Convergence of Population Dynamics in Symmetric Routing Games with a Finite Number of Players[‡]

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Abstract— **Routing games, as introduced in the pioneering work of Orda, Rom and Shimkin (1993), are very closely related to the traffic assignment problems as already studied by Wardrop and to congestion games, as introduced by Rosenthal. But they exhibit more complex behavior: often the equilibrium is not unique, and computation of equilibria is typically harder. They cannot be transformed in general into an equivalent global optimization problem as is the case with congestion games and in the traffic assignment problem which possess a potential under fairly general conditions. In this paper we study convergence of various learning schemes to an equilibrium in the problem of routing games. We are able to considerably extend previous published results [1] that were restricted to routing into two parallel links. We study evolutionary-based learning algorithms and establish their convergence for general topologies.**

I. Introduction

Routing games are concerned with the question of how each of several non-cooperative sources of traffic (say a service provider) should split (or route) its demand among several available paths in a network. Each service provider wishes to minimize its own cost, which is however influenced by the routing decisions of other service providers. A routing policy vector (where entry i specifies the routing policy for provider i) is called an equilibrium if no provider can reduce strictly its costs by changing its routing policy unilaterally.

In this paper we are concerned with the question of how the equilibrium can be attained from non-equilibrium starting points. This question was already raised in the very paper [5] by Orda, Rom and Shimkin that introduced these games (in the communication networks context). They have been able to show that adjustments based on best responses converge to equilibrium in the case of a network of two parallel links shared by two providers. They point out however that "this convergence result is not readily extendible to more general cases". Indeed, when extending to more than two providers, some asynchronous as well as synchronous best response schemes have been showed in [1] not to converge to the equilibrium. It is shown, however, in [1] that a round robin scheme converges (for the case of two parallel links and linear costs).

This type of formulation of a routing game, already studied in the context of road traffic [4], turns out to be much more difficult and does not enjoy in general the structure of a potential game. In particular, counterexamples are given in [5] for non-uniqueness of the equilibrium. It is therefore of interest to identify conditions on the cost structure that allow one to obtain a potential game in the setting of [5]. In [2] it was shown that the case of linear link costs provides such conditions. The authors then exploit the potential game structure to obtain convergence to equilibrium of schemes based on best responses. In this paper we follow a similar approach, this time for a general

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topology and cost function, but for symmetric systems.

II. Model

Consider a network with a general topology given by $G = (L, E)$ where L is a set of $|L|$ links and E a set of edges. The links are assumed to be directed. Consider K classes of users where class i has a total flow demand r^i to ship from a source s to a destination d.

Assume that we have a given set P of paths from the source to the destination over which we can send the traffic. Assume that each player i sends a flow of y^i_p over path p. Let $y_p = \sum_i y^i_p$.

Let x^i_l be the flow that player i ships over link l. With some abuse of notation we write $x_l = \sum_{i \in I} x^i_l$. We have

$$x^i_l = \sum_{p: l \in p} y^i_p$$

which we write in matrix form as $x^i = Ay^i$ where A is the incidence matrix of size $|L| \times |P|$ that has 1 at its (l, p) entry if and only if l is in path p.

Introduce for each link l a cost density function $T_l(.)$ assumed to be convex increasing and to depend on the total flow through link l. Let $J^i(x) = \sum_{l \in L} x^i_l T_l(x_l)$ be the total cost for player i.

Player i wishes to minimize $J^i(x)$ subject to $x^i_l \geq 0$ for all l, and to the standard flow conservation constraints which should hold at each node:

$$\sum_{e \in E} \sum_{l \in In(e)} x^i_l + r^{i,in}_e = \sum_{e \in E} \sum_{l \in Out(e)} x^i_l + r^{i,out}_e \quad (1)$$

where $In(e)$ are the links directed into edge e, where $Out(e)$ are the links directed out of edge e, $r^{i,in}_e$ equals r^i if e is the source node s_i of class i and equals $-r^i$ if e is its destination node d_i.

