Telecommunications Network Equilibrium with Price and Quality-of-Service Characteristics*

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Abstract

We present a competitive model that describes the interaction between competing telecommunications service providers (SPs), their subscribers, and a network owner. Competition takes place in pricing as well as in terms of the quality of service (QoS) they offer. The subscribers' demand depends not only on the price and QoS of that SP but also upon those of its competitors. We establish conditions for existence and uniqueness of the equilibria, compute them explicitly and characterize their properties.

1. INTRODUCTION

Competitive routing and equilibrium models for networks generally assume a single characteristic through which an equilibrium is computed, such as delay [6], prices [7], or even loss probabilities [2]. However, in order to take into account Quality of Service (QoS) on a network, it is necessary to incorporate more than one parameter. An example is to include both prices and some measure of QoS. We present a general model for computing a bi-criteria equilibrium of several providers. We then examine particular functional forms, and the resulting properties, when the two parameters of each providers are prices and a measure of QoS. For surveys on competitive games in telecommunication, see [1,3].

The model can be considered an aggregate one; the network itself (sources and destinations of requests) are not taken into account. We assume that the demand for the services of a given service provider (SP) is a function of the vectors of prices and QoS of *all service providers*. Our methodology is inspired by [4] who studied a dynamic competitive inventory control model that includes both pricing as well as a quality parameter. The authors reduced the problem to a static game of a form similar to that which we study.

In the next section, we provide the model setting and theorems for existence and uniqueness of equilibria. Sec. 2.2 presents the demand model and establishes properties of the equilibrium when the QoS are fixed and competition occurs through prices. The next two sections concern characterizations of the equilibria and analyze competition both in prices and in QoS. Two specific QoS measures are considered: delay (Sec. 3) and loss probabilities (Sec. 4). Sec. 4 includes several ways of modeling loss probabilities including packet loss versus session rejection probability, and the influence of incorporating large-deviation scaling.

2. Model and general properties

2.1. Model setting

Let us consider a game in which there are N service providers, $\mathbf{SP}$, denoted by $\mathcal{I} = \{1, 2, .., N\}$. Each provider sets two service parameters: $(\mathbf{p}, \mathbf{f}) \in \mathbb{R}_+^{N+N}$. (We use bold face to denote a vector, e.g. $\mathbf{p}$; its ith component is p_i.) Although the two parameters can be general, we shall regard the first as prices (p_i is the price that $\mathbf{SP}$ i charges his subscribers per unit demand) and the

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second as a QoS measure (delay, loss or rejection probability, etc). Then, **SP** i experiences a demand for its service, $D_i : \mathbb{R}_+^{N+N} \mapsto \mathbb{R}_+$ which depends not only on its own parameters, p_i and f_i, but also on the prices and QoS offered by its competitors; that is, D_i depends upon the entire price vector $\mathbf{p} = (p_1, p_2, .., p_N)$ and the entire QoS vector $\mathbf{f} = (f_1, ..., f_N)$. In this work, we shall use a linear demand model, which will be defined subsequently.

The utility functions of the service providers are given by $U : \mathbb{R}_+^{N+N} \mapsto \mathbb{R}_+$. We suppose that all service providers' strategies are defined by the following orthogonal constraints; then, the strategy space, $\mathcal{R}_i$, of provider i is given by the subset of $\mathbb{R}^2$:

$$\mathcal{R}_i = \{(p_i, f_i) : \ 0 \leq \hat{g}_i(f_i) \leq p_i \leq p^{\max}; \quad 0 \leq f_i^{\min} \leq f_i \leq f_i^{\max}\}, \tag{1}$$

where $p_i^{\min} = \hat{g}_i(f_i)$. (The dependence of $p_i^{\min}$ on f_i through the function $\hat{g}_i$ will allow us to eliminate explicitly policies that will result in negative profits for a service provider). The fact that there is an upper bound on price reflects the fact that beyond some price, demand will be zero (what ever the prices and QoS levels of competitors are). In addition, $\hat{g}_i$ is assumed convex so that the strategy space of player i is convex, for all $i \in \mathcal{I}$. We further define:

$$\mathcal{R}_{pi}(f_i) = \{p_i : \ 0 \leq \hat{g}_i(f_i) \leq p_i \leq p^{\max}\}, \ \ \mathcal{R}_{fi} = \{f_i : \ 0 \leq f_i^{\min} \leq f_i \leq f_i^{\max}\}, \tag{2}$$

$$\mathcal{R} = \mathcal{R}_1 \times \cdots \mathcal{R}_N, \ \ \mathcal{R}_p(\mathbf{f}) = \mathcal{R}_{p1}(\mathbf{f}) \times \cdots \mathcal{R}_{pN}(\mathbf{f}), \ \ \mathcal{R}_f = \mathcal{R}_{f1} \times \cdots \mathcal{R}_{fN}. \tag{3}$$

Along with the model of the service providers, we include a model of the (single) network owner, who may charge each service provider, i, a cost per unit of bandwidth requested as a function of the two criteria it requires, p_i and f_i. Then, the total price charged to the service provider will be a function of that per-unit price and of the demand that the provider i experiences. That is, given the aggregate demand $d_i = D_i(\mathbf{p}, \mathbf{f})$ of **SP** i, we associate with **SP** i a cost function given by $R_i(p_i, f_i, d_i)$, representing the fee that i pays to the network owner.

