first and last equalities follow from the definition of $K^i_{(l)}$. The other equality as well as the first inequality follow from the Kuhn Tucker conditions, whereas the last inequality follows from the monotonicity assumption ($\mathcal{F}3$). Using again ($\mathcal{F}3$), this time for the first argument, we conclude from the fact $K^i_l(\tilde{x}^i_{(l)}, \tilde{\rho}_l) \geq K^i_l(x^i_{(l)}, \tilde{\rho}_l)$ that (6.9.2) holds. Thus we try instead of (a):

(b) $\tilde{\alpha}^i_{\nu(l)} \leq \alpha^i_{\nu(l)}$. (6.9.2) holds trivially if $\tilde{x}^i_{(ck)} = 0$. So it remains to check the case $\tilde{x}^i_{(ck)} > 0$. Recall that $K^i_{(ck)}(x^i_{(ck)}, \boldsymbol{\rho}) = K^i_c(x^i_{(ck)}, \rho_c) + K^i_k(x^i_{(ck)}, \rho_k)$. We then have for $i \in \mathcal{N}$

$$\begin{aligned} K^i_c(x^i_{(ck)}, \rho_c) + K^i_k(x^i_{(ck)}, \rho_k) \geq \alpha^i_{\nu(l)} \geq \tilde{\alpha}^i_{\nu(l)} &= K^i_c(\tilde{x}^i_{(ck)}, \tilde{\rho}_c) + K^i_k(\tilde{x}^i_{(ck)}, \tilde{\rho}_k) \\ &\geq K^i_c(\tilde{x}^i_{(ck)}, \rho_c) + K^i_k(\tilde{x}^i_{(ck)}, \rho_k). \end{aligned}$$

Here, the first inequality and the equality follow from the Kuhn Tucker conditions, whereas the last inequality follows from ($\mathcal{F}3$). Using again ($\mathcal{F}3$), we conclude that (6.9.2) holds in case (b) as well.

Then (6.9.2) is established.

Third Step: We shall next prove that our two mixed equilibria have the same utilization of links.

(I) If there exists some $i \in \mathcal{W}$ such that $\tilde{x}^i_{(ck)} > 0$ and $\tilde{A}^i \leq A^i$, it has already been proved (see case 2 in the first step of the proof of our Lemma).

(II) If there exists some $i \in \mathcal{W}$ such that $x^i_{(l)} > 0$ and $\tilde{A}^i \geq A^i$, then $\tilde{\rho}_k = \rho_k$ and $\tilde{\rho}_l = \rho_l$ (This is the first case of the first step of the proof of the Lemma). It then remains to show that $\tilde{\rho}_c = \rho_c$.

Suppose that $\tilde{\rho}_c > \rho_c$, thus there exists $i_0 \in \mathcal{N} \cup \mathcal{W}$ such that $\tilde{x}_c^{i_0} > x_c^{i_0}$. (i) if $i_0 \in \mathcal{N}$, then $\tilde{x}_{(ck)}^{i_0} + \tilde{x}_{(cl)}^{i_0} > x_{(ck)}^{i_0} + x_{(cl)}^{i_0}$, which implies, due to (6.9.2), that $\tilde{x}_{(cl)}^{i_0} > x_{(cl)}^{i_0}$ and $x_{(k)}^{i_0} > \tilde{x}_{(k)}^{i_0}$.
We obtain

$$K_k^{i_0}(x_{(k)}^{i_0}, \rho_k) = \alpha_{\nu(k)}^{i_0} \leq K_c^{i_0}(x_{(cl)}^{i_0}, \rho_c) + K_l^{i_0}(x_{(cl)}^{i_0}, \rho_l)$$

$$< K_c^{i_0}(\tilde{x}_{(cl)}^{i_0}, \tilde{\rho}_c) + K_l^{i_0}(\tilde{x}_{(cl)}^{i_0}, \tilde{\rho}_l) = \tilde{\alpha}_{\nu(k)}^{i_0} \leq K_k^{i_0}(\tilde{x}_{(k)}^{i_0}, \tilde{\rho}_k),$$

where the strict inequality follows from $(\mathcal{F}3)$ and the equality and the others inequality follow from the Kuhn Tucker conditions. Using again $(\mathcal{F}3)$, we conclude that $x_{(k)}^{i_0} < \tilde{x}_{(k)}^{i_0}$, which is a contradiction, then $\tilde{\rho}_c = \rho_c$.
(ii) if $i_0 \in \mathcal{W}$, then we shall show that this implies that

there exists $i_1 \in \mathcal{W}$ such that $s(i_1) = \nu(l)$ and $\tilde{x}_{(ck)}^{i_1} > x_{(ck)}^{i_1}$.

Indeed, if $i_0 \in \mathcal{W}$ and $s(i_0) = \nu(k)$, we have $\tilde{x}_{(cl)}^{i_0} > x_{(cl)}^{i_0}$ but since $\tilde{\rho}_k = \rho_k$, then

– either there exists $j \in \mathcal{W}$ such that $\tilde{x}_{(ck)}^{j} > x_{(ck)}^{j}$, which implies $s(j) = \nu(l)$, and hence (6.9.3) is established,
– or there exists $j \in \mathcal{N}$ such that $\tilde{x}_{(ck)}^{j} + \tilde{x}_{(k)}^{j} > x_{(ck)}^{j} + x_{(k)}^{j}$. In this case (6.9.2) implies that $\tilde{x}_{(k)}^{j} > x_{(k)}^{j}$ or equivalently $\tilde{x}_{(cl)}^{j} < x_{(cl)}^{j}$. Then there must exist at least one other class j such that $\tilde{x}_c^{i_0} > x_c^{i_0}$ (so that $\tilde{\rho}_c > \rho_c$).

Then we can conclude from (i) that there exists $i_1 \in \mathcal{W}$ such that $s(i_1) = \nu(l)$ and $\tilde{x}_{(ck)}^{i_1} > x_{(ck)}^{i_1}$. If $\tilde{A}^{i_0} \leq A^{i_0}$, we are faced with the second case of the first step and we have seen that $\tilde{\rho}_c = \rho_c$. Then assume that $\tilde{A}^{i_0} \geq A^{i_0}$, therefore $x_{(l)}^{i_0} > 0$ ($x_{(l)}^{i_0} > \tilde{x}_{(l)}^{i_0}$) and we have

$$\min_{p \in P^{i_0}} F_{(p)}^{i_0}(\mathbf{x}) = F_{(l)}^{i_0}(\mathbf{x}) = F_{(l)}^{i_0}(\tilde{\mathbf{x}}) \leq F_{(ck)}^{i_0}(\mathbf{x}) < F_{(ck)}^{i_0}(\tilde{\mathbf{x}}),$$

where the first equality is due to $x_{(l)}^{i_0} > 0$, the second one to $\tilde{\rho}_l = \rho_l$ and the last inequality to $\tilde{\rho}_k = \rho_k$ and our first assumption $\tilde{\rho}_c > \rho_c$.
Hence $\tilde{x}_{(ck)}^{i_0} = 0$, which is a contradiction.

(III) If $\forall i \in \mathcal{W},\ \tilde{x}_{(l)}^{i} \geq x_{(l)}^{i}$, then according to (6.9.2), we conclude that

$$\sum_{i \in \mathcal{I}} \tilde{\rho}_k^{i_{\nu(l)}} \leq \sum_{i \in \mathcal{I}} \rho_k^{i_{\nu(l)}}. \tag{6.9.3}$$

Combining this with $\tilde{\rho}_k \geq \rho_k$, we obtain that $\sum_{i \in \mathcal{I}} \tilde{\rho}_k^{i_{\nu(k)}} \geq \sum_{i \in \mathcal{I}} \rho_k^{i_{\nu(k)}}$.

However, since for $i \in \mathcal{N} \cup \mathcal{W},\ \tilde{x}_{(l)}^{i} + \tilde{x}_{(ck)}^{i} = \phi_{\nu(l)}^{i}$ and $\tilde{x}_{(k)}^{i} + \tilde{x}_{(cl)}^{i} = \phi_{\nu(k)}^{i}$ (note that if $i \in \mathcal{W}$ and $s(i) = \nu(l)$ (*resp.* $s(i) = \nu(k)$), then $\phi_{\nu(k)}^{i} = 0$ (*resp.* $\phi_{\nu(l)}^{i} = 0$)) which is equivalent to $\tilde{\rho}_k^{i_{\nu(l)}} + \tilde{\rho}_l^{i_{\nu(l)}} = \frac{\phi_{\nu(l)}^{i}}{\mu^i}$ and $\tilde{\rho}_k^{i_{\nu(k)}} + \tilde{\rho}_l^{i_{\nu(k)}} = \frac{\phi_{\nu(k)}^{i}}{\mu^i}$ by our assumption on μ^i. It follows that

$$\sum_{i \in \mathcal{I}} \tilde{\rho}_l^{i_{\nu(k)}} \leq \sum_{i \in \mathcal{I}} \rho_l^{i_{\nu(k)}}. \tag{6.9.4}$$

But this, combined with (6.9.3), implies that $\tilde{\rho}_c \leq \rho_c$ and then $\tilde{\rho}_c = \rho_c$ (since $x^i_c = x_l^{i_{\nu(k)}} + x_k^{i_{\nu(l)}}$).
Suppose now that $\tilde{\rho}_l < \rho_l$ then due to (6.9.2) and our assumption $\forall i \in \mathcal{W}_l\ \tilde{x}^i_{(l)} \geq x^i_{(l)}$, it follows that $\exists i \in \mathcal{I}$ such that $\tilde{x}^i_{(cl)} < x^i_{(cl)}$, *i.e.*, $\tilde{x}^i_{(k)} > x^i_{(k)}$.
If $i \in \mathcal{N}$, then we have

$$K^i_k(x^i_{(k)}, \rho_k) < K^i_k(\tilde{x}^i_{(k)}, \tilde{\rho}_k) = \tilde{\alpha}^i_{\nu(k)} \leq K^i_l(\tilde{x}^i_{(cl)}, \tilde{\rho}_l) + K^i_c(\tilde{x}^i_{(cl)}, \tilde{\rho}_c)$$

$$< K^i_l(x^i_{(cl)}, \rho_l) + K^i_c(x^i_{(cl)}, \rho_c) = \alpha^i_{\nu(k)} \leq K^i_k(x^i_{(k)}, \rho_k),$$

where the first and the third inequalities are due to the strict increase in each argument of the K's and the equalities to $\tilde{x}^i_{(k)} > 0$, $x^i_{(cl)} > 0$ and (6.8.11). But this is impossible.
If $i \in \mathcal{W}$, therefore we have

$$F^i_{(cl)}(\tilde{\mathbf{x}}) < F^i_{(cl)}(\mathbf{x}) = \min_{p \in P^i} F^i_{(p)}(\mathbf{x}) \leq F^i_{(k)}(\mathbf{x}) < F^i_{(k)}(\tilde{\mathbf{x}}),$$

where the first equality is due to $x^i_{(cl)} > 0$, the first inequality to $\tilde{\rho}_l < \rho_l$ and $\tilde{\rho}_c = \rho_c$ and the last inequality to $\tilde{\rho}_k > \rho_k$. This implies that $\tilde{x}^i_{(k)} = 0$ and this is a contradiction.

Then $\forall i \in \mathcal{I} \quad \tilde{x}^i_{(cl)} \geq x^i_{(cl)}$ and finally $\tilde{\rho}_l = \rho_l$. Hence the lemma is established.□

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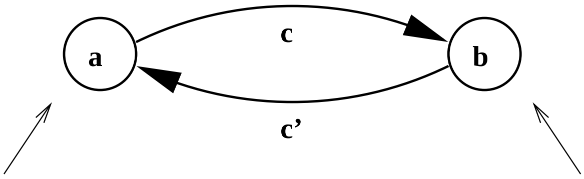

Figure 6.1: Two processors and two unidirectional links between them

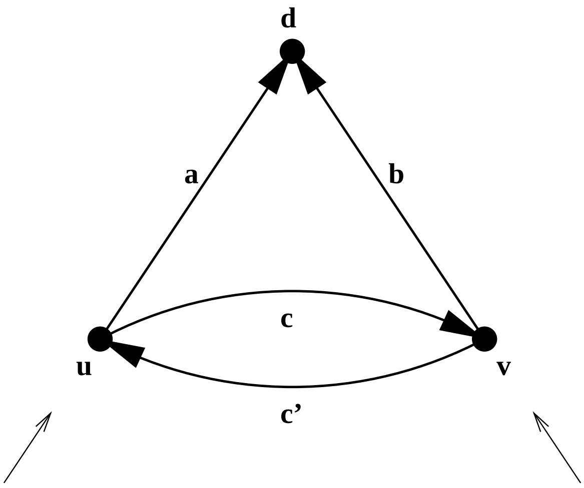

Figure 6.2: The corresponding network

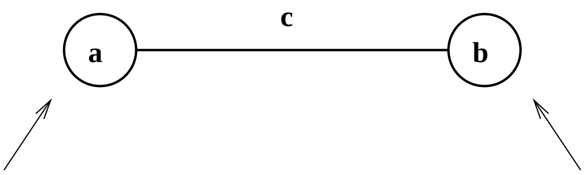

Figure 6.3: Two processors and a communication bus

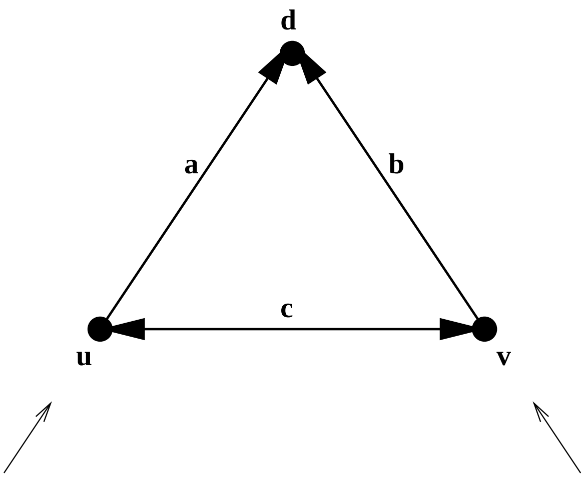

Figure 6.4: The corresponding network

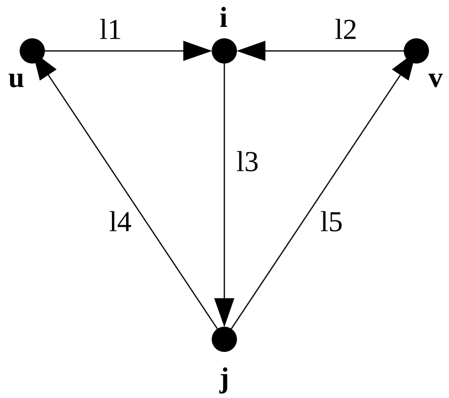

Figure 6.5: the corresponding network

Chapter 7

On the convergence to Nash equilibrium in problems of distributed computing [1]

Abstract **This paper studies two problems that arise in distributed computing. We deal with these problems from a game theoretical approach. We are interested in the convergence to the Nash equilibrium of algorithms based on the best reply strategy in a special case of linear costs. We present three specific types of algorithm that converge to the equilibrium. In our first model, composed of two processors, the convergence is established through monotonicity of the sequence of updates generated by each of the three algorithms. In the second model, made up of N processors, the convergence is due to the contraction of the algorithms.**

7.1 Introduction

The purpose of this paper is to study the convergence to the Nash equilibrium in two specific cases of distributed computing. In both cases we will show that models for user's behaviour in non-equilibrium converges to the unique *Nash Equilibrium*. We consider several greedy type best response algorithms that are natural in describing possible behavior of users in information technology networks. We shall focus on the *Elementary Stepwise System* (ESS) [9], in which players update their actions one after the other, in round robin, and where at each update a player uses the best response action against the actions of the other players. We shall further extend the convergence result to some other related algorithms.

The problems we consider can be modelled as a special case of a multi-user routing game for which the existence of equilibrium has been established in e.g. [5, 9]. Its uniqueness has been established in [6] for our first network. For the second model, uniqueness is obtained using an equivalent parallel link problem and applying an extension of [9, Thm 2.1].