The existence of an equilibrium has been established in [3].

We write the Lagrangian that corresponds to the optimization problem faced by player i at equilibrium, where we relax the constraints in (1):

$$L(\lambda, y) = J^i(x) + \sum_{e \in E} \lambda^i_e \Big(\sum_{e \in E} \sum_{l \in In(e)} x^i_l + r^{i,in}_e$$

$$- \sum_{e \in E} \sum_{l \in Out(e)} x^i_l + r^{i,out}_e \Big)$$

where $x^i = Ay^i$. The Kuhn Tucker conditions at equilibrium state that there exists for every link (n, m) Lagrange multipliers λ_n and λ_m such that

$$T_{nm}(x_{mn}) + x^i_{mn} T'_{nm}(x_{mn}) \geq \lambda_n - \lambda_m,$$

$$T_{nm}(x_{mn}) + x^i_{mn} T'_{nm}(x_{mn}) = \lambda_n - \lambda_m \quad \text{for } x^i_{mn} > 0$$

III. Relation to Wardrop equilibrium

Introduce the following assumption:

A1. There exists an equilibrium such that for each link l, $x^i_l > 0$ for some i if and only if $x^i_l > 0$ for all i.

We say that users are symmetric if they have the same demands, same sources, destinations and authorized links. A direct extension of the proof in [5] implies:

Lemma 3.1: There is at most one equilibrium satisfying A1. If users are symmetric then there exists a single equilibrium, it satisfies A1, and at that equilibrium each user sends over each link 1/I of the whole link's flow.

Summing over all users we get under A1

$$IT_{nm}(x_{mn}) + x_{mn} T'_{nm}(x_{mn}) \geq \lambda_n - \lambda_m \quad (2)$$

$$IT_{nm}(x_{mn}) + x_{mn} T'_{nm}(x_{mn}) = \lambda_n - \lambda_m \quad \text{for } x_{mn} > 0 \quad (3)$$

where $\lambda_n = \sum_i \lambda^i_n$. Note that if A1 does not hold then the last equality need not hold.

Define $\lambda_p = \min_{l \in p} x_l$ and

$$c_l(x_l) = IT_l(x_l) + x_l T'_l(x_l)$$

By summing (2) we get for $\alpha = \lambda_s - \lambda_d$:

$$\sum_{l \in p} c_l(x_l) \geq \alpha \tag{4}$$

and summing (3) we get

$$\sum_{l \in p} c_l(x_l) = \alpha$$

if $\lambda_p > 0$. Since $y_p > 0$ implies that $\lambda_p > 0$ we conclude that if y^* is an equilibrium then it satisfies (4) with $x = Ay$ and

$$\sum_{l \in p} c_l(x_l) = \alpha \quad \text{ if } y_p > 0. \tag{5}$$

These are known to be the variational inequalities that determine the Wardrop equilibrium for link costs given by c_l.

Next, introduce the link cost

$$H_l(x_l, I) = \frac{x_l T_l(x_l)}{I} + \int_0^{x_l} T_l(s) ds$$

and the total cost

$$H(x, I) = \sum_l H_l(x_l, I)$$

Then $H(x)$ is the potential for this Wardrop equilibrium. Note that when users are symmetric then there is a unique equilibrium and A1 holds for that equilibrium, see [5]. Thus the following theorem holds.

Theorem 3.1: (i) There is a unique (in terms of total link flows) Wardrop equilibrium to the problem with transformed link costs given by c.
(ii) The link flows at that equilibrium are the unique solution to the convex optimization problem of minimizing the total expected cost $H(x, I)$.
(iii) Under A1, there exists a Nash equilibrium to the original problem such that the link flows are a Wardrop equilibrium with the transformed costs c. In particular, if the users are symmetric, A1 holds, and the Wardrop equilibrium is a Nash equilibrium for our original problem where, each user sends over each link 1/I of the whole link's flow.