We next define the Nash equilibria that we are interested in computing. The first involves each service provider setting a single parameter, as a function of those of all other providers, and the second involves simultaneous setting of both parameters by all providers, as a function of those of all other providers. As mentioned, our approach is inspired by [4] who studied a dynamic competitive inventory control model including both pricing and a quality parameter.

Definition 1
1- [Single-parameter Nash equilibrium] *Let $U_i(\mathbf{p}, \mathbf{f})$ be the net revenue*[2] *of **SP** i, when the vector of prices set by all providers is $\mathbf{p}$, and the vector of QoS parameters, $\mathbf{f}$, of all providers is fixed at some point, $\hat{\mathbf{f}}$. Then, a single-parameter Nash equilibrium in $\mathbf{p}$ at $\hat{\mathbf{f}}$ is the vector $\mathbf{p}^*$ that solves the for all $i \in \mathcal{I}$:*

$$U_i(\mathbf{p}^*, \hat{\mathbf{f}}) = \max_{p_i \in \mathcal{R}_{pi}(\hat{\mathbf{f}})} U_i(p_1^*, ..., p_{i-1}^*, p_i, p_{i+1}^*, .., p_N^*, \hat{\mathbf{f}}). \tag{4}$$

2- [Two-parameter Nash equilibrium] *Let $U_i(\mathbf{p}, \mathbf{f})$ be the net revenue of **SP** i, when both parameters are set simultaneously by all providers: $(\mathbf{p}, \mathbf{f})$. A two-parameter Nash equilibrium in $(\mathbf{p}, \mathbf{f})$ is the vector couple $(\mathbf{p}^*, \mathbf{f}^*)$ that solves for all $i \in \mathcal{I}$:*

$$U_i(\mathbf{p}^*, \mathbf{f}^*) = \max_{(p_i, f_i) \in \mathcal{R}_i} U_i(p_1^*, ..., p_{i-1}^*, p_i, p_{i+1}^*, .., p_N^*, f_1^*, ..., f_{i-1}^*, f_i, f_{i+1}^*, .., f_N^*). \tag{5}$$

2.2. Demand model

We suppose that the average demand, $D_i(\mathbf{p}, \mathbf{f})$, is linear in all prices and QoS levels. For a particular **SP** i, the demand for its service, D_i decreases in its price, p_i, but increases in its competiitors' prices, p_j, $j \neq i$. The analogous relationship holds in QoS, but then, D_i is increases in f_i and decreases in f_j, for $j \neq i$. We have:

$$D_i(\mathbf{p}, \mathbf{f}) = a_i - b_i p_i + \sum_{j \in \mathcal{I}, j \neq i} c_{ij} p_j + \beta_i f_i - \sum_{j \in \mathcal{I}, j \neq i} \gamma_{ij} f_j, \tag{6}$$

with b_i, c_{ij}, β_i, γ_{ij} positive constants. We next make the following blanket assumptions:

[2] The net revenue can be expressed as the income minus the cost R_i. Examples will be given in later sections.

Assumption 1 [3] *We assume that the demand function of each **SP** satisfies, for all $i \in \mathcal{I}$, $D_i(p^{\max} - p_i^{\min}, \mathbf{0}_{-i}, \mathbf{f}) \leq 0$, $\forall \mathbf{f} \in \mathcal{R}_f$, where $p_i^{\min} = \hat{g}_i(f_i)$, and $\mathbf{0_{-i}}$ is the $N-1$-dim 0-vector.*

Assumption 2 *The constants b and c satisfy : $b_i > \sum_{j \neq i} c_{ij}$, $i \in \mathcal{I}$. Moreover, assume that D_i is non-negative over the strategy space*[4]

Assumption 2 is needed to ensure uniqueness of the resulting equilibrium. It is furthermore a reasonable condition; it implies that the influence of an **SP**'s price is significantly greater on its own demand than are the prices of its competitors. This condition represents the presence of customer loyalties and/or imperfect knowledge of competitors' prices.

2.3. General QoS model

Along with the model of the **SP**s, the (single) network owner charges each service provider, i, a cost per unit of bandwidth requested. We assume that the amount of bandwidth μ_i requested by **SP** i depends on its demand and on the QoS it wishes to offer (the higher the demand and the better the QoS, the higher μ_i will be). For example, if the QoS is linear in the standard Kleinrock delay function (obtained from an M/M/1 queuing model), i.e. $f_i = c - 1/(\mu_i - d_i)$, then $\mu_i(f_i, d_i) = d_i + 1/(c - f_i)$.

Thus, given the aggregate demand $d_i = D_i(p_i, f_i)$, the fee paid by **SP** i can be written as: $R_i = v_i \mu_i(f_i, d_i)$, where $\mu_i(f_i, d_i) \in [0, B]$ is again the bandwidth required in order to guarantee the QoS, f_i, at a demand level of d_i for **SP** i, and v_i is the price to be paid per unit of bandwidth.