[1] This work was done in collaboration with E. Altman and O. Pourtallier and published in *Annals of Operation Research*, vol 109, no 1-4, pp. 279-291, Jan 2002.

The question of whether policies based on repeated optimal responses converge to the equilibrium is quite important in practice, since dynamics based on optimal responses is quite a plausible behaviour of users, who in practice do not know the utility of other users and thus cannot be expected to be able to compute and to use the equilibrium immediately. Results on the convergence of such schemes to the equilibrium (in the context of routing games, which we also use here) have been limited to the very special case of two parallel links and with either two users [9] or with $N \geq 2$ users but with linear link costs [1]. As discussed in that reference, such costs are useful as they can well approximate a wide range of other costs under a light load regime.

We would like to mention that other learning approaches have been considered in the past with other types of assumptions on updating. For example, in [8], Monderer and Shapley consider n-person game in strategic form with identical payoff functions and *finite number of available actions* for each player. They show that if the players update their beliefs according to the actions of others in previous stages and choose a pure best responses against these beliefs, then the players' strategy will converge to the equilibrium, this is called the *fictitious play property*. This approach requires more knowledge of the history of plays than ours. Moreover, the fictitious play approach can neither be applied to distributed computing games (and to our model in particular) nor to routing games, since the players (or users) have an infinite set of actions.

The structure of this chapter is as follows. In Section 7.2, we define our model and introduce the concept of Nash equilibrium and the best reply algorithm or ESS. In Section 7.3, we consider a network made up of two processors and a communication bus between them, this network is shared by two users, each one being associated to a processor, this model was introduced in [7]; we show that the sequence generated by the algorithm is monotone, then we establish the convergence to Nash equilibrium of this sequence. In Section 7.4, we consider a network of N processors arranged in loop, each processor is in communication with its successor, as in the first model to each processor we associate a user, hence N users share this network. We give a uniqueness result and then establish the convergence of the ESS due to the contraction of this algorithm.

7.2 Model

We consider load balancing problems, that is, networks made of processors. The set of processors is denoted by $\mathcal{I}$. The processors are connected by links, $\mathcal{L}$ denotes the set of links. Such a network is shared by a finite number of users. In fact, to each processor corresponds a user, hence the set of users is also denoted by $\mathcal{I}$. Each user $i \in \mathcal{I}$ has an amount of jobs ϕ^i to process at each point in time. It can either process its jobs in "its" processor or ship them to other processors. The aim of each user is to minimize its cost (which depends of the decision of the others users), to do this the user has to decide how to split its amount of jobs into the different processors. Each user has infinitely many jobs to process, hence we approximate them by a continuous flow.

7.2.1. Remark. Since the index $i \in \mathcal{I}$ denotes a user as well as a processor, the superscript ^{i} will denote the user and the subscript $_i$ the processor.

The rate of jobs that user $i \in \mathcal{I}$ processes in processor j is denoted by $\mathbf{x}_j^i$ and $\mathbf{x}_l^i$ denotes the flow that i sends through link $l \in \mathcal{L}$, the strategy vector of user i is $\mathbf{x}^i = (\mathbf{x}_j^i, \mathbf{x}_l^i)_{j\in\mathcal{I}, l\in\mathcal{L}}$ and the total strategy vector (or in other words the flow configuration) is $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{I}}$. For $j \in \mathcal{I}$, $x_j = \sum_{i\in\mathcal{I}} \mathbf{x}_j^i$ denotes the aggregate flow processed in j and for $l \in \mathcal{L}$, $x_l = \sum_{i\in\mathcal{I}} \mathbf{x}_l^i$ denotes the aggregate flow which goes through link l.

The cost function of user i, $J^i(\mathbf{x})$, is the summation of the cost of the processing and the cost of communication, that is

$$J^i(\mathbf{x}) = \sum_{j\in\mathcal{I}} \mathbf{x}_j^i f_j(x_j) + \sum_{l\in\mathcal{L}} \mathbf{x}_l^i f_l(x_l)$$

where $\forall l \in \mathcal{L}$, f_l (*resp.* $\forall j \in \mathcal{I}$, f_j) is the cost of link l (*resp.* processor j) and depends only on the flow which goes through (*resp.* is processed in) it .

The set of all admissible flow configurations is denoted by $\mathcal{X}$, we have $\mathcal{X} = \Pi_{i\in\mathcal{I}}\mathcal{X}^i$ where $\mathcal{X}^i$ is the set of admissible flow configuration of user i, $\mathcal{X}^i = \{\mathbf{x}^i \,|\, \forall j \in \mathcal{I}\ \mathbf{x}_j^i \geq 0, \sum_{i\in\mathcal{I}} \mathbf{x}_j^i = \phi^i\}$. Note that $\mathcal{X}$ is convex, compact and non-empty.

This type of networks can be modelled as routing games, to do so we have just to transform the processors into unidirectional links which all have the same terminating node d, whereas the beginning node corresponds to the processor (thus we give the same index to a processor and the corresponding node). Therefore the set of nodes is $\mathcal{N} = \mathcal{I} \cup d$ and the set of links is $\mathcal{L}' = \mathcal{L} \cup \mathcal{P}$, where $\mathcal{P} = \{id, \forall i \in \mathcal{I}\}$. Figures 7.1, 7.2, 7.3 and 7.4 show the previous transformation for the two specific networks studied in this paper.

The cost of a processor i becomes the cost of the corresponding link id. Therefore a user who was associated to processor i, has in this new network to ship its jobs from the node i to the node d. Its decision is how to route these jobs into the different possible sequences of links in order to minimize its cost.

7.2.1 Nash equilibrium

Since each user wants to minimize its cost separately of the others, we are faced with a N-person non-cooperative game. In such a game the concept of optimization is the Nash equilibrium.

A Nash equilibrium is a flow configuration where no user has an interest in unilateral deviation. It is a necessary condition in order that a flow configuration will be a steady state. A formal definition is

Definition: $\mathbf{x} \in \mathcal{X}$ is a Nash Equilibrium if and only if

$$\forall i \in \mathcal{I}, \quad J^i(\mathbf{x}) \leq J^i(\mathbf{x}^1, \ldots, \mathbf{x}^{i-1}, \mathbf{y}^i, \mathbf{x}^{i+1}, \ldots, \mathbf{x}^N) \quad \forall \mathbf{y}^i \in \mathcal{X}^i$$

When the strategy set is convex, compact and nonempty, and when for each user $i \in \mathcal{I}$, for all $\mathbf{x} \in \mathcal{X}$, the cost function $J^i(\mathbf{x})$ is continuous in $\mathbf{x}$ and convex in $\mathbf{x}^i$ for each fixed value of $\mathbf{x}^{-i}$ (the vector $\mathbf{x}$ without the i^{th} component), then a Nash equilibrium exists (see for example [10, Thm 1]).

In order to obtain uniqueness of the equilibrium and the convergence of the ESS, we introduce the following assumption:

Assumption ($\mathcal{A}$):

- $f_l(\cdot) : [0, \infty) \to [0, \infty)$ is linear and strictly increasing, more precisely, $f_l(x) = p_l x + q_l$, $p_l > 0$.

Under this assumption, a Nash equilibrium exists (in [9], one can find a specific proof of existence of equilibria for routing games).

7.2.2 The Elementary Stepwise System

The Elementary Stepwise System (ESS) is an algorithm based on the best reply, where the users update their flow one after the other.

Relabel the users in order that user 1 is the first to update, user 2 the second and so on. The system starts with some (non-equilibrium) flow configuration $\mathbf{x}(0)$. From time to time, each user measures the current load on each link, and (after performing the necessary calculations) adjusts its own flow to minimize its cost function. We assume that exact minimization is achieved at each stage, and that all the above sequence (measuring, calculating and adjusting) are done instantly. Essentially, the system can be modelled as a sequence of steps in each of which a user updates its routing decisions.

At first step user 1 updates its flow and the resulting flow configuration is $\mathbf{x}(1) = (\mathbf{x}^1(1), \mathbf{x}^2(0), \ldots, \mathbf{x}^N(0))$ where $\mathbf{x}^1(1)$ is the optimal flow (or in other words the best reply) for user 1 against $\mathbf{x}^{-1}(0)$. At the i^{th} step user i updates its flow, and the resulting flow configuration is $\mathbf{x}(i) = (\mathbf{x}^1(1), \ldots, \mathbf{x}^i(1), \mathbf{x}^{i+1}(0), \ldots, \mathbf{x}^N(0))$, where $\mathbf{x}^i(i)$ is the optimal flow for user i against $\mathbf{x}^{-i}(i-1)$. At $(kN+j)^{th}$ user j updates and the resulting flow configuration is $\mathbf{x}(kN+j) = (\mathbf{x}^1(k+1), \ldots, \mathbf{x}^j(k+1), \mathbf{x}^{j+1}(k), \ldots, \mathbf{x}^N(k))$.

In the following sections, we present two specific networks and show that the ESS converges to Nash equilibrium in these networks.

7.3 A network with two processors

We consider a network made up of two processors 1 and 2 and a communication bus c between them (Figure 7.1). Two users 1 and 2, associated to the processors, share this network. When modelling this problem as a routing game as seen previously, the two processors can be represented as two directional links between two different sources 1 and 2 and a unique destination d. An additional bidirectional link connects

1 and 2 and represents the bus c (Figure 7.2). (Note that in [9] only unidirectional links are considered. However, it is easy to see that a bidirectional link can be represented by several unidirectional ones, see [3, Appendix B].)

In this new network, there exists a unique Nash equilibrium (the uniqueness is established in [6]).

Denoting y^1 the rate of jobs that user 1 processes in 1 and y^2 the rate of jobs that user 2 processes in 2, we can see that the vector $\mathbf{x}$ is uniquely determined by y^1 and y^2; indeed we have $y^1 = \mathbf{x}_1^1,\ y^2 = \mathbf{x}_2^2,\ x_1 = y^1 + \phi^2 - y^2$,
$x_2 = y^2 + \phi^1 - y^1$ and $x_c = \phi^1 + \phi^2 - y^1 - y^2$.

Therefore in the rest of the section we will write $J^i(y^i, y^j)$ instead of $J^i(\mathbf{x})$, $i, j \in \{1, 2\},\ j \neq i$.

The minimization program of user i is:

$$\min_{y^i \in [0, \phi^i]} y^i f_i(x_i) + (\phi^i - y^i) f_j(x_j) + (\phi^i - y^i) f_c(x_c)$$

We denote the derivative of $J^i(y^i, y^j)$ with respect to y^i by $K^i(y^i, y^j)$.

Since the constraints are linear, $\mathbf{x}$ is a Nash equilibrium if and only if it satisfies the following Kuhn-Tucker conditions:

for $i, j \in \{1, 2\},\ j \neq i,\ \exists \alpha^i \geq 0,\ \beta^i \geq 0$ such that

$$K^i(y^i, y^j) + \alpha^i - \beta^i = 0 \tag{7.3.1}$$

and

$$\alpha^i \cdot (y^i - \phi^i) = \beta^i \cdot y^i = 0 \tag{7.3.2}$$

7.3.1. Remark. Given y^j, if y^i satisfies (7.3.1) and (7.3.2) then y^i is an optimal flow against (*i.e.*, a best reply to) y^j.

We recall that user 1 is the first to update its flow, therefore
$\forall n \geq 1,\ y^1(n)$ is a best reply to $y^2(n-1)$ $(y^1(n) \in BR(y^2(n-1)))$ and $y^2(n)$ is a best reply to $y^1(n)$ $(y^2(n) \in BR(y^1(n)))$.

Our purpose is to show that $(y^1(n), y^2(n))$ converges to the Nash equilibrium.

7.3.2 Lemma. *If $y^2(n-1) \geq y^2(n)$, then $y^2(n) \geq y^2(n+1)$.*

Proof: Assume that $y^2(n-1) \geq y^2(n)$.

For $i, j \in \{1, 2\},\ j \neq i$, we have

$$K^i(y^i, y^j) = Ay^i + By^j + C^i$$

where $A = 2(p_1 + p_2 + p_c) > 0$,
$B = -p_1 - p_2 + p_c$,
and $C^i = p_i \phi^j - 2p_j \phi^i - p_c(2\phi^i + \phi^j) + q_i - q_j - q_c$.

We are faced with three cases:

- $B = 0$: the minimization of $J^1(y^1, y^2)$ (*resp.* $J^2(y^2, y^1)$) does not depend on y^2 (*resp.* y^1), therefore $y^1(n) = y^1(n+1)$ and $y^2(n) = y^2(n+1)$.

- $B < 0$:
  - If $y^1(n) = 0$ (then $K^1(y^1(n), y^2(n-1)) - \beta^1(n) = 0$) and $y^1(n) \leq y^1(n+1)$, thus, since K^1 is increasing in y^1 and decreasing in y^2 and since by assumption of the Lemma $y^2(n-1) \geq y^2(n)$, we have

    $$0 \leq K^1(y^1(n), y^2(n-1)) \leq K^1(y^1(n), y^2(n)) \leq K^1(y^1(n+1), y^2(n)).$$

    Hence $y^1(n+1) = 0$; indeed, if this were not the case, then $0 < K^1(y^1(n+1), y^2(n))$ and then we would have $\beta^1(n+1) > 0$ by the Kuhn-Tucker condition (7.3.1). But then the Kuhn-Tucker condition (7.3.2) would be violated.
  - If $0 < y^1(n) < \phi^1$ and $y^1(n) < y^1(n+1)$ then $\beta^1(n+1) = 0$ and

    $$0 = K^1(y^1(n), y^2(n-1)) \leq K^1(y^1(n), y^2(n)) < K^1(y^1(n+1), y^2(n))$$

    which is impossible according to (7.3.1)-(7.3.2). In the above, the first equality follows from Kuhn-Tucker condition (7.3.1), since the Kuhn-Tucker condition (7.3.2) implies that $\alpha^1(n) = \beta^1(n) = 0$. The first inequality follows since by assumption of the Lemma, $y^2(n-1) \geq y^2(n)$.
  - If $y^1(n) = \phi^1$, then clearly $y^1(n) \geq y^1(n+1)$.

  We conclude that if $B < 0$, $y^2(n-1) \geq y^2(n)$ implies that $y^1(n) \geq y^1(n+1)$, where $y^1(n) \in BR(y^2(n-1))$ and $y^1(n+1) \in BR(y^2(n))$.

  Since K^1 and K^2 are the same function up to a constant, in particular both of them are strictly increasing in their first argument and strictly decreasing in the second one (due to assumption $B < 0$), therefore we can apply the same proof to K^2 and $y^1(n) \geq y^1(n+1)$ in order to show that $y^2(n) \geq y^2(n+1)$ where $y^2(n) \in BR(y^1(n))$ and $y^2(n+1) \in BR(y^2(n+1))$.

- $B > 0$:
  - If $y^1(n) = \phi^1$, then $K^1(y^1(n), y^2(n-1)) + \alpha^1(n) = 0$ and $y^1(n+1) \leq y^1(n)$, thus, since K^1 is increasing in y^1 and in y^2, we have

    $$0 \geq K^1(y^1(n), y^2(n-1)) \geq K^1(y^1(n), y^2(n)) \geq K^1(y^1(n+1), y^2(n))$$

    hence $y^1(n+1) = \phi^1$(if not $0 > K^1(y^1(n+1), y^2(n))$ and then $\alpha^1(n+1) > 0$, which is a contradiction).
  - If $0 > y^1(n) > \phi^1$ and $y^1(n) > y^1(n+1)$ then

    $$0 = K^1(y^1(n), y^2(n-1)) \geq K^1(y^1(n), y^2(n)) > K^1(y^1(n+1), y^2(n))$$

    then $\beta^1(n+1) > 0$ which is a contradiction.

– If $y^1(n) = 0$, then $y^1(n) \leq y^1(n+1)$.

Then if $B > 0$, $y^2(n-1) \geq y^2(n)$ implies that $y^1(n) \leq y^1(n+1)$. Applying the same proof to K^2 and $y^1(n) \leq y^1(n+1)$,we can conclude that $y^2(n) \geq y^2(n+1)$.