IV. CONVERGENCE

The Wardrop equilibrium is unchanged if we divide the link costs by I. It then follows from Theorem 3.1 that for symmetric users, the path flows at the unique Nash equilibrium is the Wardrop equilibrium corresponding to the transformed link costs

$$c_l(x_l) = T_l(x_l) + \frac{x_l T_l'(x_l)}{I}$$

We see that as $I \to \infty$, this converges to the costs that give the Wardrop equilibrium for the original (non-transformed) link costs $T_l(x_l)$. This suggests that for symmetric users, the Nash equilibrium link flows converge to the Wardrop equilibrium (that corresponds to the original link costs). We next establish this.

Consider the convex optimization problem of minimizing the total expected cost $H(x, I)$ over the convex set

$$\{(x, y) : x = Ay, y \geq 0, \sum_p y_p = r\}.$$

This is equivalent to the problem $P(I)$ of minimizing $\widehat{H}(y, I) = H(Ay, I)$ over the convex set

$$\Delta = \{y \geq 0, \sum_p y_p = r\}.$$

Denote

$$\widehat{H}(y, \infty) = \int_0^{x_l} T_l(s) ds, \quad \text{ where } x = Ay.$$

Then $\widehat{H}(\cdot, I)$ converges to $\widehat{H}(\cdot, \infty)$ uniformly over Δ. Since T_l are convex increasing, $\widehat{H}(\cdot, I)$ and $\widehat{H}(\cdot, \infty)$ are convex functions. It then follows (see e.g. [8, Appendix]) that the argument that minimizes $\widehat{H}(\cdot, I)$ over Δ converges to the one that minimizes $\widehat{H}(\cdot, \infty)$ over Δ. But $\widehat{H}(\cdot, \infty)$ is the potential that corresponds to the Wardrop equilibrium for the original cost. We thus conclude that the link flows at Nash equilibrium converge to the link flows at the Wardrop equilibrium as the number of players goes to infinity.

Note that this convergence is already established in [4] using a variational approach without the restriction on symmetrical users, but with more restrictive conditions on the cost function.

V. Population Dynamics

Denote $Z_p(y) = \sum_{l \in p} c_l(x_l)$ where $x = Ay$. Introduce the following evolutionary dynamics:

- Replicator Dynamics:

$$\dot{y}_p^i = -y_p^i \left(Z_p(y) - \frac{1}{r^i} \sum_q y_q^i Z_q(y) \right)$$

- Brown – Von Neumann (BNN) Dynamics: Define

$$\gamma_p^i = \max \left[\frac{1}{r^i} \sum_q y_q^i Z_q(y) - Z_p(y), 0 \right]$$

Then

$$\dot{y}_p^i = \left(r^i \gamma_p^i - y_p^i \sum_q \gamma_q^i \right)$$

A symmetric flow configuration is the one in which the individual flows of each user on any path are the same. Since the dynamics are defined in terms of the individual path flows y_p^i rather than y_p, it is necessary to start from from a symmetric point so that the final configuration is symmetric, which according to theorem (3.1) is a necessary condition for it to be a Nash equilibrium. Using [6] that applies to convergence of BNN to the Wardrop equilibrium, we see that BNN converges to the original Nash equilibrium for the symmetric case. Similar result holds for the replicator dynamics provided we start at an interior point, see [6],[7]. We thus have the following theorem.

Theorem 5.1: Assume that the users are symmetric. Then A1 holds and

- The BNN dynamics converges to the Nash equilibrium provided we start from a symmetric point.
- The Replicator dynamics converges to Nash equilibrium provided we start from an interior symmetric point.