2.4. Utility model

The revenue of **SP** i is given by $p_i D_i(\mathbf{p}, \mathbf{f})$, whereas the profit (or net revenue) of **SP** i is the difference between the revenue and the fee it pays to the network owner,

$$U_i(\mathbf{p}, \mathbf{f}) = p_i D_i(\mathbf{p}, \mathbf{f}) - R_i(f_i, d_i). \tag{7}$$

Later, we prove uniqueness of the Nash equilibrium for the price game (4) in which the capacity μ_i required by user i has the following form:

$$\mu_i(f_i, d_i) = d_i g_i(f_i) + h_i(f_i), \tag{8}$$

where g_i and h_i are positive functions. Hence, the profit function becomes $U_i(\mathbf{p}, \mathbf{f}) = D_i(\mathbf{p}, \mathbf{f})(p_i - v_i g_i(f_i)) - v_i h(f_i)$.

In the single-parameter Nash game, the strategy space of each **SP** i is given by (1) and in the two-parameter game, the strategy space is given by (2) where $p^{\min} = \hat{g}_i(f_i) = v_i g_i(f_i)$. Note that the sets $\mathcal{R}_i$, and $\mathcal{R}_{pi}$, $i \in \mathcal{I}$ are convex when g_i is a convex function.

2.5. Properties of the general model

The question is then, under general assumptions on the model functions and data, when can we ensure the existence and uniqueness of the resulting equilibrium across service providers. To this end, we assume the following of the utility functions.

Assumption 3 *Without loss of generality, express the net revenue (or utility) of service provider i as $U_i = \hat{U}_i(\mathbf{p}, \mathbf{f}, d_i) = U_i(\mathbf{p}, \mathbf{f})$, $i \in \mathcal{I}$ where, as before, $d_i = D_i(\mathbf{p}, \mathbf{f})$. Assume that U_i is continuous in $(\mathbf{p}, \mathbf{f})$ and concave in (p_i, f_i), for all $i \in \mathcal{I}$.*

Under Assumption 3, the existence of a Nash equilibrium follows from [8].

We first consider the single-parameter price game (4) which arises when the vector of QoS levels of all providers is fixed. Next, we consider the two-parameter price and QoS game.

The following theorem is a generalization of [4, Thrm. 5] that established uniqueness of an equilibrium for a similar setting but when $h_i(.)$ are all zero and for some specific functions $g_i(.)$. By allowing for general g_i and h_i we shall be able to handle in the next sections a large number of quality of service functions that are typical in networks.

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[3] The above assumption should be understood as imposing a relation between the parameters that define the demand functions. It could seem unrealistic, since it allows demand to be negative. However, the nonpositivity of the demand function is required for prices outside the feasible strategie set $\mathcal{R}_i$'s. Therefore we are saying that if we extrapolated the demand outside the feasible region, allowing other prices to be zero, we would get nonpositive demand.

[4] Sufficient condition for Assumption 1: Let assumption 2 hold. In view of (6), if we choose $p^{\max}$ large enough then Assumption 1 will hold.

Theorem 1 [Uniqueness of Price-based Nash equilibrium] *Consider the price game (4) which arises when the vector of quality of service levels* $\mathbf{f}$ *is fixed, and assume that* $\mathcal{R}_i(\mathbf{f})$ *are nonempty,* $i \in \mathcal{I}$. *Define* $L_{ii} = 2b_i$, $L_{ij} = -c_{ij}$ *for* $i \neq j$ *and* $q_i = b_i v_i g_i(f_i) + a_i + \beta_i f_i - \sum_{j\neq i} \gamma_{ij} f_j$. *Then*

1. *The price game has a unique equilibrium* $\mathbf{p}^*(\mathbf{f})$, *with* $\mathbf{p}^*(\mathbf{f}) = (I-T)^{-1}\Gamma^{-1}q$, *where* $\Gamma = diag(2b_1, ..., 2b_N)$, $T_{ii} = 0$, $T_{ij} = \frac{c_{ij}}{2b_j}$ *for* $i \neq j$ *and* $q_i = b_i v_i g_i(f_i) + a_i + \beta_i f_i - \sum_{j\neq i} \gamma_{ij} f_j$.
2. *The equilibrium prices* p_i^* *are increasing in each of the unit prices* v_j.
3. *The equilibrium prices* p_i^* *are decreasing in* f_j, $j \in \mathcal{I}$ *when* $g_j'(f_j) < \frac{\sum_{l\in\mathcal{I}}(L^{-1})_{il}\gamma_{jl} - (L^{-1})_{ij}\beta_j}{(L^{-1})_{ij} b_j v_j}$ *and increasing otherwise.*