□

7.3.3. Remark. Note that this lemma is also true for user 1, that is if $y^1(n-1) \geq y^1(n)$, then $y^1(n) \geq y^1(n+1)$.

7.3.4 Proposition. *Assume that the ESS is initialized with a feasible system configuration* $\mathbf{x}(0)$ *induced by the vector* $(y^1(0), y^2(0))$*. Then the system configuration converges over time to the (unique) Nash equilibrium* $\mathbf{x}$ *(induced by* (y^1, y^2)*), i.e.,*

$$\lim_{n\to\infty} (y^1(n), y^2(n)) = (y^1, y^2)$$

Proof: From the above lemma, we know that each component of $(y^1(n), y^2(n))$ $(n \geq 1)$ increases or decreases monotonically in n. Since the flows are bounded, this implies that $y^1(n)$ and $y^2(n)$ converge as n goes to infinity. Denote these limits y^1 and y^2. Due to the continuity of the cost functions it follows that y^1 is optimal for user 1 against y^2 and y^2 is optimal for user 2 against y^1, such that (y^1, y^2) is the Nash equilibrium. □

7.3.5. Remark. We can observe that this two users routing game is a S-modular game (for the definition of such a game see, e.g.,[11]). More precisely if $B \geq 0$ then it is a supermodular game (cases 1 and 3 of the proof of our Lemma), if $B < 0$ then it is a submodular game (case 2 of our Lemma), which implies the convergence of the ESS.

7.3.6. Remark. An alternative way to obtain convergence to the Nash equilibrium would be as follows. Define the matrix

$$T(u, w, v) := [\nabla^2_{y^1} J^1(u, v)]^{-1} \nabla_{y^1 y^2} J^1(u, v) \cdot [\nabla^2_{y^2} J^2(v, w)]^{-1} \nabla_{y^2 y^1} J^2(v, w)$$

as in [2, Ch 4.3], where $w \in [0, \phi^1]$, $v \in BR(w)$ and $u \in BR(z)$. Then $||T|| = (B/A)^2 < 1$ and we can apply Proposition 4.1 of [2] in order to obtain the uniqueness and the stability of the Nash equilibrium. Note however, that our method to establish the convergence gives in addition the **monotonicity** of the convergence, which is not a direct corollary of the approach in Proposition 4.1 of [2].

Next we consider the parallel update approach.

Parallel update: Consider a dynamic model of best reply where both users update at the same moment : $(y^1(0), y^2(0))$ is the original flow configuration, at the first step the flow configuration becomes $(y^1(1), y^2(1))$ where $y^1(1)$ is the best reply to $y^2(0)$ and $y^2(1)$ is the best reply to $y^1(0)$, and so on. Then the system also converges over time. Indeed we can divide this model into two parts:

- the sequence $\{(y^1(2p), y^2(2p+1))\}_{p\in\mathbb{N}}$

- and the sequence $\{(y^1(2p+1), y^2(2p))\}_{p\in\mathbb{N}}$

both sequences behave like our first model, then we obtain the convergence of this model but the convergence goes twice less fast than the previous algorithm, since we make twice more computations.

Convergence of a greedy algorithm with relaxation An algorithm frequently used in information technology, of which ESS is a special case is the greedy algorithm with relaxation (see e.g. [1, 4]). The users update their flow with a convex combination between the best reply and the previous action. More precisely, if $\overline{y}^1(n)$ is the best reply of player 1 to $y^2(n-1)$ and $\overline{y}^2(n)$ is the best reply of player 2 to $y^1(n)$, then the nth action used by player i is

$$y^i(n) = \lambda^i \overline{y}^i(n) + (1-\lambda^i) y^i(n-1)$$

where λ^i is a relaxation factor within $(0,1]$. *Lemma 7.3.2* also holds since the updates of a user will still increase or decrease monotonically. Hence this algorithm also converges to equilibrium.

Other cost functions
By computing numerically the different steps of our dynamic model, we have observed that the ESS seems to converge for all the cost functions satisfying condition B in [9], we have tried. Nevertheless we have not been able to apply the same type of monotonic argument we use here, to obtain the proof for more general cost functions.

In the following section, we give the result on uniqueness of the equilibrium and show the convergence of the ESS for a network made up of N processors.

7.4 Network with N users

Before to deal with a specific network with N processors, we present a result of uniqueness of Nash equilibrium.

7.4.1 Uniqueness of Nash equilibrium

We present in this section an extension of the *Theorem 2.1* of [9] on the uniqueness of Nash equilibrium in networks made up of parallel links.

We consider a model where N users share a network with parallel links and where any two users may have a different set of links available. We denote the set of links available for user $i \in \mathcal{I}$, by $\mathcal{L}^i$. Obviously we have $\mathcal{L} = \cup_{i\in\mathcal{I}} \mathcal{L}^i$.

We consider cost functions which satisfy the following assumptions:

Assumptions:

- $J^i(\mathbf{x}) = \sum_{l\in\mathcal{L}^i} J^i_l(\mathbf{x}^i_l, x_l)$,
- $J^i_l : [0,\infty)^2 \to [0,\infty)$ is increasing in each of its two arguments and is continuously differentiable.

7.4.1 Lemma. *In a network of parallel links, where each user may have a different set of links, there exists a unique Nash equilibrium*

Proof: A direct extension of [9, Thm 2.1]. □

Due to this lemma we will have the uniqueness of the equilibrium in the network presented below.

7.4.2 A network with N processors: uniqueness

We consider a network of N processors arranged in loop, each processor is in communication with its successor (Figure 7.3). To each processor we associate a user, then this network is shared by N users. Again, index i refer to the user i, and the processor i to which jobs of user i arrive. As previously, each user i has a certain amount of jobs ϕ^i to process. These jobs can be processed either in their "arrival" processor i or in the following one. Then the decision of a user is the quantity of jobs to process in "its processor" and the quantity to process in the following one. Therefore each link between two processors is used only by one user, and each processor is used by two users.

The successor of a processor i is denoted by $succ(i)$ and $isucc(i)$ denotes the link between i and its successor.

7.4.2. Remark. Note that when restricting to two processors only, we get a network that resembles the one studied at the previous section, yet the networks do not coincide. The difference is that in the previous section the link between the processors is bi-directional (it may represent a bus) and thus the flow in one direction influences the cost of traffic in the other direction. In this section the links between processors are all unidirectional (which is typical for fiber optics communications).

As in the previous sections, we can transform this network into an equivalent routing game (Figure 7.4). It can also be seen as a parallel link network. Indeed consider a network made up of N parallel links between a source and a destination. As previously we keep the same indices for the links and the users. Any link $i \in \mathcal{I}$ may be borrowed by only two users i and $i-1$ (if $i = 1$, then $i - 1 := N$), the cost on link i for user i is

$$J_i^i(\mathbf{x}_i^i, x_i) = \mathbf{x}_i^i f_i(x_i)$$

and for user $i-1$,

$$J_i^{i-1}(\mathbf{x}_i^{i-1}, x_i) = \mathbf{x}_i^{i-1}(f_i(x_i) + f_{(i-1)i}(\mathbf{x}_i^{i-1})).$$

The total cost incurred by user i is then $J^i(\mathbf{x}) = J_i^i(\mathbf{x}_i^i, x_i) + J_{i+1}^i(\mathbf{x}_{i+1}^i, x_{i+1})$.

In order to apply *Lemma 7.4.1* to this network and obtain the uniqueness of the equilibrium, for all $i \in \mathcal{I}$, the f_i's and the $f_{(i-1)i}$'s have to be increasing and continuously differentiable.

7.4.3 Convergence of ESS

To deal with the convergence of the ESS, we assume again Assumption ($\mathcal{A}$) to which we add the following one:

Assumption $(\mathcal{B})$ **:**

- p^i and $p_{i,succ(i)}$ do not depend on i, i.e. there are some constants p and p' such that $\forall i \in \mathcal{I}$, $p = p^i$, $p' = p_{i,succ(i)}$.

As previously the vector $\mathbf{x}$, denotes the flow configuration, $\mathbf{x} = (\mathbf{x}^i_l)_{i\in\mathcal{I}, l\in\mathcal{I}\cup\{ij, j=succ(i)\}}$. Denoting y^j the rate of jobs that user j processes in processor j, the rate of jobs that the user processes in processor $succ(j)$ is $\phi^j - y^j$. Hence the flow configuration $\mathbf{x}$ is uniquely determined by the vector $y = (y^i)_{i\in\mathcal{I}}$. Hence we will express the cost functions only through the vector y.

The cost of a player j (its predecessor being i and its successor being k) is

$$J^j(y^i, y^j, y^k) = y^j f^j_j(x_j) + (\phi^j - y^j) f^j_{jk}(x_{jk}) + (\phi^j - y^j) f^j_k(x_k)$$

where x_j (resp. x_k) denotes the load of processor j (resp. k), that is $x_j = y^j + \phi^i - y^i$ (resp. $x_k = y^k + \phi^j - y^j$) and x_{jk} the flow going through link jk, *i.e.*, $x_{jk} = \phi^j - y^j$.

Note that the constraints are still linear, therefore in order that $\mathbf{x}$ is a Nash equilibrium, the vector y has to satisfy the Kuhn-Tucker conditions:

$$\forall j \in \mathcal{I},\ \exists \alpha^j \geq 0,\ \beta^j \geq 0 \text{ such that}$$

$$K^j(y^i, y^j, y^k) + \alpha^j - \beta^j = 0 \tag{7.4.1}$$

and

$$\alpha^j \cdot (y^j - \phi^j) = \beta^j \cdot y^j = 0$$

replacing $K^j(y^i, y^j, y^k)$ by its expression in (7.4.1) gives

$$y^j 2(2p + p') - y^i p - y^k p + C^j + \alpha^j - \beta^j = 0 \tag{7.4.2}$$

where $C^j = \phi^i p - \phi^j(p' + p) + q_j - q_{jk} - q_k$.

7.4.3. Remark. Note that a symmetric network fulfils this assumption. However, the assumption is also satisfied when the demands are different and when the q_k's and q_{jk}'s are different.

7.4.4 Lemma. *Assume Conditions* ($\mathcal{A}$) *and* ($\mathcal{B}$) *hold true. For any initial flow configuration, a best reply algorithm converges to the equilibrium.*

Proof: Recall that the vector $(y^i)_{i\in\mathcal{I}}$ uniquely determines the flow configuration $\mathbf{x}$. Denote $(y^1(0), \ldots, y^j(0), y^{j+1}(0), \ldots, y^N(0))$ the initial flow configuration. At first step, the first player chooses his best reply to the flow configuration $(y^2(0), \ldots, y^j(0), y^{j+1}(0), \ldots, y^N(0))$ of the other players, and the flow configuration at the end of this step is, $(y^1(1), y^2(0), \ldots, y^j(0), y^{j+1}(0), .$ At step 2, user 2 chooses his best reply, we get the flow configuration $(y^1(1), y^2(1), y^3(0) \ldots, y^j(0), y^{j+1}(0), \ldots, y^N(0))$, and so one. At step $kN + j$, user j chooses his best reply, y^j_k.

Let $\delta^i(n)$ be the difference $y^i(n) - y^i(n-1) =: \delta^i(n)$, let $j = succ(i)$ and $k = succ(j)$.

Due to Kuhn-Tucker conditions (7.4.2), we have for $j > 1$

$$\begin{aligned} K^j(y^i(n), y^j(n), y^k(n-1)) \quad &- \quad K^j(y^i(n-1), y^j(n-1), y^k(n-2)) \\ &= \quad \delta^j(n)2(2p+p') - \delta^i(n)p - \delta^k(n-1)p \\ &\qquad +\alpha^j(n) - \alpha^j(n-1) - \beta^j(n) + \beta^j(n-1) \\ &= \quad 0 \end{aligned}$$

if $\delta^j(n) > 0$, then $\alpha^j(n-1) = \beta^j(n) = 0$ therefore

$$\delta^j(n) \leq \frac{p}{2(2p+p')}(\delta^i(n) + \delta^k(n-1))$$

if $\delta^j(n) < 0$, then $\alpha^j(n) = \beta^j(n-1) = 0$ therefore

$$\delta^j(n) \geq \frac{p}{2(2p+p')}(\delta^i(n) + \delta^k(n-1))$$

it follows that in both cases

$$|\delta^j(n)| \leq \frac{p}{2(2p+p')}(|\delta^i(n)| + |\delta^k(n-1)|)$$

for $j = 1$, proceeding in a similar way, we obtain

$$|\delta^1(n)| \leq \frac{p}{2(2p+p')}(|\delta^N(n-1)| + |\delta^2(n-1)|)$$

Let $\delta = \max_{i\in\mathcal{I}} |\delta^i(1)|$
Hence we have

$$|\delta^1(2)| \leq \frac{p}{2p+p'}\delta$$

and

$$|\delta^2(2)| \leq \frac{p}{2(2p+p')}\delta(\frac{p}{2p+p'} + 1) < \frac{p}{2p+p'}\delta$$

by induction we obtain that $\forall i > 1$, $|\delta^i(2)| < \frac{p}{2p+p'}\delta$ and finally that

$$\forall i \in \mathcal{I},\ |\delta^i(n)| < \left(\frac{p}{2p+p'}\right)^{n-1}\delta$$

We conclude that $\lim_{n\to\infty} y(n) = y$, where $\forall i \in \mathcal{I},\ y^i$ is a best reply to y^{-i}, and then the vector $\mathbf{x}$ induced by y is the unique Nash equilibrium of the game. □

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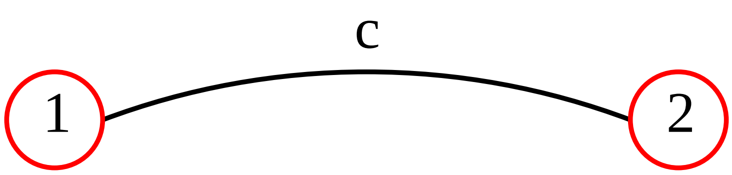

Figure 7.1: Two processors and a communication bus

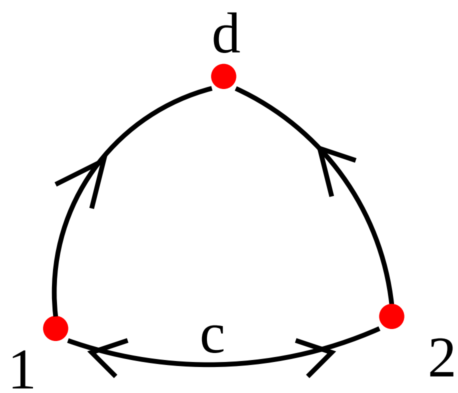

Figure 7.2: The corresponding network

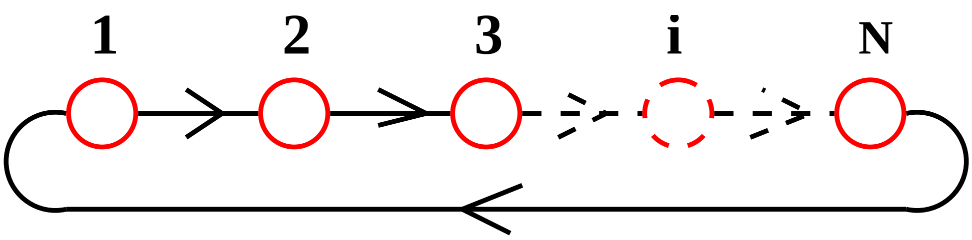

Figure 7.3: N processors in loop

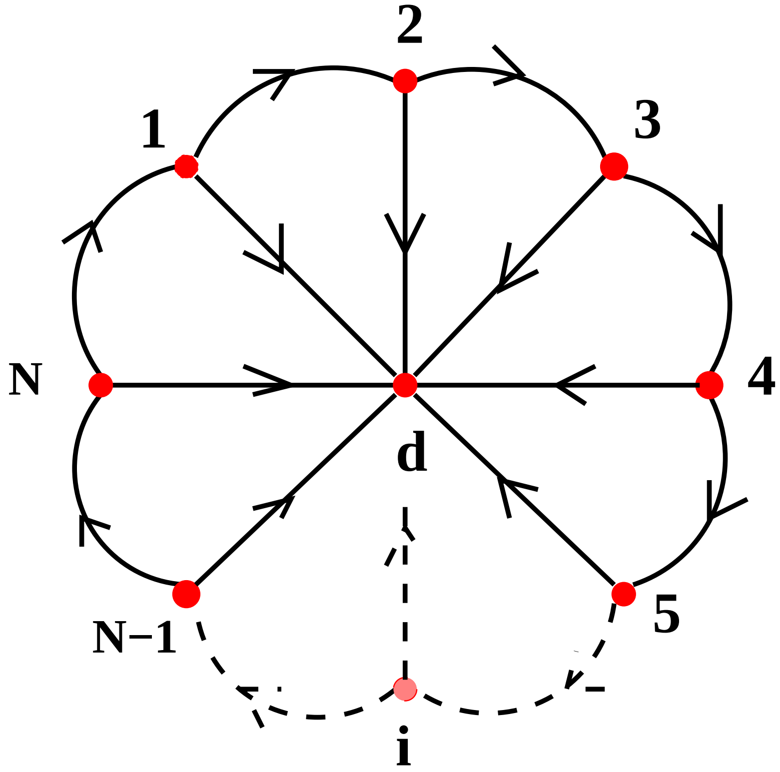

Figure 7.4: The corresponding network

Chapter 8

Competitive routing in multicast communications [1]

Abstract We consider competitive routing in multicast networks from a noncooperative game theoretical perspective. N users share a network, each has to send an amount of packets to a different set of addressees (each address must receive the same packets), to do this it has only to send one copy of a packet, the network making the duplications of the packets at appropriate nodes (depending on the chosen trees). The routing choice of a user is how to split its flow between different multicast trees. We present different criteria of optimization for this type of games. We treat two specific networks, establish the uniqueness of the Nash equilibrium in these networks, as well as the uniqueness of links' utilization at Nash equilibria for specific cost functions in networks with general topology. We also present a result of convergence to equilibria from an initial non equilibrium state.