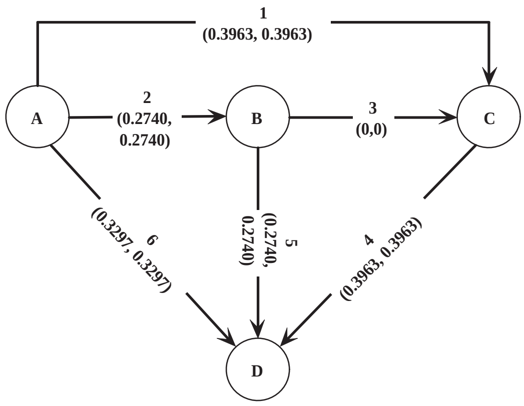

Fig. 1. Link flows at Nash equilibrium

VI. Numerical Example

Consider the network topology shown in Fig.1. There are two users who wish to route traffic flows from source node A to destination node D with demands $r^1 = r^2 = 1$. There are four possible paths that connect the source to the destination. Path 1 is A-C-D, path 2 is A-B-D, path 3 is A-B-C-D and path 4 is A-D. For $l = 1, \cdots, 6$, we use an exponential cost function for the links marked in the diagram, given by $T_l(x_l) = k_l e^x$ where $k = (k_1, \cdots, k_6) = (1, 2, 3, 4, 5, 6)$. We use the two distributed algorithms, replicator dynamics and BNN dynamics to reach the equilibrium path flow profiles of the users: $y^{*i} = (y_1^{*i}, y_2^{*i}, y_3^{*i}, y_4^{*i})$ starting from the symmetric point $y^1 = y^2 = (0.25, 0.25, 0.25, 0.25)$. Both algorithms converge to the following Nash equilibrium flow profiles:

$$y^{*1} = y^{*2} = (0.3963, 0.2740, 0.0000, 0.3297)$$

The equilibrium flows of the players on each of the links are shown in the diagram. Following are the trajectories of individual user path flows with respect to time under both the dynamics. As the graphs show, convergence under replicator dynamics is much faster as compared to the BNN dynamics.

VII. Concluding Comments

We have shown in this paper how population dynamics can be used for convergence to the unique Nash

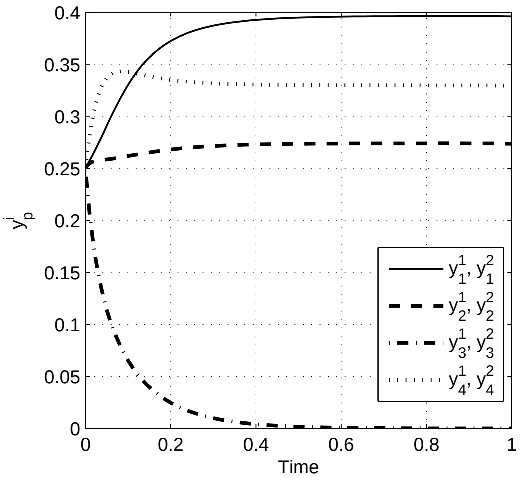

Fig. 2. Convergence to Nash equilibrium under Replicator dynamics

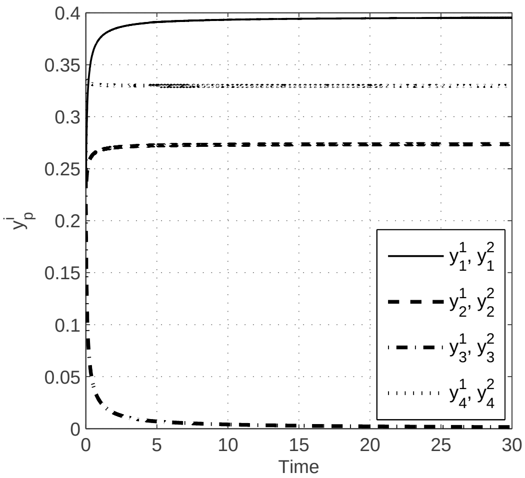

Fig. 3. Convergence to Nash equilibrium under BNN dynamics

equilibrium in some symmetric routing games with general topology and costs. The convergence followed by showing that a potential exists in a large class of games that include those with symmetrical players. This potential allows us to obtain only the total amount of link flows at equilibrium, at symmetricity is used to deduce from it the share of each player.

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