Proof: Now we show the uniqueness of the equilibrium point. Since $\mathbf{p}^*(\mathbf{f})$ is a Nash equilibrium, it follows that p_i^* maximizes $U_i(\mathbf{p}_i, \mathbf{p}_{-i}^*, \mathbf{f})$, where $\mathbf{p}_{-i} = (p_1, .., p_{i-1}, p_{i+1}, .., p_N)$. Moreover, we have

$$\begin{aligned}
\lim_{p_i \to v_i g_i(f_i)} \frac{\partial U_i(p_i, p_{-i}^*, \mathbf{f})}{\partial p_i} &= d_i(v_i g_i(f_i), p_{-i}^*, \mathbf{f}) \geq 0 && (9)\\
\lim_{p_i \to p^{max}} \frac{\partial U_i(p_i, p_{-i}^*, \mathbf{f})}{\partial p_i} &= -b_i p^{max} + b_i v_i g_i(f_i) + D_i(p^{\max}, \mathbf{p}_{-i}^*, \mathbf{f}) \\
&= a_i - 2b_i p^{\max} + \sum_{j\neq i} c_{ij} p_j^* + b_i v_i g_i(f_i) + \beta_i f_i - \sum_{j\neq i} \gamma_{ij} f_j \\
&\leq a_i - 2b_i p^{\max} + p^{\max} \sum_{j\neq i} c_{ij} + b_i g_i(f_i) + \beta_i f_i - \sum_{j\neq i} \gamma_{ij} f_j && (10)\\
&< a_i - b_i p^{\max} + b_i v_i g_i(f_i) + \beta_i f_i - \sum_{j\neq i} \gamma_{ij} f_j && (11)\\
&= d_i(p^{\max} - v_i g_i(f_i), \mathbf{0}_{-i}, \mathbf{f}) \leq 0 && (12)
\end{aligned}$$

where the transition (10)-(11) is from Assumption 2 and the last transition is from Assumption (1). Thus, p_i^* is the unique solution of the optimality conditions:

$$0 = \frac{\partial U_i(\mathbf{p}, \mathbf{f})}{\partial p_i} = -b_i(p_i - v_i g_i(f_i)) + d_i(\mathbf{p}, \mathbf{f}). \quad (13)$$

Then, $\mathbf{p}^*$ is a solution to the following linear system : $Lp = q$, where $L_{ii} = 2b_i$, $L_{ij} = -c_{ij}$ for $i \neq j$ and $q_i = b_i v_i g_i(f_i) + a_i + \beta_i f_i - \sum_{j\neq i} \gamma_{ij} f_j$.
To show the uniqueness of Nash equilibrium, it suffices to show that the above linear system admits a unique solution. We observe that the matrix L can be written as: $L = \Gamma(I-T)$ with $\Gamma = diag(2b_1, ..., 2b_N)$, and $T = \begin{pmatrix} 0 & \vdots & \frac{c_{1N}}{2b_1} \\ & \ddots & \\ \frac{c_{N1}}{2b_N} & \vdots & 0 \end{pmatrix}$. The matrix T is substochastic since $\frac{\sum_{j\neq i} c_{ij}}{2b_i} \leq \frac{b_i}{2b_i} < 1$. Thus L^{-1} exists and is equal to $(I-T)^{-1}\Gamma^{-1}$ and $p^*(\mathbf{f}) = L^{-1}q = (I-T)^{-1}\Gamma^{-1}q$. Writing explicitly one term from the equilibrium price vector, $p_i^*(\mathbf{f})$, and rearranging terms to identify dependencies on f_i and f_j independently, $j \in \mathcal{I}$, $j \neq i$, we have that

$$\begin{aligned}
p_i^*(\mathbf{f}) &= (L^{-1})_{ii}(a_i + (b_i v_i g_i(f_i)) + f_i((L^{-1})_{ii}\beta_i - \sum_{j\neq i}(L^{-1})_{ij}\gamma_{ji}) + \\
&\quad \sum_{j\neq i}(L^{-1})_{ij}(a_j + b_j v_j g_j(f_j) + \beta_j f_j) - \sum_{j\in\mathcal{I}}\sum_{l\neq i}(L^{-1})_{ij}\gamma_{jl} f_l && (14)
\end{aligned}$$

Then, taking partial derivatives, we obtain: $\frac{\partial p_i^*(\mathbf{f})}{\partial f_j} = (L^{-1})_{ij} b_j v_j g_j'(f_j) + (L^{-1})_{ij}\beta_j - \sum_{l\in\mathcal{I}}(L^{-1})_{il}\gamma_{jl}$,, which is negative when $g_j'(f_j) < \frac{\sum_{l\in\mathcal{I}}(L^{-1})_{il}\gamma_{jl} - (L^{-1})_{ij}\beta_j}{(L^{-1})_{ij} b_j v_j}$. $\square$

Next we make the following assumptions on the functions g_i and h_i from Definition (8) of μ_i. This will guarantee the uniqueness of the equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$ in the price-QoS game (5). Several examples will be given in Sections 3 and 4.

Assumption 4 *Let $h_i(.) = 0$. Moreover, suppose that $g(.) \in C^2$ is convex increasing, and its derivative satisfies for all $i \in \mathcal{I}$: $\lim_{f_i \to f_i^{\max}} g_i'(f_i) = +\infty$.*

This is satisfied in most queuing systems we study. Next we prove uniqueness of the equilibrium when prices and QoS levels are set by all players simultaneously.