8.1 Introduction

In view of the deregulation of the telecommunication market, it has been recognized that optimal decision making concerning the network operation (such as routing) at the level of service providers cannot be modelled in the framework of centralized optimization. The natural framework to study this issue is non-cooperative game theory, and the optimality concept is the Nash equilibrium (see e.g. [15, 17]).

Within this framework, we consider in this paper the optimal routing problems, in which each service provider (that will be called "user") has to determine which paths to use and how to split the flow of its subscribers between these paths. This problem, known as "competitive routing" has received much attention in the framework of point-to-point communications, see e.g. [10, 11, 15]. Related competitive models have also been studied in the context of road traffic even earlier, see e.g. [8].

The unit entity that is routed is called a *packet*. There are infinitely many packets that we model by a continuous flow. A finite number of users share the network.

[1] This work was done in collaboration with E. Altman and is under revision to appear in *Networks*.

Each one has to send an amount of packets (possibly of different sessions) from one or more sources to a set of (possibly source dependent) destinations at each point in time. The routing decision of a user consists of how to split its flow between various multicast virtual paths which are represented as trees. We consider in this paper not only the point-to-multipoint situation but also the case of multipoint-to-multipoint. Yet even in the latter, we shall assume that the routing from each source is performed using virtual paths which are represented as trees.

Into a tree, a user has only to send one copy of each packet, and the network will duplicate the information at appropriate nodes: At each node of the tree the network will duplicate the packets so that a copy of the packet goes through each outgoing link (which belong to the tree) of this node. This feature makes it impossible to use standard methods from games that arise in road traffic or in point-to-point communications (in which there is a single source and a single destination per each communications) that are based on flow conservation at each node (such as [15]).

Objective functions to be minimized in multicast competitive routing are also different from those that arise in unicast communications and in road traffic. We treat the case when the objective is to minimize a cost function that is obtained through the sum of link costs. In that case, our analysis can be used to relate the pricing of links (as a function of congestion) to the cost obtained at equilibrium. We also analyze two different types of cost related to delays. Surprisingly, in the case of multicast, the cost representing delay cannot be taken as a simple special case of the previous type of cost, as will be discussed.

Uniqueness of the equilibrium is of interest to forecast the flow configuration of a network. But, even in unicast networks, equilibria are often nonunique (see [15] for a simple example of multiple equilibria) so we cannot expect uniqueness to hold in multicast scenari neither. The challenge, as with the unicast case, is then to identify cases in which the equilibrium is unique. We obtain uniqueness of equilibrium in this paper for general cost functions for two specific topologies and for a general topology but specific cost functions.

We also establish in this paper the dynamic convergence to equilibrium from an initial non equilibrium state under some conditions on the costs and topology.

The structure of the paper is as follows. After introducing the mathematical model, we define in Section 8.2 the optimality concept of Nash equilibrium, establish its existence in our networking routing game and present explicitly three different criteria of optimization. We then study two networks with specific topologies (Sections 8.3.1 and 8.3.2), in which we establish the uniqueness of the Nash equilibrium for the three criteria presented; in Section 8.3.1, we also show the convergence to equilibrium of a best reply algorithm in the case where the network is shared by two users and give an example where there exist several distinct equilibria if the orthogonality of the strategy sets does not hold. In Section 8.4, we present an extended version of the well known Wardrop equilibria [20], relate it to Nash equilibrium with specific cost functions. We then obtain uniqueness of the equilibrium for a general topology. In Section 8.5 we present a numerical example. Finally, in Section 8.6 we apply the relation between Wardrop equilibria and Nash equilibria presented in Section 8.4 to a subclass of multicast networks and obtain convergence to equilibria for a class of dynamics. We conclude with a section that discusses further extensions and some remaining open problems.

8.2 Model

We consider a general network. We denote $\mathcal{N}$ the set of nodes, and $\mathcal{L} \subset \mathcal{N} \times \mathcal{N}$ the set of unidirectional links. The unit entity that is routed through the network is called a *packet*. Each packet j has a source and a set of addressees, we call this pair an origin-destinations pair (*sD pair*). We denote the origin, or the source by $s(j)$ and the set of destinations by $D(j)$. The network is shared by N users, we denote $\mathcal{I}$ the set of users. Each user $i \in \mathcal{I}$ has a set of sD pairs and for each sD pair a certain amount of packets to route from s to D, we call this amount the flow demand of user i for the pair sD and we denote it by ϕ^i_{sD}. A user with a single source (and several destinations) represents the, so called, session of point-to-multipoint. When a user has several sources and several destinations, the scenario is called a multipoint-to-multipoint communication. This can be treated as a multiple singlepoint-to-multipoint communication.

A tree a from $s \in \mathcal{N}$ to $D \subset \mathcal{N}$ is a subnetwork, constituting by a set of nodes and a set of unidirectional links; an example of a tree belonging to the pair sD is $a = (su, uv, vd_1, vd_2, uw, wx, xd_3, xd_4, xd_5)$, where $D = \{d_1, d_2, d_3, d_4, d_5\}$ and u, v, w and x are intermediate nodes, in this tree a duplication is made in u, in v and two in x, Figure 8.1.

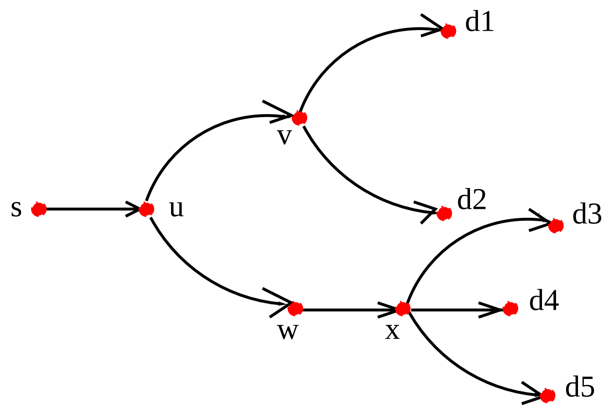

Figure 8.1: Example of a tree

$\mathcal{A}$ is the set of all possible trees. Each node of a tree (except for the source) has only one in-going link belonging to this tree. For a user $i \in \mathcal{I}$ we denote by $\mathcal{E}^i$ its set of sD pairs, by $\mathcal{A}^i$ its set of possible trees and by $\mathcal{A}^i_{sD}$ its set of possible trees which go from s to D. Then we have $\mathcal{A} = \cup_{i \in \mathcal{I}} \mathcal{A}^i$ and $\mathcal{A}^i = \cup_{sD \in \mathcal{E}^i} \mathcal{A}^i_{sD}$. (Note that we can have $\mathcal{A}^i_{sD_1} \cap \mathcal{A}^i_{sD_2} \neq \emptyset$, for some $i \in \mathcal{I}$, sD_1, $sD_2 \in \mathcal{E}^i$.)

Each user has to choose a (set of) tree(s) to route its packets. For $i \in \mathcal{I}$ and $a \in \mathcal{A}^i$, we denote by $\mathbf{x}^i_{(a)}$ the amount of flow sent into tree a by user i.

Denoting $x^i_{(a)l}$ the flow that user i sends into a tree a which goes through link l (this notation will only be used once), then by definition of our model we have

$$\forall i \in \mathcal{I},\ \forall a \in \mathcal{A}^i,\ \forall l \in a, \quad x^i_{(a)l} = \mathbf{x}^i_{(a)}$$

We introduce the incident indicator δ_{la}, $\delta_{la} = \begin{cases} 1 & \text{if } l \in a, \\ 0 & \text{otherwise.} \end{cases}$

The flow on a link $l \in \mathcal{L}$ sent by user $i \in \mathcal{I}$ is $x_l^i = \sum_{a \in \mathcal{A}^i} \delta_{la} \mathbf{x}^i_{(a)}$ and the aggregate flow on link l is

$$x_l = \sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \delta_{la} \mathbf{x}^i_{(a)} = \sum_{i \in \mathcal{I}} x_l^i$$

For every user $i \in \mathcal{I}$, its routing decision $\mathbf{x}^i \in \mathbb{R}^{n^i}$ $(n^i = \#\mathcal{A}^i)$ has to satisfy

$$\mathbf{x}^i \in \tilde{\mathcal{X}}^i = \{\mathbf{x}^i \in \mathbb{R}^{n^i} \mid \forall sD \in \mathcal{E}^i, \forall a \in \mathcal{A}^i_{sD}, \mathbf{x}^i_{(a)} \geq 0, \sum_{a \in \mathcal{A}^i_{sD}} \mathbf{x}^i_{(a)} = \phi^i_{sD}\}$$

We denote the set of all admissible flow configuration $\mathbf{x} = (\mathbf{x}^i)_{i \in \mathcal{I}}$ by $\mathcal{X}$, we call it the total strategy set. $\mathcal{X} \subset \tilde{\mathcal{X}} = \tilde{\mathcal{X}}^1 \times \tilde{\mathcal{X}}^2 \times \ldots \times \tilde{\mathcal{X}}^N$ is convex, compact and nonempty. $\tilde{\mathcal{X}}$ is called an orthogonal policy space.

Usually $\mathcal{X} = \tilde{\mathcal{X}}$, but we may add some extra constraints as link capacities and then $\mathcal{X} \subset \tilde{\mathcal{X}}$. [2]

Definition of Nash equilibrium The cost function of a user $i \in \mathcal{I}$ is denoted J^i, $J^i : \mathbb{R}^n \to [0, \infty]$ $(n = \sum_{i \in \mathcal{I}} n^i)$.

The aim of each user $i \in \mathcal{I}$ is to minimize its cost function (according to the strategy set), that is find a $\mathbf{x}^i$ such that

$$\mathbf{x}^i \in \min_{\mathbf{y}^i \in \tilde{\mathcal{X}}^i} \{J^i(\mathbf{x}^1, \ldots, \mathbf{y}^i, \ldots, \mathbf{x}^N) \mid (\mathbf{x}^1, \ldots, \mathbf{y}^i, \ldots, \mathbf{x}^N) \in \mathcal{X}\}$$

Let $(\mathbf{x}^{-i}, \mathbf{y}^i)$ be the flow configuration where user j $(j \neq i)$ uses strategy $\mathbf{x}^j$ and user i uses strategy $\mathbf{y}^i$.

A Nash equilibrium is a flow configuration where any unilateral deviation will not be profitable for the deviator, then it is the less restrictive condition in order that the flow configuration will be stable. A formal definition is

Definition: $\mathbf{x} \in \mathcal{X}$ is a *Nash equilibrium* if and only if

$$\forall i \in \mathcal{I},\ \forall \mathbf{y}^i\ s.t.\ (\mathbf{x}^{-i}, \mathbf{y}^i) \in \mathcal{X}, \quad J^i(\mathbf{x}) \leq J^i(\mathbf{x}^{-i}, \mathbf{y}^i) \tag{8.2.1}$$

Cost functions We denote the cost function of a link $l \in \mathcal{L}$ for a user $i \in \mathcal{I}$ by f_l^i, $f_l^i : [0, \infty) \to [0, \infty]$ depends only on the flow which goes through link l.

In the kind of model presented in this paper, we may have different criteria of optimization. Then we have different types of cost functions.

[2] Note that constraints such as link capacities may always be represented in the orthogonal policy space as well by introducing infinity for costs of policies that do not satisfy the constraints. But when doing so, we may loose the continuity of the cost due to jumps to infinity that may occur. As we shall see, such constraints may result in nonuniqueness of the equilibrium. However, there are some specific cost functions, such as that obtained from the expected delay in an M/M/1 queue, that include capacity constraints (by imposing infinite delays when capacity is attained) but have the property that the cost remains continuous everywhere. For such costs, we may often obtain the uniqueness of equilibria, see e.g. [15].

We consider two types of cost functions:
(A) Cost functions similar to tolls in road traffic
(B) Cost functions as delay

In case *(A)* each user wants to minimize its total cost (which is the sum of the cost functions on links over all the links which it uses). In the unicast case (one destination per source) or even in the case of multiple sources and single destination, there is no essential difference between "cost" and "delay": the cost can be taken as the delay.

This is not the case in multicast problems, and delay has to be treated differently. To see that, consider a single user in a simple network consisting of nodes (s, x, d_1, d_2); s is the source, d_1 and d_2 are two destinations. The links are sx, xd_1, xd_2 and there is a single tree that contains these three links. Assume that the (load dependent) cost of using each link is 1 unit. The total cost is then 3 since there are three links. But if the cost corresponds to the average delay then it is 2 units, since the delay between the source and each destination is 2 units. If we consider the total delay then it is 4 units (2 units between the source and each destination).

In case *(B)* we consider two different criteria of optimization, either each of the users wishes to minimize its total delay *(B.1)* (as in the example above) or it wishes to minimize its maximum delay over paths *(B.2)* (a path being a sequence of unidirectional links between a source and a destination, any path belongs to a tree). The latter type of criterion has been advocated and used in the point-to-point framework for ad-hoc networks, see [7].

For any tree a and all nodes $u \in a$, we denote by $a_{|u}$ the subtree of of a which begins at node u and by τ_{ua} the number of destinations of the subtree $a_{|u}$, for $a \in \mathcal{A}$, $\tau_{ua} = \#\{d \in D(a) \mid d \in D(a_{|u})\}$.

The following three cases of users' cost functions are considered:
(A) $J^i(\mathbf{x}) = \sum_{a \in \mathcal{A}^i} \mathbf{x}^i_{(a)} \sum_{l \in a} f^i_l(x_l)$
(B.1) $J^i(\mathbf{x}) = \sum_{a \in \mathcal{A}^i} \mathbf{x}^i_{(a)} \sum_{uv \in a} \tau_{va} f^i_{uv}(x_{uv})$
(B.2) $J^i(\mathbf{x}) = \max_{a \in \mathcal{A}^i, \mathbf{x}^i_{(a)} > 0, p \in a} \sum_{l \in p} f^i_l(x_l)$

Existence of Nash quilibrium When the strategy set is convex, compact and nonempty, and when for each user $i \in \mathcal{I}$, for all $\mathbf{x} \in \mathcal{X}$, the cost function $J^i(\mathbf{x})$ is continuous in $\mathbf{x}$ and convex in $\mathbf{x}^i$ for each fixed value of $\mathbf{x}^{-i}$, then a Nash equilibrium exists. Rosen [17, Thm 1] establishes the existence of a Nash equilibrium under similar conditions when the range of the cost functions does not include the value ∞. Orda *et al.* [15] give an extension of the proof when the value ∞ is allowed. Nevertheless, in order that at equilibrium the cost of each user be finite, we require the following assumption (assumption G5 in [15]), which we will assume all through the paper:

For every system flow configuration $\mathbf{x}$, *if not all costs are finite then at least one user with infinite cost can change its own flow configuration to make its cost finite.*

This assumption implies that at a Nash equilibrium the cost of a user is not ∞.

Note that in order to obtain the existence of a Nash equilibrium, the differentiability of the J^i's is not required. Then, if the f_l^i's are continuous and increasing (resp. convex) for cases *(A)* and *(B.1)* (resp. *(B.2)*), conditions of existence of equilibria are satisfied:

the J^i's are continuous in $\mathbf{x}$ and convex in $\mathbf{x}^i$ for each fixed value of $\mathbf{x}^{-i}$.

We shall furthermore impose frequently the following assumption to obtain uniqueness of equilibrium in the cases studied.
Assumption ($\mathcal{G}$): $f_l^i : [0,\infty) \to [0,\infty]$ is continuously differentiable (wherever finite), strictly increasing and convex (wherever finite).

Characterization of equilibria For cases *(A)* and *(B.1)*, for all $i \in \mathcal{I}$, $a \in \mathcal{A}$, we denote the partial derivative of $J^i(\mathbf{x})$ with respect to $\mathbf{x}^i_{(a)}$, $\frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x})$, $\nabla_i J^i$ is the gradient of J^i with respect to $\mathbf{x}^i$, it is a $\#\mathcal{A}^i$-dimensional vector and f' denotes the derivative of a function f which has only one argument. Hence

$$\frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x}) = \sum_{uv \in a} c^i_{uv} \left(x^i_{uv} {f^i_{uv}}'(x_{uv}) + f^i_{uv}(x_{uv}) \right), \text{ where } c^i_{uv} = \begin{cases} 1 & \text{if } J^i \in \textit{(A)} \\ \tau_{va} & \text{if } J^i \in \textit{(B.1)} \end{cases}$$

In case *(B.2)*, remark that for any $i \in \mathcal{I}$, J^i is convex in $\mathbf{x}^i$, continuous in $\mathbf{x}$ but no more differentiable.

8.2.1 Proposition. *For cases* (A) *and* (B.1)*, if $\mathcal{X} = \tilde{\mathcal{X}}$, the equilibrium condition is equivalent to for all $i \in \mathcal{I}$*

$$\exists \alpha^i = \alpha^i(\mathbf{x}),\ \alpha^i = (\alpha^i_{sD})_{sD \in \mathcal{E}^i}$$

such that for all $sD \in \mathcal{E}^i$ and $a \in \mathcal{A}^i_{sD}$

$$\mathbf{x}^i_{(a)} \geq 0; \quad \frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x}) - \sum_{sD | a \in sD} \alpha^i_{sD} \geq 0; \quad \left(\frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x}) - \sum_{sD | a \in sD} \alpha^i_{sD} \right) \mathbf{x}^i_{(a)} = 0 \tag{8.2.2}$$

These are Kuhn-Tucker conditions where α^i_{sD} is the Lagrange multiplier associated to the constraint $\phi^i_{sD} = \sum_{a \in \mathcal{A}^i_{sD}} \mathbf{x}^i_{(a)}$.
Proof: If $\mathcal{X} = \tilde{\mathcal{X}}$, equation (8.2.1) is equivalent to: $\forall i \in \mathcal{I}$, $\mathbf{x}^i$ is solution of

$$\min_{\mathbf{y}^i \in \tilde{\mathcal{X}}^i} J^i(\mathbf{x}^{-i}, \mathbf{y}^i) \tag{8.2.3}$$

Since $\tilde{\mathcal{X}}^i$ consists only of linear constraints, then Slater condition is trivially satisfied, moreover for all $i \in \mathcal{I}$, J^i is differentiable and convex. Therefore $\mathbf{x}$ is a solution of (8.2.3) if and only if $\mathbf{x}$ satisfies the Kuhn-Tucker conditions (8.2.2). □

We may also have a variational inequalities characterization of the equilibrium in case (A) and $(B.1)$ as in Gabay and Moulin [6] or Altman and Kameda [1]. Nevertheless it wont help us to resolve our problems since in neither of the networks we present, we do not have the diagonal dominance of the Jacobian matrix of the cost functions derivatives as required in [6] to obtain uniqueness of equilibrium. In contrast, the Kuhn-Tucker conditions will turn out to be useful in establishing the uniqueness.

8.2.2. Remark. Case *(B.2)*. For a user $i \in \mathcal{I}$, the decision of others ($\mathbf{x}^{-i}$) being fixed, for any tree $a \in \mathcal{A}^i$ and for any path $p \in a$, the cost of p ($\sum_{l\in p} f_l^i(x_l)$) is continuous and increasing in $\mathbf{x}^i_{(a)}$. If $\mathcal{X} = \tilde{\mathcal{X}}$ and $\#\mathcal{E}^i = 1$, then the strategy of user i which minimizes $J^i(\cdot, \mathbf{x}^{-i})$ will be a $\mathbf{x}^i$ such that
$\forall a, b \in \mathcal{A}^i, \text{s.t. } \mathbf{x}^i_{(a)} > 0$,

$$\max_{p\in a} \sum_{l\in p} f_l^i(x_l) \leq \max_{q\in b} \sum_{l\in q} f_l^i(x_l) \tag{8.2.4}$$

(if $\mathcal{X} \neq \tilde{\mathcal{X}}$, this may not hold anymore, since we can have a link saturated, for example). For this $\mathbf{x}$, typically $J^i(\mathbf{x})$ will not be differentiable.

Nevertheless, a characterization similar to Proposition 8.2.1 exists for the case *(B.2)* when dealing with subdifferentials.

8.2.3. Remark. If we replace the cost *(B.2)* by
$J^i(\mathbf{x}) = \max_{a\in\mathcal{A}^i, \mathbf{x}^i_{(a)}>0} \sum_{uv\in a} \tau_{va} f^i_{uv}(x_{uv})$, that is instead of minimizing its maximum delay over paths, a user will minimize its maximum delay over trees, then the equation (8.2.4) becomes $\forall a, b \in \mathcal{A}^i, \text{s.t. } \mathbf{x}^i_{(a)} > 0$,

$$\sum_{uv\in a} \tau_{va} f^i_{uv}(x_{uv}) \leq \sum_{uv\in b} \tau_{va} f^i_{uv}(x_{uv}) \tag{8.2.5}$$

A proof is given in Section 8.4, Proposition 8.4.5.

The equation (8.2.5) looks to be a natural extension of the characterization of the so called Wardrop [20] equilibrium to the context of trees in a multicast network. This is an equilibrium notion adapted from the context of road traffic in which a decision maker is a single infinitesimal packet (as opposed to our original definition in which the decision maker for a large number of packets is the so called "user"). More details and a result of uniqueness for these cost functions are given in Sections 8.4 and 8.6.

Comments on bidirectional links When we consider a bidirectional link uv, we assume that for every user $i \in \mathcal{I}$, $f^i_{uv} = f^i_{vu}$ and moreover the cost of this function depends only on the aggregative flow which goes trough this link, $x_{uv} = x_{uv\rightarrow} + x_{vu\rightarrow}$ ($x_{uv\rightarrow}$ represents the flow from u to v).

With this assumption, we know that a bidirectional link may be transformed into a network of unidirectional ones where some are of null cost, see [3, Appendix B]. Hence the existence of Nash equilibria also holds for networks with both unidirectional and bidirectional links (the conditions of existence allow links with constant cost).

8.3 Uniqueness of equilibrium: specific topologies

In this section we will establish the uniqueness of equilibrium for two specific networks, moreover in one case we present a result of convergence toward equilibrium of a simple dynamic when there are only two users.

8.3.1 A three nodes network

Consider a network with three nodes u, v and d, a bidirectional link between u and v and two unidirectional ones, one between u and d and the other between v and d (Figure 8.2). N users share this network. $\mathcal{I} = \mathcal{I}_I \cup \mathcal{I}_{II}$, where every user $i \in \mathcal{I}_I$ has to ship an amount ϕ^i of packets from u to v and d, and every user $i \in \mathcal{I}_{II}$ has to ship an amount ϕ^i of packets from v to u and d. To do this each user has two possible trees. A user $i \in \mathcal{I}_I$ has the tree (uv, ud) where the duplication is made in u and the tree (uv, vd) where the duplication is made in v. A user $i \in \mathcal{I}_{II}$ has the tree (vu, vd) where the duplication is made in v and the tree (vu, ud) where the duplication is made in u.

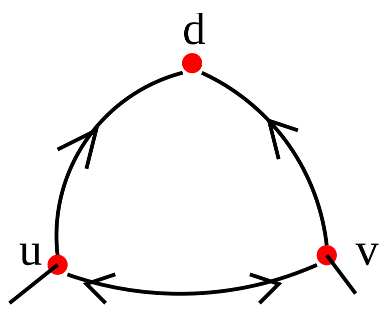

Figure 8.2: A three nodes network

With the notations previously introduced, this gives
$\mathcal{I} = \mathcal{I}_I \cup \mathcal{I}_{II}$, for $i \in \mathcal{I}_I$, $s^i = u$, $D^i = \{v, d\}$, $\mathcal{A}^i = \mathcal{A}^{\mathcal{I}_I} = \{(uv, ud), (uv, vd)\}$, for $i \in \mathcal{I}_{II}$, $s^i = v$, $D^i = \{u, d\}$, $\mathcal{A}^i = \mathcal{A}^{\mathcal{I}_{II}} = \{(vu, vd), (vu, ud)\}$ for $i \in \mathcal{I}$, $\mathcal{E}^i = \{s^i D^i\}$, $\phi^i_{s^i D^i} = \phi^i$

8.3.1. Remark. No matter which strategy a user $i \in \mathcal{I}_I$ (resp. $i \in \mathcal{I}_{II}$) chooses, it has to send an amount of flow ϕ^i from u to v (resp. from v to u), hence the flow which goes through the link connecting u and v will be the constant $\phi = \sum_{i \in \mathcal{I}} \phi^i$.

Uniqueness of Nash equilibrium

8.3.2 Lemma. *Assume $\mathcal{G}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Then in cases* (A) *and* (B.1)*, there is a unique Nash equilibrium, and in case* (B.2) *for any two Nash equilibria,* $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$*, we have* $\forall i \in \mathcal{I}$, $J^i(\tilde{\mathbf{x}}) = J^i(\bar{\mathbf{x}})$*, and moreover the links' utilization is unique (at equilibrium).*

Proof:
(A) From the *Remark* 8.3.1 it follows that our networking game is equivalent to a

classical routing game in a unicast network with two parallel links between a source s and a destination d, where a user $i \in \mathcal{I}_I$ (resp. $i \in \mathcal{I}_{II}$) has to ship a amount of flow ϕ^i from s to d with the cost functions on links $F_1^i(x) = f_{ud}^i(x)$, $F_2^i(x) = f_{vd}^i(x)$.

In such a network, under the assumption $\mathcal{G}$ it is known that there is a unique Nash equilibrium $\mathbf{x}$ (see [15, Thm 2.1]).

Then at equilibrium, the cost for a user $i \in \mathcal{I}_I$ is:

$$\begin{aligned} J^i(\mathbf{x}) &= \mathbf{x}^i_{(uv,ud)}(f^i_{uv}(x_{uv}) + f^i_{ud}(x_{ud})) + \mathbf{x}^i_{(uv,vd)}(f^i_{uv}(x_{uv}) + f^i_{vd}(x_{vd})) \\ &= \phi^i f^i_{uv}(\phi) + \mathbf{x}^i_{(uv,ud)} f^i_{ud}(x_{ud}) + \mathbf{x}^i_{(uv,vd)} f^i_{vd}(x_{vd}) \end{aligned}$$

and for a user $i \in \mathcal{I}_{II}$

$$J^i(\mathbf{x}) = \phi^i f^i_{vu}(\phi) + \mathbf{x}^i_{(vu,vd)} f^i_{vd}(x_{vd}) + \mathbf{x}^i_{(vu,ud)} f^i_{ud}(x_{ud})$$

(B.1) In this case, the game is also equivalent to a classical routing game in a network of two parallel links, but due to the specificity of the total delay, we are faced to a change in the cost functions on links of the network of parallel links, they have to be for $i \in \mathcal{I}_I$ $F_1^i(x) = f_{ud}^i(x) + f_{uv}^i(\phi)$, $F_2^i(x) = f_{vd}^i(x) + 2f_{uv}^i(\phi)$ and for $i \in \mathcal{I}_{II}$ $F_1^i(x) = f_{ud}^i(x) + 2f_{vu}^i(\phi)$, $F_2^i(x) = f_{vd}^i(x) + f_{vu}^i(\phi)$.

Therefore, in this case again, the equilibrium is unique, and at equilibrium $\mathbf{x}$ the total delay for user i is:
if $i \in \mathcal{I}_I$

$$J^i(\mathbf{x}) = \mathbf{x}^i_{(uv,ud)}(f^i_{uv}(\phi) + f^i_{ud}(x_{ud})) + \mathbf{x}^i_{(uv,vd)}(2f^i_{uv}(\phi) + f^i_{vd}(x_{vd}))$$

if $i \in \mathcal{I}_{II}$

$$J^i(\mathbf{x}) = \mathbf{x}^i_{(vu,vd)}(f^i_{vu}(\phi) + f^i_{vd}(x_{vd})) + \mathbf{x}^i_{(vu,ud)}(2f^i_{vu}(\phi) + f^i_{ud}(x_{ud}))$$

(B.2) The cost function of a user $i \in \mathcal{I}$ is

$$J^i(\mathbf{x}) = max_{a \in \mathcal{A}^i, \mathbf{x}^i_{(a)} > 0, p \in a} \sum_{l \in p} f^i_l(x_l)$$

According to *Remark* 8.2.2 at equilibrium $\mathbf{x}$ for $i \in \mathcal{I}_I$ we have
if $\mathbf{x}^i_{(uv,ud)} > 0$

$$\max\{f^i_{uv}(\phi), f^i_{ud}(x_{ud})\} \le f^i_{uv}(\phi) + f^i_{vd}(x_{vd}) \tag{8.3.1}$$

and if $\mathbf{x}^i_{(uv,vd)} > 0$

$$f^i_{uv}(\phi) + f^i_{vd}(x_{vd}) \le \max\{f^i_{uv}(\phi), f^i_{ud}(x_{ud})\} \tag{8.3.2}$$

Firstly we prove that for any two equilibria, $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$, we have for all $i \in \mathcal{I}$, $J^i(\bar{\mathbf{x}}) = J^i(\tilde{\mathbf{x}})$. Suppose that there exists $i \in \mathcal{I}_I$ such that $J^i(\bar{\mathbf{x}}) > J^i(\tilde{\mathbf{x}})$.

If $\tilde{\mathbf{x}}^i_{(a)} > 0$ for all $a \in \mathcal{A}^i$, we obtain that

$$f^i_{uv}(\phi) + f^i_{vd}(\tilde{x}_{vd}) < f^i_{uv}(\phi) + f^i_{vd}(\bar{x}_{vd})$$

and

$$max\{f^i_{uv}(\phi), f^i_{ud}(\tilde{x}_{ud})\} < max\{f^i_{uv}(\phi), f^i_{ud}(\bar{x}_{ud})\}$$

from which it follows that $\bar{x}_{vd} > \tilde{x}_{vd}$ and $\bar{x}_{ud} > \tilde{x}_{ud}$. This is impossible since $\bar{x}_{vd} + \bar{x}_{ud} = \tilde{x}_{vd} + \tilde{x}_{ud}$.