Theorem 2 [Uniqueness of Price and QoS-based Nash equilibrium] *Let Assumption 4 hold. The price-QoS problem (5) has a unique two-parameter Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* the unique solution to the following equations, for every $i \in \mathcal{I}$,*

$$g_i'(f_i^*) \quad = \quad \frac{\beta_i}{v_i b_i} \ \textit{if}\ \frac{\beta_i}{v_i b_i} \geq g_i'(f_i^{\min}) \ \ \textit{and}\ f_i^* = 0 \ \textit{if}\ \frac{\beta_i}{v_i b_i} < g_i'(f_i^{\min})$$

Proof: Let $\tilde{U}_i(\mathbf{p}, \mathbf{f}) = \log[p_i - v_i g_i(f_i)] + \log[D_i(\mathbf{p}, \mathbf{f})]$. Note that $(\mathbf{p}^*, \mathbf{f}^*)$ is an equilibrium in the original utilities iff it is an equilibrium in their logarithms, i.e. of $\tilde{U}_i$. Since g_i is convex by assumption, U_i is concave in $(\mathbf{p}, \mathbf{f})$. Under Assumption 4, we have

$$\lim_{f_i \to f_i^{\max}} \frac{\partial \tilde{U}_i(\mathbf{p}, \mathbf{f})}{\partial f_i} = \lim_{f_i \to f_i^{\max}} \frac{-v_i g'(f_i)}{p_i - v_i g_i(f_i)} + \frac{\beta_i}{D_i(\mathbf{p}^*, \mathbf{f})} = -\infty. \tag{15}$$

This implies there is no Nash equilibrium at the constraint boundary where $f_i = f_i^{\max}$, $i \in \mathcal{I}$. To show the uniqueness of the Nash solution, we must thus examine the interior of the feasible region with respect to the upper bound on the f_i and show that there is precisely one point, $(\mathbf{p}^*, \mathbf{f}^*)$, which solves simultaneously the utility maximization problems of each provider, $i \in \mathcal{I}$. Note from (15) that the lower limit, $\lim_{f_i \to f_i^{\min}} \partial \tilde{U}_i(\mathbf{p}, \mathbf{f})/\partial f_i$ may be either positive or negative, and as such, the Nash equilibrium may arise at a point for which $f_i^* = f_i^{\min}$ for some $i \in \mathcal{I}$.

We do not need Lagrange multipliers for the constraints on p_i since p_i^* is obtained at a point where the partial derivative (13) is zero, possibly at the lower boundary, (9) and (12).

Consider the following Kuhn-Tucker conditions. Lagrange multiplier, α_i is on the lower bound constraint on f_i for each i. In other words, $(\mathbf{p}^*, \mathbf{f}^*)$ is a Nash equilibrium if for every $i \in \mathcal{I}$, there exist non-negative Lagrange multipliers α^i s.t.

$$0 = \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial p_i} \quad = \quad \frac{1}{(p_i - v_i g_i(f_i))} + \frac{-b_i}{D_i(p^*, \mathbf{f})} \tag{16}$$

$$-\alpha^i \quad = \quad \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial f_i} = \frac{-v_i g_i'(f_i)}{p_i - v_i g_i(f_i)} + \frac{\beta_i}{D_i(p^*, \mathbf{f})} \tag{17}$$

$$\alpha^i (f_i - f_i^{\min}) \quad = \quad 0, \ \ \alpha^i \geq 0 \tag{18}$$

Multiplying (16) by $1/b_i$ and (17) by $1/\beta_i$ and substituting, we obtain $\frac{\alpha^i}{\beta_i} = \frac{v_i g_i'(f_i)/\beta_i - 1/b_i}{p_i - v_i g_i(f_i)}$, and $\alpha_i(f_i - f_i^{\min}) = 0$. It follows that $g_i'(f_i^*) = \frac{\beta_i}{v_i b_i}$ if $\frac{\beta_i}{v_i b_i} \geq g_i'(f_i^{\min})$ and $f_i^* = f_i^{\min}$ if $\frac{\beta_i}{v_i b_i} < g_i'(f_i^{\min})$. Fixing $f = f^*$, we obtain p^* through Theorem 1. Thus $(\mathbf{p}^*, \mathbf{f}^*)$ is the unique Nash equilibrium. □

Next are two applications of this framework, with two different QoS measures.

3. Case study I: expected delay as QoS

We shall suppose in this section that the measure defining the QoS, f, corresponds to some function of the expected delay. We shall use the Kleinrock delay function[5] which is a common delay function used in networking games [6]. Instead of Minimizing delay, we consider the maximization of the reciprocal of its square-root (in that way the QoS f_i indeed increases as the delays decrease (this is similar to [7]):

$$f_i = \frac{1}{\sqrt{Delay}} = \sqrt{\mu_i - d_i} \tag{19}$$

[5]This function corresponds to a queuing delay in an $M/M/1$ queue with first-in-first-out discipline or to the more general $M/G/1$ queue under processor sharing delay.

Thus $\mu_i(f_i, d_i) = d_i g(f_i) + h_i(f_i)$ where $g_i(f_i) = 1$ is indeed convex and where $h_i(f_i) = f_i^2$. The cost that **SP** i pays the network owner is $v_i[(f_i^2) + d_i]$. Then, the net revenue of **SP** i is

$$U_i(\mathbf{p}, \mathbf{f}) = D_i(\mathbf{p}, \mathbf{f})(p_i - v_i) - v_i f_i^2. \tag{20}$$

We observe that this profit function is special case of the one defined in (8), by taking $g_i(f_i) = 1$ and $h_i(f_i) = f_i^2$. We shall make the following assumption:

Assumption 5 [6] *The constants b, v, and β satisfy:* $2b_i v_i > \beta_i^2, \quad i \in \mathcal{I}$

We now proceed with the analysis of the price-QoS game (5) (two-parameter) in which prices and services levels are set by all **SP**s simultaneously.

Theorem 3 *Let Assumption 5 hold. The price-QoS game (5) with QoS,* $\mathbf{f}$ *given by (19) has a unique Nash equilibrium* $(\mathbf{p}^*, \mathbf{f}^*)$*, with* p_i^* *increasing in each* v_j*,* $j \in \mathcal{I}$*.*

Proof: Assumption 3 is satisfied by (5) with (19) and (20). Hence there is a Nash equilibrium.

We next show that the Nash equilibrium is unique. We have seen, in the previous section, that the partial derivatives of the profit functions with respect to price variables p_i of each providers i will be zero. In this section, the upper and lower bounds on the quality of service variables, f_i may or may not be active, and so we take a Lagrangian relaxation of those constraints and examine the Kuhn Tucker conditions of the resulting Lagrangian.

To show the uniqueness of the Nash equilibrium solution, we show first that the profit functions U_i are jointly strictly concave in (p_i, f_i) for all providers, $i \in \mathcal{I}$, so that the solution to system defined by the Kuhn Tucker conditions is uniquely defined.

To this end, note first that

$$\begin{aligned}
\frac{\partial U_i(p, \mathbf{f})}{\partial p_i} &= -b_i(p_i - v_i) + D_i(p^*, \mathbf{f}) \text{ and } \frac{\partial U_i(p, \mathbf{f})}{\partial f_i} = \beta_i(p_i - v_i) - 2v_i f_i \\
\frac{\partial^2 U_i(p, \mathbf{f})}{\partial p_i^2} &= -2b_i < 0, \ \frac{\partial^2 U_i(p, \mathbf{f})}{\partial p_i \partial f_i} = \beta_i, \text{ and } \frac{\partial^2 U_i(p, \mathbf{f})}{\partial f_i^2} = -2v_i < 0.
\end{aligned}$$

Since $\frac{\partial^2 U_i}{\partial f_i^2} = -2v_i < 0$ and $\frac{\partial^2 U_i}{\partial p_i^2} = -2b_i < 0$, it suffices to show that the determinant of the Hessian matrix H is positive, where $H = \begin{pmatrix} \frac{\partial^2 U_i}{\partial p_i^2} & \frac{\partial^2 U_i}{\partial p_i \partial f_i} \\ \frac{\partial^2 U_i}{\partial f_i \partial p_i} & \frac{\partial^2 U_i}{\partial f_i^2} \end{pmatrix}$. We have $\det H[U_i(\mathbf{p}, \mathbf{f})] = \frac{\partial^2 U_i}{\partial f_i^2}\frac{\partial^2 U_i}{\partial p_i^2} - \left(\frac{\partial^2 U_i}{\partial p_i \partial f_i}\right)^2 = 4b_i v_i - \beta_i^2 > 0$. The latter inequality follows from Assumption 5.

We note that any solution to the following Kuhn-Tucker conditions is a Nash equilibrium: For every $i \in \mathcal{I}$, there exist non-negative Lagrange multipliers α^i and λ^i such that

$$0 = \frac{\partial U_i(\mathbf{p}, \mathbf{f})}{\partial p_i} = -b_i(p_i - v_i) + D_i(\mathbf{p}^*, \mathbf{f}) \tag{21}$$

$$\alpha^i - \lambda^i = \frac{\partial U_i(\mathbf{p}, \mathbf{f})}{\partial f_i} = \beta_i(p_i - v_i) - 2v_i f_i \tag{22}$$

$$\alpha^i(f_i - f_i^{\min}) = 0, \ \lambda^i(f_i^{\max} - f_i) = 0 \tag{23}$$

where α^i and λ^i are the Lagrange multipliers corresponding to the min and max constraints on the values of f_i. As before, we do not need Lagrange multipliers for the constraints on p_i since p_i^* is obtained at a point where the partial derivative (21) is zero, possibly at the lower boundary, (9) and (12). It follows from (22) that if $\lambda^i = 0$ and $\alpha^i = 0$ we have $f_i^* = \frac{\beta_i}{2v_i}(p_i - v_i)$, if $\alpha^i > 0$ we have $f_i^* = f_i^{\min}$ and if $\lambda^i > 0$ we have $f_i^* = f_i^{\max}$. Hence, it follows that $\mathbf{p}^*$ is solution of the following linear system : $Ap = r$, where