If there exists $a \in \mathcal{A}^i$ such that $\tilde{\mathbf{x}}^i_{(a)} = 0$, assume that $a = (uv, vd)$ (the other case is similar), then we obtain that

$$max\{f^i_{uv}(\phi), f^i_{ud}(\tilde{x}_{ud})\} < max\{f^i_{uv}(\phi), f^i_{ud}(\bar{x}_{ud})\}$$

Then $\bar{x}_{ud} > \tilde{x}_{ud}$, $f^i_{ud}(\bar{x}_{ud}) > f^i_{uv}(\phi)$ and there exists $j \in \mathcal{I}$ such that $\bar{x}^j_{ud} > \tilde{x}^j_{ud}$, which implies that $\tilde{x}_{vd} > \bar{x}_{vd}$ and $\tilde{x}^j_{vd} > \bar{x}^j_{vd}$.

If $j \in \mathcal{I}_{II}$, therefore $\tilde{\mathbf{x}}^j_{(vu,vd)} > \bar{\mathbf{x}}^j_{(vu,vd)}$ (we have also $\bar{\mathbf{x}}^j_{(vu,ud)} > \tilde{\mathbf{x}}^j_{(vu,ud)}$). Finally we obtain that

$$J^j(\tilde{\mathbf{x}}) \geq \max\{f^j_{vu}(\phi), f^j_{vd}(\bar{x}_{vd})\} \geq J^j(\bar{\mathbf{x}})$$

and

$$J^j(\bar{\mathbf{x}}) > f^j_{vu}(\phi) + f^j_{ud}(\tilde{x}_{ud}) \geq J^j(\tilde{\mathbf{x}})$$

A contradiction. Then $\forall a \in \mathcal{A}^i$, $\tilde{\mathbf{x}}^i_{(a)} > 0$.

If $j \in \mathcal{I}_I$, we obtain a similar result replacing $\mathbf{x}^j_{(vu,ud)}$ by $\mathbf{x}^j_{(uv,ud)}$ and $\mathbf{x}^j_{(vu,vd)}$ by $\mathbf{x}^j_{(uv,vd)}$.

Therefore for all $i \in \mathcal{I}_I$, $J^i(\bar{\mathbf{x}}) \leq J^i(\tilde{\mathbf{x}})$. Interchanging $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$ we obtain that $J^i(\bar{\mathbf{x}}) = J^i(\tilde{\mathbf{x}})$, a similar result holds for users $i \in \mathcal{I}_{II}$.

The uniqueness of the links utilization at equilibrium is a trivial implication of $J^i(\bar{\mathbf{x}}) = J^i(\tilde{\mathbf{x}})$ for all $i \in \mathcal{I}$. □

8.3.3. Remark. For the cases *(A)* and *(B.1)*, if the cost functions on links are the same for all users ($f^i_l = f_l$) and if all users have the same amount of packets to ship ($\phi^i = \phi$), it follows from [15, Lem 3.1] applied to the equivalent unicast networks defined in the proof of *Lemma 8.3.2*, that $\mathbf{x}^i_{(a)} = x_{(a)}/\#\mathcal{I}_m$, $\forall m \in \{I, II\}$, $i \in \mathcal{I}_m$, $a \in \mathcal{A}^{\mathcal{I}_m}$.

$\mathcal{X} \neq \tilde{\mathcal{X}}$: an example with distinct equilibria Consider the previous network shared by two users, I and II, each user i has an amount of flow of 1 to send from s^i to D^i and has the following cost functions:
$f^i_{uv}(x) = f_{uv}(x) = x$, $f^i_{ud}(x) = f_{ud}(x) = x$ and $f^i_{vd}(x) = f_{vd}(x) = 2x$.

Moreover the capacity of the link ud is limited: $x_{ud} \leq 1$. Then we have

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \forall i \in \mathcal{I}, \forall a \in \mathcal{A}^i,\ \mathbf{x}^i_{(a)} \geq 0,\ \sum_{a \in \mathcal{A}^i} \mathbf{x}^i_{(a)} = \phi, \quad x_{ud} \leq 1\}$$

We define the flow configuration $\tilde{\mathbf{x}}$ by $\tilde{\mathbf{x}}_{(uv,ud)} = 1$, $\tilde{\mathbf{x}}_{(uv,vd)} = 0$, $\tilde{\mathbf{x}}_{(vu,vd)} = 1$ and $\tilde{\mathbf{x}}_{(vu,ud)} = 0$ and the flow configuration $\bar{\mathbf{x}}$ by $\bar{\mathbf{x}}_{(uv,ud)} = 0$, $\bar{\mathbf{x}}_{(uv,vd)} = 1$, $\bar{\mathbf{x}}_{(vu,vd)} = 0$ and $\bar{\mathbf{x}}_{(vu,ud)} = 1$.

Obviously $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$ are both Nash equilibria, since in the first case user I has no interest to change its flow configuration, and user II has no other choice that send all its flow into the tree (vu, vd) and inversely for the second case (in fact every convex combination of these two flow configurations is a Nash equilibrium).

Then *Lemma 8.3.2* is false if $\mathcal{X} \neq \tilde{\mathcal{X}}$.

Convergence to Nash equilibrium In this paragraph we present a classical algorithm of updating flow configuration based on best reply strategy (see [15, Sec. 2.4] or [21, Sec. 2] for a detailed description of this algorithm). We show, under the restriction of twice-differentiability of the f_l^i's, that this algorithm converges to the unique equilibrium.

In the networking game presented previously, each user has only one decision which is the quantity of flow to send through tree (uv, vd), $\mathbf{x}_{(uv,vd)}$, since the other variable is $(\phi^I - \mathbf{x}_{(uv,vd)})$, similarly the decision of user II is $\mathbf{x}_{(vu,vd)}$. We denote y the strategy of user I and z this of user II.

Let the sequence $(y_n, z_n)_{n\geq 0}$ be defined as follows:
- step 0: (y_0, z_0) is a given initial flow configuration
- step 1: I updates its flow in order to minimize its cost function the strategy of II, z_0, being given, the resulting flow configuration is (y_1, z_1), where y_1 is a best reply to z_0 $(= z_1)$
- step 2: II updates its flow in order to minimize its cost function the strategy of I, y_1, being given, the resulting flow configuration is (y_2, z_2), where z_2 is a best reply to y_1 $(= z_2)$

...
- step $2n-1$: I updates, the resulting flow configuration is (y_{2n-1}, z_{2n-1}), where y_{2n-1} is a best reply to z_{2n-2} $(= z_{2n-1})$
- step $2n$: II updates, the resulting flow configuration is (y_{2n}, z_{2n}), where z_{2n} is a best reply to y_{2n-1} $(= y_{2n})$.

8.3.4. Remark. A user does not have necessarily a unique best reply to a strategy of the other user, it can choose any of its best reply strategies.

8.3.5. Remark. We note that at each step of the update, all constraints are satisfied. Although we assumed a round robin update, the convergence of this algorithm will imply that of any greedy algorithm in which best responses of the users are update infinitely often at different times. Indeed, as long as one user does not update its action, if the other user updates its actions several times then all updates will be the same. Therefore this update algorithm allows to describe natural greedy asynchronous behavior.

Assumption ($\mathcal{C}$): $\forall i \in \mathcal{I}$, $l \in \mathcal{L}$, f_l^i is twice-differentiable wherever finite.

8.3.6 Lemma. *Assume $\mathcal{G}$, $\mathcal{C}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Given any initial flow configuration, the sequence generated by a succession of best reply strategies will converge to equilibrium in cases (A), (B.1) and (B.2).*

Proof:
(A) It is sufficient to remark that the networking game is supermodular, *i.e.*,

$$\frac{\partial^2}{\partial y \partial z} J^i(y, z) \geq 0 \quad \forall (y, z) \in [0, \phi^I] \times [0, \phi^{II}]$$

and the result follows from Yao [21, Thm 2.3].

The supermodularity of G_1 is trivial. Indeed we have

$$J^I(y,z) = y(f^I_{uv}(\phi) + f^I_{vd}(y+z)) + (\phi^I - y)(f^I_{uv}(\phi) + f^I_{ud}(\phi - y - z))$$

Then

$$\begin{aligned}\frac{\partial^2}{\partial y \partial z} J^I(y,z) &= {f^I_{vd}}'(y+z) + {f^I_{ud}}'(\phi - y - z) \\ &\quad + y{f^I_{vd}}''(y+z) + (\phi^I - y){f^I_{ud}}''(\phi - y - z) \\ &\geq 0 \qquad \forall (y,z) \in [0,\phi^I] \times [0,\phi^{II}]\end{aligned}$$

where the inequality is due to assumptions $\mathcal{G}$ and $\mathcal{C}$ (we obtain the result for J^{II} by a similar way).

(B.1) similar to *(A)*.

(B.2) The proof is based on the fact that the sequence $(y_n, z_n)_{n\geq 2}$ is monotone, more precisely we have $z_n \geq z_{n-1},\ y_n \geq y_{n+1}$ or $z_{n-1} \geq z_n,\ y_{n+1} \geq y_n$. The boundedness of the flows will imply the convergence of the sequence to a limit which will be an equilibrium since the cost functions are continuous.

We will only show

$$z_n \geq z_{n-1} \implies y_n \geq y_{n+1} \quad \forall n \geq 2 \text{ (n even)} \tag{8.3.3}$$

1) $\phi^I > y_n = y_{n-1} > 0$, due to *Remark* 8.2.2, we have

$$f^I_{uv}(\phi) + f^I_{vd}(y_{n-1} + z_{n-1}) = \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n-1} - z_{n-1})\} \tag{8.3.4}$$

Recall that $y_n = y_{n-1}$. By hypothesis, we obtain that

$$f^I_{uv}(\phi) + f^I_{vd}(y_n + z_n) \geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_n - z_n)\}$$

If $y_{n+1} = 0$, (8.3.3) is checked. Suppose that $y_{n+1} = \phi^I$, then we have

$$\begin{aligned} f^I_{uv}(\phi) &+ f^I_{vd}(y_{n+1} + z_{n+1}) > f^I_{uv}(\phi) + f^I_{vd}(y_n + z_n) \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_n - z_n)\} \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n+1} - z_{n+1})\}\end{aligned}$$

but in order to $y_{n+1} = \phi^I$ be a best reply to z_n, it has to satisfy (cf. *Remark* 8.2.2)

$$f^I_{uv}(\phi) + f^I_{vd}(y_{n+1} + z_{n+1}) \leq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n+1} - z_{n+1})\}$$

A contradiction.

It remains to consider the case where $\phi^I > y_{n+1} > 0$, in this case in order to y_{n+1} be a best reply to z_n, it is necessary that (8.3.4) holds also for $n+1$ (where $z_{n+1} = z_n$). Hence (8.3.3) is checked.

2) $y_n = \phi^I$, (8.3.3) is always true.

3) $y_n (= y_{n-1}) = 0$, we have

$$\begin{aligned} f^I_{uv}(\phi) + f^I_{vd}(y_{n+1} + z_{n+1}) &\geq f^I_{uv}(\phi) + f^I_{vd}(y_{n-1} + z_{n-1}) \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n-1} - z_{n-1})\} \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n+1} - z_{n+1})\} \end{aligned} \quad (8.3.5)$$

where the second inequality comes from the fact that $y_{n-1} = 0$ is a best reply to z_{n-1}. If $y_{n+1} > 0$, then (8.3.5) holds with a strict inequality, and y_{n+1} is not a best reply to z_{n+1}, then (8.3.3) is checked.

Implication (8.3.3) holds also when $\leq$ is substituted to $\geq$, and when we interchange y and z for n odd. The result follows. □

8.3.2 A four nodes network

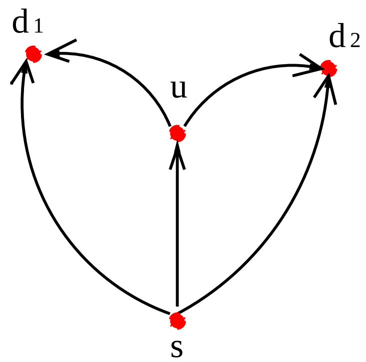

Figure 8.3: A four nodes network

Consider a network with four nodes s, u, d_1 and d_2 and unidirectional links between them, N users share this network, each user $i \in \mathcal{I}$ has an amount of packets ϕ^i to ship from the source s to the destination d_1 and also from s to d_2. To do this each user has two possible trees either the tree $t_d = (sd_1, sd_2)$ (d for direct) where the duplication is made in s or the tree $t_s = (su, ud_1, ud_2)$ (s for split) where the duplication is made in u, we call this network N_1 (Figure 8.3).

According to our mathematical notations we have
$\mathcal{I} = \{1, \ldots, N\}$, $\forall i \in \mathcal{I}$, $s^i = s$, $D^i = \{d_1, d_2\}$, $\mathcal{E}^i = \{s^i D^i\}, \phi^i_{s^i D^i} = \phi^i$ and $\mathcal{A}^i = \mathcal{A} = \{t_d, t_s\}$.

Uniqueness of Nash equilibrium

8.3.7 Lemma. *Assume $\mathcal{G}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Then in cases* (A) *and* (B.1)*, there is a unique Nash equilibrium, and in case* (B.2) *for any two Nash equilibria, $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$, we have*

$$\forall i \in \mathcal{I}, \quad J^i(\tilde{\mathbf{x}}) = J^i(\bar{\mathbf{x}})$$

and moreover the links' utilization is unique (at equilibrium).

Proof: We only present the proof for case *(A)*, the one for case *(B.1)* is similar, and those of *(B.2)* is identical to the one of *Lemma 8.3.2* replacing (8.3.1) and (8.3.2) by if $\mathbf{x}^i_{(t_s)} > 0$

$$f^i_{su}(x_{su}) + \max\{f^i_{ud_1}(x_{ud_1}), f^i_{ud_2}(x_{ud_2})\} \le \max\{f^i_{sd_1}(x_{sd_1}), f^i_{sd_2}(x_{sd_2})\}$$

and if $\mathbf{x}^i_{(t_d)} > 0$

$$\max\{f^i_{sd_1}(x_{sd_1}), f^i_{sd_2}(x_{sd_2})\} \le f^i_{su}(x_{su}) + \max\{f^i_{ud_1}(x_{ud_1}), f^i_{ud_2}(x_{ud_2})\}$$

where $x_{sd_1} = x_{sd_2}$ and $x_{su} = x_{ud_1} = x_{ud_2}$.

Suppose that there exist two distinct equilibria $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$, then both of them have to satisfy the conditions: for $\mathbf{x} = \tilde{\mathbf{x}},\ \mathbf{x} = \bar{\mathbf{x}}$

$$\exists \alpha = \alpha(\mathbf{x}),\ \alpha^T = (\alpha^i)_{i\in\mathcal{I}}$$

such that for all $i \in \mathcal{I}$ and $a \in \mathcal{A}$

$$\frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x}) - \alpha^i \ge 0; \quad \left(\frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x}) - \alpha^i\right) \mathbf{x}^i_{(a)} = 0 \tag{8.3.6}$$

Firstly we prove that $\forall a \in \mathcal{A},\ \bar{x}_{(a)} = \tilde{x}_{(a)}$. Suppose that there exists a tree $a \in \mathcal{A}$ such that $\tilde{x}_{(a)} > \bar{x}_{(a)}$, therefore there exists a user $i \in \mathcal{I}$ such that $\tilde{\mathbf{x}}^i_{(a)} > \bar{\mathbf{x}}^i_{(a)}$, denoting by b the other tree we have that $\bar{x}_{(b)} > \tilde{x}_{(b)}$ and $\bar{\mathbf{x}}^i_{(b)} > \tilde{\mathbf{x}}^i_{(b)}$ (since $\mathbf{x}^i_{(a)} + \mathbf{x}^i_{(b)} = \phi^i,\ \forall i \in \mathcal{I}$). But by construction of our model we have that

$$\forall l \in a, \quad x_{(a)} = x_l \quad \text{and} \quad \mathbf{x}^i_{(a)} = x^i_l$$

(and similarly for tree b). Therefore from (8.3.6) it follows that

$$\tilde{\alpha}^i = \frac{\partial}{\partial \mathbf{x}^i_{(a)}} J^i(\mathbf{x})(\tilde{\mathbf{x}}) = \sum_{l\in a}\left(\tilde{x}^i_l f^{i\prime}_l(\tilde{x}_l) + f^i_l(\tilde{x}_l)\right) > \sum_{l\in a}\left(\bar{x}^i_l f^{i\prime}_l(\bar{x}_l) + f^i_l(\bar{x}_l)\right) \ge \bar{\alpha}^i \tag{8.3.7}$$

due to the strict increase of f^i_l and the increase of its derivative $f^{i\prime}_l$.