[6] Note that, for price-sensitive demands, i, this assumption is likely to hold, since in that case, $b_i >> \beta_i$. For relatively price-*in*-sensitive demands, who value above all QoS, Assumption 5 may still hold if the unit price for capacity charged by the network owner to **SP** i, v_i **is high.**

$$
\begin{aligned}
A_{ii} &= 2b_i - \delta_{\{\lambda^i-\alpha^i=0\}}\frac{\beta_i^2}{2v_i}, \quad A_{ij} = -c_{ij} + \delta_{\{\lambda^j-\alpha^i=0\}}\gamma_{ij}\frac{\beta_j}{2v_j}, \\
R_i &= b_i v_i + a_i + \delta_{\{\lambda^i-\alpha^i=0\}}\frac{\beta_i^2}{2} - \sum_{j\neq i}\delta_{\{\lambda^j-\alpha^j=0\}}\gamma_{ij}\frac{\beta_j}{2} + \beta_i(\delta_{\{\lambda^i>0\}}f_i^{\max} + \delta_{\{\alpha^i>0\}}f_i^{\min}) \\
&\quad - \sum_{j\neq i}\gamma_{ij}(\delta_{\{\lambda^i>0\}}f_j^{\max} + \delta_{\{\alpha^j>0\}}f_i^{\min}).
\end{aligned}
$$

To show the uniqueness of Nash equilibrium, it suffices to show that the above linear system admits a unique solution. We observe that the matrix A can be written as: $A = \Gamma_1(I - T_1)$ with $\Gamma_1 = diag(2b_1 - \delta_{\{\lambda^i-\alpha^i=0\}}\frac{b_1^2}{2v_1}, ..., 2b_N - \delta_{\{\lambda^i-\mu^i=0\}}\frac{b_N^2}{2v_N})$, and

$$
T_1 = \begin{pmatrix}
0 & \vdots & \frac{c_{1N} - \delta_{\{\lambda^i-\mu^i=0\}}\gamma_{1N}\frac{\beta_N}{2v_N}}{2b_1 - \delta_{\{\lambda^i-\mu^i=0\}}\frac{\beta_1^2}{2v_1}} \\
 & \ddots & \\
\frac{c_{N1} - \delta_{\{\lambda^i-\mu^i=0\}}\gamma_{N1}\frac{\beta_1}{2v_1}}{2b_N - \delta_{\{\lambda^i-\mu^i=0\}}\frac{\beta_N^2}{2v_N}} & \vdots & 0
\end{pmatrix}
$$

We observe that the matrix T is substochastic since

$$
\frac{\sum_{j\neq i} c_{ij} - \delta_{\{\lambda^i-\mu^i=0\}}\gamma_{ij}\frac{\beta_j}{2v_j}}{2b_i - \delta_{\{\lambda^i-\mu^i=0\}}\frac{\beta_i^2}{2v_i}} \leq \frac{\sum_{j\neq i} c_{ij}}{2b_i - \frac{\beta_i^2}{2v_i}} \leq \frac{b_i}{2b_i - \frac{\beta_i^2}{2v_i}} < \frac{b_i}{b_i} = 1.
$$

The third inequality follows from Assumption 5. Thus $A^{-1} = (I - T_1)^{-1}\Gamma_1^{-1}$ and

$$
\begin{aligned}
p_i^* &= \sum_{j\in\mathcal{I}}(A^{-1})_{ij}(b_j v_j + a_j + \delta_{\{\lambda^j-\alpha^j=0\}}\frac{\beta_j^2}{2} - \sum_{k\neq j}\delta_{\{\lambda^k-\alpha^k=0\}}\gamma_{jk}\frac{\beta_k}{2} \\
&\quad + \beta_j(\delta_{\{\lambda^j>0\}}f_j^{\max} + \delta_{\{\alpha^j>0\}}f_j^{\min}) - \sum_{k\neq i}\gamma_{jk}(\delta_{\{\lambda^j>0\}}f_k^{\max} + \delta_{\{\alpha^k>0\}}f_j^{\min})).
\end{aligned}
$$

We conclude that $(\mathbf{p}^*, \mathbf{f}^*)$ is the unique solution of Kuhn-Tucker conditions (21)-(22) and therefore the unique Nash equilibrium to the price-QoS game (5). $\square$

4. Case study II: Loss or rejection probability as QoS

In this section, QoS corresponds to the loss probability, either packet loss probability (e.g. at some common input queue) or rejection probability in a loss network (such as a circuit switched network): $f_i = (1 - P^i_{loss})^{1/s}$, where P^i_{loss} is the loss probability. i.e,. $P^i_{loss} := G(\rho_i)$, where $\rho_i = d_i/\mu_i$ is the traffic intensity, and $s \geq 1$ a scaling coefficient that adjusts the relative importance of the QoS parameter with respect to price. Note that when $s = 1$, QoS enters linearly in the demand function, D, whereas for $s > 1$, the QoS increases as a concave function of its parameters; in other words, for $s > 1$, we allow for *decreasing rates of return* on the QoS provided.[7]

We use the notation $G'(\rho)$ to denote the derivative with respect to ρ and we assume that $G'(\rho)$ is positive function over the interval $(0, +\infty)$. Hence, the loss probability is strictly increasing in ρ, and the function G^{-1} exists. Consequently, for all $i \in \mathcal{I}$,

$$
g_i(f_i) = \frac{1}{G^{-1}(1 - f_i^s)} \tag{24}
$$

Indeed, since G^{-1} exists, then $d_i/\mu_i = \rho_i = G^{-1}(1 - f_i)$, it follows that $\mu_i = \frac{1}{G^{-1}(1-f_i)}d_i$. Hence from (8), the function g_i is given by (24). Note that $f_i^{\min} = 0$ and $f_i^{\max} = 1$ for every $i \in \mathcal{I}$.