But from the inequalities on tree b we obtain

$$\bar{\alpha}^i = \sum_{l\in b}\left(\bar{x}^i_l f^{i\prime}_l(\bar{x}_l) + f^i_l(\bar{x}_l)\right) > \sum_{l\in b}\left(\tilde{x}^i_l f^{i\prime}_l(\tilde{x}_l) + f^i_l(\tilde{x}_l)\right) \ge \tilde{\alpha}^i \tag{8.3.8}$$

(8.3.8) contradicts (8.3.7), hence for $a = t_d,\ t_s$ we have that

$$\tilde{x}_{(a)} = \bar{x}_{(a)}$$

It remains us to show that $\forall i \in \mathcal{I}$, $a \in \mathcal{A}$, $\tilde{\mathbf{x}}^i_{(a)} = \bar{\mathbf{x}}^i_{(a)}$, which is trivial. Indeed suppose that there exists a user $i \in \mathcal{I}$ such that $\tilde{\mathbf{x}}^i_{(a)} > \bar{\mathbf{x}}^i_{(a)}$, then (8.3.7) and (8.3.8) are still valid and the conclusion follows. $\square$

Equivalent unicast network The four nodes network presented in this section may be transformed in a equivalent parallel links unicast network. One more time we will only treat the case *(A)*, case *(B.1)* being similar.

Indeed consider the network N_2 shared by N users, with two nodes, a source s and a destination d, and two parallel links between them l_s and l_d, each user $i \in \mathcal{I}$ has to ship an amount of packets ϕ^i from s to d. The cost functions of the links are

$$F^i_{l_s}(x_{l_s}) = f^i_{su}(x_{l_s}) + f^i_{ud_1}(x_{l_s}) + f^i_{ud_2}(x_{l_s})$$

$$\text{and } F^i_{l_d}(x_{l_d}) = f^i_{sd_1}(x_{l_d}) + f^i_{sd_2}(x_{l_d})$$

in order that the cost of the link l_s (resp. l_d) be the same as the cost of the tree t_s (resp. t_d) in the original model.

The total cost function for a user $i \in \mathcal{I}$ is

$$J^i(x) = x^i_{l_s} F^i_{l_s}(x_{l_s}) + x^i_{l_d} F^i_{l_d}(x_{l_d})$$

In such a network it is known that there is a unique Nash equilibrium (see Orda, Rom and Shimkin [15, Thm 2.1]) and moreover if the users are symmetric, that is the cost functions on links are the same for all users ($f^i_l = f_l$) and all users have the same amount of packets to ship from s to d ($\phi^i = \phi$) then this equilibrium is symmetric, that is $\mathbf{x}^i_l = \mathbf{x}^j_l$ for all $i,\ j \in \mathcal{I}$ and $l \in \mathcal{L}$ (then we have that $\mathbf{x}^i_l = \frac{x_l}{N}$) ([15, Lem 3.1]).

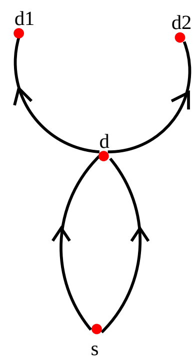

Figure 8.4: Equivalent unicast network

Now we construct a new network, N_3, by adding to this network two extra nodes d_1 and d_2 and two links dd_1 and dd_2 of null cost (Figure 8.4). Assume that the packets which arrive in d are duplicated in order that one copy goes into dd_1 and the other one into dd_2. The results obtained for N_2 are still valid in this new network. Since this network (N_3) is equivalent to N_1, therefore the results (uniqueness of the Nash equilibrium and symmetry of this equilibrium for symmetrical users) are valid for the original network.

8.4 Uniqueness of equilibrium: general topology

8.4.1 Extended Wardrop equilibrium

Wardrop equilibrium is the concept of optimization in a network shared by an infinity of users where the decision of any user has a negligible influence on the others' decisions (it is typically the case in road traffic). We call this model a large routing game.

We assume that the cost functions on the links are the same for any user and we group the users in classes according to their sD pair, therefore into a class $i \in \mathcal{I}$, any user $u \in i$ has the same sD pair and the same set of possible trees $\mathcal{A}^i$. We denote the cost (or delay) for a user $i \in \mathcal{I}$ of a tree $a \in \mathcal{A}^i$ when the strategy $\mathbf{x}$ is used by $F^i_{(a)}(\mathbf{x})$.

We define the extended Wardrop equilibrium through the condition
$\forall i \in \mathcal{I}, \forall a, b \in \mathcal{A}^i$,

$$\mathbf{x}^i_{(a)} > 0 \implies F^i_{(a)}(\mathbf{x}) \leq F^i_{(b)}(\mathbf{x})$$

We know that, under assumption $\mathcal{G}$, in unicast networks links' utilization is unique at Wardrop equilibrium, this can be proved in two different ways (1) a variational inequalities approach, or (2) a transformation of the problem into another equivalent one in which we first transform the cost, and then consider a single entity that optimizes for everybody; the optimal routing is then equal to the equilibrium. For more details see e.g. [16].

We can still use the variational inequalities approach to show the uniqueness of links' utilization at Wardrop equilibrium in multicast networks, yet we cannot apply the second approach due to the presence of a factor depending of trees (τ_{va}) in the cost functions. We give here the proof of the uniqueness based on the variational inequalities approach for general topology.

Assumption: For any $l \in \mathcal{L}$, f_l is strictly increasing.
Notations:

Given the flow configuration $\mathbf{x} \in \mathcal{X}$, we denote the delay for a user of class $i \in \mathcal{I}$ of a tree $a \in \mathcal{A}^i$ by $F^i_{(a)}(\mathbf{x})$, that is

$$F^i_{(a)}(\mathbf{x}) = \sum_{uv \in a} \tau_{va} f_{uv}(x_{uv})$$

and the vector of delay by

$$F(\mathbf{x}) = \left[F^i_{(a)}(\mathbf{x})\right]_{i \in \mathcal{I}, a \in \mathcal{A}^i}$$

Let Γ be the incident matrix (see Appendix) and A be the n-dimensional vector $A(\mathbf{x}) = [A^i(\mathbf{x})]_{i \in \mathcal{I}}$, where $A^i(\mathbf{x})$ denotes the minimal delay over trees for a user of class i given the flow configuration $\mathbf{x}$, that is

$$A^i(\mathbf{x}) = \min_{a \in \mathcal{A}^i} F^i_{(a)}(\mathbf{x})$$

8.4.1 Lemma. $\mathbf{x} \in \mathcal{X}$ *is an extended Wardrop equilibrium if and only if* $\mathbf{x}$ *satisfies*

$$F(\mathbf{x}) - \Gamma A(\mathbf{x}) \geq 0, \qquad (F(\mathbf{x}) - \Gamma A(\mathbf{x})) \cdot \mathbf{x} = 0$$

$$\Gamma^T \mathbf{x} = \Phi, \qquad \mathbf{x} \geq 0 \tag{8.4.1}$$

Proof: We have just to note that the conditions (8.4.1) are equivalent to $\mathbf{x} \in \mathcal{X}$. □

8.4.2 Lemma. $\mathbf{x} \in \mathcal{X}$ *is an extended Wardrop equilibrium if and only if*

$$F(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0 \quad \forall \mathbf{y} \in \mathcal{X}. \tag{8.4.2}$$

Proof: Similar to the proof of [1, Lem. 3.2] ((8.4.2) holds if and only if $\mathbf{x}$ is solution of the linear program $\min_{\mathbf{y}} F(\mathbf{x}) \cdot \mathbf{y}$, $s.t.\, \Gamma^T \mathbf{y} = \Phi,\ \mathbf{y} \geq 0$). □

8.4.3 Lemma. *For arbitrary* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *(*$\mathbf{x} \neq \tilde{\mathbf{x}}$*), if* $F(\mathbf{x})$ *is finite or* $F(\tilde{\mathbf{x}})$ *is finite and if* $\exists l \in \mathcal{L}$ *such that* $x_l \neq \tilde{x}_l$*, then*

$$(\mathbf{x} - \tilde{\mathbf{x}}) \cdot [F(\mathbf{x}) - F(\tilde{\mathbf{x}})] > 0$$

Proof: Assume that $\exists l \in \mathcal{L}$ such that $x_l \neq \tilde{x}_l$. Then

$$\begin{aligned}
(\mathbf{x} &- \tilde{\mathbf{x}}) \cdot [F(\mathbf{x}) - F(\tilde{\mathbf{x}})] \\
&= \sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \left(x^i_{(a)} - \tilde{x}^i_{(a)}\right) \left(F^i_{(a)}(\mathbf{x}) - F^i_{(a)}(\tilde{\mathbf{x}})\right) \\
&= \sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \left(x^i_{(a)} - \tilde{x}^i_{(a)}\right) \times \left(\sum_{uv \in \mathcal{L}} \delta_{uva} \tau_{va} \left(f_{uv}(x_{uv}) - f_{uv}(\tilde{x}_{uv})\right) \right) \\
&= \sum_{i \in \mathcal{I}} \sum_{uv \in \mathcal{L}} \left[\left(x^i_{uv} - \tilde{x}^i_{uv}\right) \left(f_{uv}(x_{uv}) - f_{uv}(\tilde{x}_{uv})\right) \right] \times \left[\sum_{a \in \mathcal{A}^i} \tau_{va} \right] \\
&= \sum_{uv \in \mathcal{L}} \left[\left(x_{uv} - \tilde{x}_{uv}\right) \left(f_{uv}(x_{uv}) - f_{uv}(\tilde{x}_{uv})\right) \right] \times \left[\sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \tau_{va} \right]
\end{aligned}$$

Since $\tau_{va} \geq 1$, the result follows from the strict increase of the f_l's. □

From the two previous lemmas it follows

8.4.4 Lemma. *For any two Wardrop equilibria* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *for which all users have finite cost, we have*

$$\forall l \in \mathcal{L}, \quad x_l = \tilde{x}_l$$

Proof: See [1, Thm. 3.5]. □

8.4.2 Nash equilibrium

We come back to the framework presented in Section 8.2.

Assume that every user $i \in \mathcal{I}$ has only one sD pair and has the following cost function:

$$J^i(\mathbf{x}) = \max_{a \in \mathcal{A}^i, \mathbf{x}^i_{(a)} > 0} F^i_{(a)}(\mathbf{x})$$

where $F^i_{(a)}(\mathbf{x})$ is the cost of tree a for user i at $\mathbf{x}$, $F^i_{(a)}(\mathbf{x}) = \sum_{uv \in a} \tau_{va} f^i_{uv}(x_{uv})$ (cf. Section 8.2, *Remark* 8.2.3).

Then

8.4.5 Proposition. *Assume $\mathcal{G}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. $\mathbf{x}$ will be a Nash equilibrium if and only if $\forall i \in \mathcal{I}, \forall a, b \in \mathcal{A}^i$,*

$$\mathbf{x}^i_{(a)} > 0 \implies F^i_{(a)}(\mathbf{x}) \le F^i_{(b)}(\mathbf{x}) \tag{8.4.3}$$

Proof:
($\Longrightarrow$) Assume that equation (8.4.3) does not hold, then there exist $i \in \mathcal{I}$, $d, b \in \mathcal{A}^i$ such that

$$\mathbf{x}^i_{(d)} > 0 \text{ and } F^i_{(d)}(\mathbf{x}) > F^i_{(b)}(\mathbf{x})$$

We will show that there exists a vector $\mathbf{y}^i \in \mathcal{X}^i$ such that for all $a \in \mathcal{A}^i$ with $\mathbf{y}^i_{(a)} > 0$ we have $J^i(\mathbf{x}) > F^i_{(a)}(\mathbf{x}^{-i}, \mathbf{y}^i)$. This will contradict the fact that $\mathbf{x}$ is a Nash equilibrium.

Let C be the set $C = \{c \in \mathcal{A}^i \mid F^i_{(c)}(\mathbf{x}) = J^i(\mathbf{x}),\ \mathbf{x}^i_{(c)} > 0\}$ and m its cardinal, $m = \#C$ ($m \ge 1$). Construct $\mathbf{y}^i(\epsilon)$ as follows:
- $\mathbf{y}^i_{(b)}(\epsilon) = \mathbf{x}^i_{(b)} + m\epsilon$,
- for all $c \in C$, $\mathbf{y}^i_{(c)}(\epsilon) = \mathbf{x}^i_{(c)} - \epsilon$ and
- for all $a \notin C \cup \{b\}$ $\mathbf{y}^i_{(a)}(\epsilon) = \mathbf{x}^i_{(a)}$.

$\forall\, 0 \le \epsilon \le \max\{\mathbf{x}^i_{(c)}(c \in C), \mathbf{x}^i_{(b)}/m\}$, $\mathbf{y}^i(\epsilon)$ belongs to $\mathcal{X}^i$ and $\forall c \in C$, $F^i_{(c)}(\mathbf{x}) > F^i_{(c)}(\mathbf{x}^{-i}, \mathbf{y}^i(\epsilon))$ (due to assumption $\mathcal{G}$-$\forall l \in c$, $x_l > (\mathbf{x}^{-i}, \mathbf{y}^i(\epsilon))_l$). So it remains to prove that there exists an $\epsilon > 0$ such that

$$\forall a \in \mathcal{A}^i \backslash C,\ J^i(\mathbf{x}) > F^i_{(a)}(\mathbf{x}^{-i}, \mathbf{y}^i(\epsilon))$$

This holds due to $\mathcal{G}$. Indeed, since all f^i_l's are continuous and strictly increasing, we can find a δ such that $\forall a \in \mathcal{A}^i \backslash C$, $J^i(\mathbf{x}) > \sum_{uv \in a} \tau_{va} f^i_{uv}(x_{uv} + \delta)$. Therefore let $\epsilon = \delta/m$, and $\mathbf{y}^i(\epsilon)$ is the $\mathbf{y}^i$ expected.

($\Longleftarrow$) Equation (8.4.3) implies that $\forall a, b \in \mathcal{A}^i$ such that $\mathbf{x}^i_{(a)}, \mathbf{x}^i_{(b)} > 0$ we have $F^i_{(a)}(\mathbf{x}) = F^i_{(b)}(\mathbf{x})$. Hence

$$J^i(\mathbf{x}) = \max_{c \in \mathcal{A}^i, \mathbf{x}^i_{(c)} > 0} F^i_{(c)}(\mathbf{x}) = F^i_{(a)}(\mathbf{x})$$

A change in the flow configuration of user i will strictly increase (due to $\mathcal{G}$) one of the $F^i_{(a)}(\mathbf{x})$'s. Then $\mathbf{x}$ is a Nash equilibrium. $\square$

Assume now that $f^i_l = f_l$, for all $l \in \mathcal{L}$ and $i \in \mathcal{I}$. It follows from the above result:

8.4.6 Proposition. *For any $i \in \mathcal{I}$, let $J^i(\mathbf{x}) = \max_{a\in\mathcal{A}^i, \mathbf{x}^i_{(a)}>0} \sum_{uv\in a} \tau_{va} f_{uv}(x_{uv})$. Assume $\mathcal{G}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Then for any two Nash equilibria $\mathbf{x}$ and $\tilde{\mathbf{x}}$, we have $\forall i \in \mathcal{I}$, $J^i(\mathbf{x}) = J^i(\tilde{\mathbf{x}})$ and $\forall l \in \mathcal{L}$, $x_l = \tilde{x}_l$.*

In this section, we established a link between Wardrop equilibria and Nash equilibria. Precisely we saw that in a game where the players are the sD pairs and want to minimize their maximal delay (over trees), a Nash equilibrium will correspond to a Wardrop equilibrium of a game where each user want to minimize its own tree delay and vice-versa. Different authors dealt with connexions between Nash and Wardrop equilibria in unicast networks. For example Haurie and Marcotte [8] show that in a unicast network the asymptotic behavior of a Nash equilibrium yield to a flow configuration corresponding to Wardrop equilibrium. They consider a N-person routing game; with a method of replication of the players they obtain a game with an infinity of players; they show that the variational inequality characterizing Nash equilibria of the limit game satisfies the one characterizing Wardrop equilibrium of a "correponding" large routing game. Our approach is more similar to the one of Devarajan [5]; he considers a finite game where players are sD-pairs and they minimize a cost function which is the summation of the integral of the link cost's functions. He shows that a Nash equilibrium of this game is a Wardrop equilibrium of the corresponding large routing game. Nevertheless players' cost functions have no direct meaning, contrarily to ours. Many other references on the relations between the Nash and the Wardrop equilibria can be found in [16] see in particular Chapter 2.6.1.

8.5 Numerical example

We consider the example of Section 8.3.2 with a single source and two destinations. The costs of all links are taken to be linear. The cost of each direct link, sd_1, sd_2, is $2(x+1)$. The cost of the common link, su, as well as that of the individual links, ud_1, ud_2, are $1+x$ each.

(We have been told that linear costs for links are often used in networks for pricing purposes in France Telecom.) We consider the problem of minimizing the total cost. The global demand is ϕ and there are N symmetric users. Thus the demand per user is ϕ/N.

We look for a symmetric Nash equilibrium. We thus assume first that all players except for, say, player 1, send the same amount x over the tree that consists of direct links, t_d, and the rest over the tree with the common link, t_s. We compute the best response of player 1, who sends y over t_d. The cost for player 1 is

$$Z(y) = 4y(1 + (N-1)x + y) + 3(\phi - y)(1 + (N-1)(\phi - x) + \phi - y)$$

We now choose $\phi = 1$. The best response is given by

$$y = \frac{1}{14N}(2N - 7xN^2 + 7xN + 3)$$

The symmetric Nash equilibrium is then obtaining by equating $x = y$, which yields

$$x = y = \frac{1}{7}\left(\frac{2N+3}{N(N+1)}\right)$$

In Figure 8.5 we present this solution (the amount sent over t_d at equilibrium) as a function of the number of players. Figure 8.6 shows the ratio of the amount sent by a player over t_d and the amount sent over t_s as a function of N. We see that there is a limit of this ratio as N goes to infinity (which would correspond to the concept of Wardrop equilibrium in the context of multicast).

We finally depict in Figure 8.7 the sum of costs for all players as a function of N. This figure shows clearly that the Nash equilibrium becomes less and less efficient as the number of players grow. The lowest global cost is obtained as expected when there is a single player.

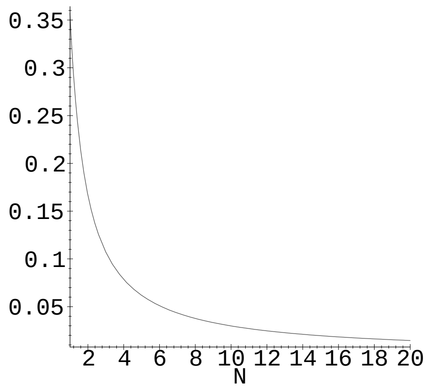

Figure 8.5: Amount sent over t_d at equilibrium as a function of N

8.6 Multicast networks with duplication at the source

We consider in this section a special class of degenerate multicast networks in which the source sends several copies of each packet: one copy for each destination, instead of duplicating packets within the network. This is useful when routers in the network do not have multicast capabilities, or when trees have a trivial form of disjoint paths with one common root.

The relation between Nash and Wardrop equilibria presented in Section 8.2 *Remark* 8.2.3 and in Section 8.4.2 appears to be of a special interest in this degenerate case.

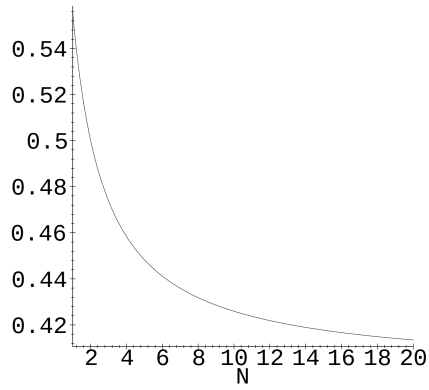

Figure 8.6: Ratio of the amount sent by a player over t_d and the amount sent over t_s as a function of N

Indeed in this context we are able to furnish results on convergence to equilibria of a class of updating algorithms (dynamics). We will briefly present the model, extract a potential function in the Wardrop case, which will allow us to apply results of convergence to equilibria due to Sandholm [18] and finally we will apply these results to the finite noncooperative routing game where cost functions are of the kind of those presented in Section 8.4.2.

The framework We consider a general network as defined in Section 8.2, except the fact that now each packet has an source-destination pair (*sd* pair), and we only restrict attention to paths to route the packets, and no more to trees. We keep the same notations as before, we only replace a tree a by a path p and the set of trees $\mathcal{A}^i$ by the set of paths $\mathcal{P}^i$, where as in Subsection 8.4.1, $i \in \mathcal{I}$ is a class of packets having the same *sd* pair; hence $\mathcal{P}^i$ denotes the set of admissible paths going from a node $s(i)$ to a node $d(i)$. Then the set of routing decision is

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \forall i \in \mathcal{I}, \forall p \in \mathcal{P}^i,\ \mathbf{x}^i_{(p)} \geq 0,\ \sum_{p \in \mathcal{P}^i} \mathbf{x}^i_{(p)} = \phi^i\}$$

where $n = \sum_{i \in \mathcal{I}} \#\mathcal{P}^i$ and ϕ^i is the total amount of flow going from $s(i)$ to $d(i)$.

The cost to route a packet of type i through the path $p \in \mathcal{P}^i$ is

$$F^i_{(p)}(\mathbf{x}) = \sum_{l \in p} f_l(x_l)$$

where l denotes a link, x_l is the amount of flow going through l : $x_l = \sum_{q \in \mathcal{P}} \delta_{lp} \mathbf{x}_{(p)}$ (δ_{lp} being define as in Section 8.2 for the paths) and the f^i_l's are continuous.

A Wardrop equilibrium $\mathbf{x}$ is a flow configuration such that $\forall i \in \mathcal{I}, \forall p, q \in \mathcal{P}^i$,

$$\mathbf{x}^i_{(p)} > 0 \implies F^i_{(p)}(\mathbf{x}) \leq F^i_{(q)}(\mathbf{x})$$

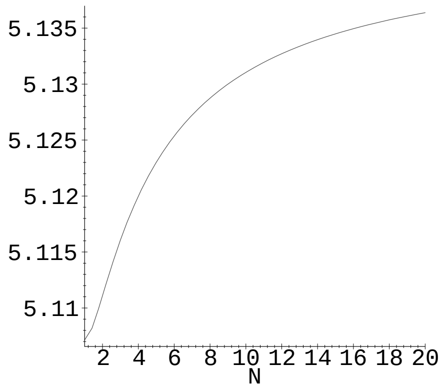

Figure 8.7: Sum of the cost for all players as a function of N

The structure of the cost functions allows us to extract a potential function of this game.

Potential function A *potential function* is a real-valued function on the decision space which measures exactly the difference in the cost that accrues to a user if he unilaterally deviates. In the case of a large routing game this definition may be restated as:

There exists a C^1 function f from $\mathcal{X}$ to $\mathbb{R}$ such that

$$\frac{\partial}{\partial x^i_{(p)}} f(\mathbf{x}) = F^i_{(p)}$$

The existence of such function establishes the existence of an (Wardrop) equilibrium and moreover its uniqueness whenever f is strictly convex. This is due to the fact that $\hat{\mathbf{x}}$ will be an equilibrium if $\hat{\mathbf{x}}$ is a solution of the minimization program:

$$\min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \tag{8.6.1}$$

the "if" may be replaced by "if and only if" when f is (strictly) convex: A point satisfying Kuhn-Tucker conditions for the Lagrangian (8.6.1) is an equilibrium and since $\mathcal{X}$ satisfies constraint qualification, Kuhn-Tucker conditions are necessary but not sufficient for local minimization of potential, these conditions are sufficient if the potential is convex. These results are presented in [18, Section 3].

The game defined in this section has the following potential function:

$$f(\mathbf{x}) = \sum_{l \in \mathcal{L}} \int_0^{x_l} f_l(t)dt$$

The large potential games admit a class of algorithms which converge toward the equilibrium.

BNN dynamics: convergence to equilibria The *Brown-von Neumann-Nash* dynamics were introduced for symmetric zero-sum games by Brown and von Neumann [4].

Let $k^i_{(p)}$ denotes the negative excess cost to path p relative to the average payoff in its class:

$$k^i_{(p)} = \max\left\{ \frac{1}{\phi^i} \sum_{p \in \mathcal{P}^i} \mathbf{x}^i_{(p)} F^i_{(p)}(\mathbf{x}) - F^i_{(p)}, 0 \right\}$$

Then the BNN dynamics are defined by

$$\dot{\mathbf{x}}^i_{(p)} = \phi^i k^i_{(p)} - \mathbf{x}^i_{(p)} \sum_{q \in \mathcal{P}^i} k^i_{(q)}$$

where $\dot{\mathbf{x}}$ denotes the derivative of $\mathbf{x}$ according to time.

8.6.1 Proposition. *Let F be a potential game. Then for any BNN dynamics D, for any $\mathbf{x} \in \mathcal{X}$ the set of accumulation points of the solution trajectory with initial condition x is a closed, connected set of rest points of D which are all equilibria.*

Proof: Follows from Proposition 4.2 and Theorem 4.5 in [18]. □

8.6.2 Corollary. *If the potential function f is strictly convex any solution trajectory will converge to the unique link's utilization induced by any equilibria.*

8.6.3. Remark. The result presented in Sandholm [18] contains a class of dynamics larger than the BNN dynamics. Note that the BNN dynamics guarantees both the flow demand and nonnegativity constraints. Other similar dynamics have been shown to converge in [14, p 271-272] where an explicit projection operator is added in order to guarantee feasibility of the constraints. In both that approach as well in BNN the time derivative of the assignment is proportional to minus the costs on the path. This can reflect the fact that users have their habits of using routes and do not change these immediately, yet the larger the profit from a change is, the larger will be the number of users who will adopt it per time unit.

Noncooperative finite game We assume in this paragraph that each class $i \in \mathcal{I}$ corresponds to a player (it can represent a service provider), which has an amount ϕ^i of flow to ship from $s(i)$ to $d(i)$, its cost function is

$$J^i(\mathbf{x}) = \max_{p \in \mathcal{P}^i, \mathbf{x}^i_{(p)} > 0} \sum_{l \in p} f_l(x_l)$$

Then the characterization of Nash equilibria is the same as the one of Wardrop (Section 8.4.2), hence, due to the preceeding paragraph, we have

8.6.4 Proposition. *If the f_l's are continuous and strictly increasing, any solution trajectory of a BNN dynamic will converge to a Nash equilibrium of the game and to the unique link's utilization induced (by any equilibrium).*

8.7 Conclusion, discussion and further extensions

In this paper, we have studied competitive routing in multicast networks using game theoretic tools. We have introduced different criteria for optimization adapted to these networks and have established for these criteria the existence of equilibrium as well as their uniqueness for two specific networks. We further introduced an extension of the notion of Wardrop equilibrium and established its uniqueness in a general topology. We have also presented results on convergence to equilibra of updating algorithms in multicast networks with duplication at the source.

The question of the uniqueness of Nash equilibrium in multicast networks with general topology is still open. In our model we assumed that the flow transmitted by a source node arrived fully to all the corresponding destinations.

An interesting extension of our model arises in the case of multirate transmission, which can be attained by hierarchically encoding real time signals. In this approach, a signal is encoded into a number of layers that can be incrementally combined to provide progressive refinements. Every layer is transmitted as a separate multicast group and receivers may choose to which groups they wish to subscribe (according to the capacity available or the congestion state). Internet protocols for adding and dropping layers can be found in [12] and [13]. Multirate transmission has applications both in video [9, 19] as well as in audio [2].

To deal with such a model we introduce a new object μ_d^{sD}, where $sD \in \mathcal{E}^i$ for some $i \in \mathcal{I}$ and $d \in D$, it represents the proportion of the flow ϕ_{sD}^i sent by user i that the address $d \in D$ wants to receive. In such an extension we cannot apply the proofs of Section 8.3.1, nevertheless the results of Section 8.3.2 are still valid.

Appendix

Incident matrix Let $a^i := \#\{a \in \mathcal{A}^i\}$, then the incident matrix Γ is

$$\Gamma = \begin{pmatrix} \gamma_1^{11} & \gamma_1^{21} & \cdots & \gamma_1^{n1} \\ \gamma_2^{11} & \gamma_2^{21} & \cdots & \gamma_2^{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{a^1}^{11} & \gamma_{a^1}^{21} & \cdots & \gamma_{a^1}^{n1} \\ \gamma_1^{12} & \gamma_1^{22} & \cdots & \gamma_1^{n2} \\ \gamma_2^{12} & \gamma_2^{22} & \cdots & \gamma_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{a^{n-1}}^{1n} & \gamma_{a^{n-1}}^{2n} & \cdots & \gamma_{a^{n-1}}^{nn} \\ \gamma_{a^n}^{1n} & \gamma_{a^n}^{2n} & \cdots & \gamma_{a^n}^{nn} \end{pmatrix}$$

whose element (i, r) is γ_k^{ij}, where $i, j \in \mathcal{I}$, $k \in \mathcal{A}^i$, $r = k + \sum_{s=1}^{i-1} a^s$ and $\gamma_k^{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

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Résumé :
Cette thèse a deux parties. La première traite des jeux stratégiques non-atomiques, la seconde propose des applications de la théorie des jeux aux réseaux de télécommunications.

Dans la première partie, les modèles de jeux non-atomiques proposés par Schmeidler (1973) et par Mas-Colell (1984) sont décrits et comparés. Nous montrons alors que ces jeux non-atomiques sont de bonnes approximations de jeux avec un nombre finis de joueurs et dans lesquels l'influence de chacun sur le paiement des autres joueurs est évanescente. Nous proposons ensuite une extension et des variations du modèle de Mas-Colell afin d'obtenir un cadre unificateur pour diverses applications des jeux non-atomiques, telles les jeux de routage, les jeux de foule et les jeux évolutionnaires. Ces trois types de jeu sont étudiés. Enfin nous étendons le concept de stratégie évolutionnairement stable au modèle de Schmeidler, ce qui donne un critère de sélection des équilibres.

La deuxième partie traite de problèmes de routage dans les réseaux. Tout d'abord nous modélisons des situations où deux types de joueur partagent un réseau, des joueurs ayant une influence certaine sur la répartition des paquets dans le réseau et des joueurs n'en ayant pas. Puis, nous étudions la convergence de dynamiques de meilleures réponses dans des réseaux d'architecture simple. Finalement, nous modélisons le problème du routage mutipoint-à-multipoint.

Abstract:
This thesis has two parts: the first is about nonatomic strategic games, and the second is about applications of game theory to telecommunication networks.

In the first part, the models of Schmeidler (1973) and of Mas-Colell (1984) of nonatomic games are described and compared. It is then shown that these models are good approximations of games having a large number of players in which the influence of each player on the other players is vanishing. An extension and some variations of Mas-Colell's model are presented in order to obtain a unifying framework for various applications of non-atomic games, such as routing games, crowding games and evolutionary games. These three types of game are analyzed in detail. Finally, the evolutionarily stable strategy is extended to Schmeidler's model, which yields an equilibrium refinement.

The second part of the thesis deals with routing in networks. Some situations in which a network is shared by two types of user are considered: some users have an influence on the state of the network while others do not. The convergence of best-reply dynamics is then studied in networks that have a simple topology. Finally, multipoint-to-multipoint routing is modelled.

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