We present special cases for which the assumption A4 is verified. In 4.1 and 4.3 we consider only the linear relationship between QoS and loss probability, that is $s = 1$, whereas in 4.2 we give results for strictly concave f as well.

[7] *Decreasing rates of return* means that the marginal benefit of QoS decreases with increasing QoS levels This phenomenon is observed often in practice; the amount that a user is willing to pay, per unit of QoS, decreases as the level increases.

4.1. Packet loss probability: a basic M/M/1/K queuing model

here, the quality of service corresponds to the loss probability, or probability of a full buffer, in the M/M/1/K queue. We note that this could be a good approximation for loss probabilities if the **SP** offers access to a network, and the bottleneck in terms of losses is in some common input buffer to that network. Thus,

$$G(\rho_i) := \begin{cases} (1-\rho_i)\rho_i^{K_i}/(1-\rho_i^{K_i+1}) & \text{if } \; 0 \le \rho_i < 1, \; K_i = 1,2,3,.., \\ \\ 1/(K_i+1) & \text{if } \quad \rho_i = 1, \; K_i = 1,2,3,.., \end{cases} \tag{25}$$

where K_i is the buffer size. Clearly, the derivative $G'(\rho)$ is positive, hence we can use the above framework. Let the concavity coefficient satisfy $s = 1$, implying that QoS measure f and loss probabilities are linearly related.

The next lemma characterizes the function g_i with respect to service levels f_i, $i \in \mathcal{I}$.

Lemma 1 *The function $g_i(.) \in C^2$ is convex, increasing, and its derivative satisfies*

$$\lim_{f_i \to 1} g_i'(f_i) = +\infty, \; \textit{and} \; \lim_{f_i \to 0} g_i'(f_i) = 1.$$

Proof: See Appendix in the full version of the paper [5] . □

The above result implies that the function g_i verifies Assumption 4. Hence, from Theorem 2, we have the following result.

Corollary 1 *The price-QoS game (5) has a unique two-parameter Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* the unique solution to the equations*

$$g_i'(f_i^*) = \frac{\beta_i}{v_i b_i} \;\; \textit{if} \;\; \frac{\beta_i}{v_i b_i} \ge 1 \;\; \textit{and} \;\; f_i^* = 0 \;\; \textit{if} \;\; \frac{\beta_i}{v_i b_i} < 1$$

and $\mathbf{p}^*$ is then obtained through Theorem 1.

4.2. Packet loss probability: incorporating large-deviation scaling

Let P_{loss} be, as above, the loss probability for a M/M/1/K queue with K the buffer size at the queue. Given the probability of loss from (25), where $\rho_i := \frac{d_i}{\mu_i}$, and letting the arrival rate, capacity and buffer size be scaled by n, as in many-source large-deviation scaling, we obtain, letting $n \to \infty$, $\lim_{n\to\infty} \frac{(1-\rho_i)\rho_i^{nK}}{1-\rho_i^{nK+1}} = \frac{(d_i-\mu_i)^+}{d_i}$, with $(\cdot)^+ = \max\{0, \cdot\}$.

Note that even though the buffer size goes to ∞, the delay remains constant, as the capacity, μ_i, also goes to ∞. In a deterministic fluid model, this has the interpretation of fraction of fluid lost when the arrival rate exceeds capacity. Using (25), the QoS is then given by:

$$f_i = \left[1 - \frac{(d_i - \mu_i)^+}{d_i}\right]^{1/s}. \tag{26}$$

Note that (26) is not a differentiable function due to the max operation. However, since the QoS measure remains constant ($f_i = 1$) when capacity exceeds the demand, it is sufficient to consider that the largest capacity implemented would be $\mu_i = d_i$. Hence, we neglect the max in (26). The bandwidth required to guarantee QoS f_i is given by: $\mu_i = g_i(f_i)d_i$ with $g_i(f_i) = f_i^s$ The profit of **SP** i is then: $U_i(\mathbf{p}, \mathbf{f}) = D_i(\mathbf{p}, \mathbf{f})(p_i - v_i f_i^s)$, where the strategy space is given by the subset $\mathcal{R}_i$ with $\mathcal{R}_i = \{(p_i, f_i) : \; p_i^{\min} \le p_i \le p_i^{\max}; \; 0 \le f_i \le 1\}$.

We require the following assumption, which is likely to hold, since none of the constants β, v or b should be zero in this setting.

Assumption 6 *The constants b, v, and β satisfy: $\beta_i - b_i v_i \neq 0$*

Theorem 4 *The price-QoS problem (5) with Quality of Service $\mathbf{f}$ as in (26) has a unique two parameter Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* is given by:*

1. *For concave QoS coefficient, $s > 1$, that is, under decreasing rate of return on the QoS,*
$$f_i^* = \begin{cases} \left(\frac{\beta_i}{s v_i b_i}\right)^{\frac{1}{s-1}} & \text{if } \frac{\beta_i}{s v_i b_i} < 1, \\ 0 & \text{if } \frac{\beta_i}{s v_i b_i} \geq 1. \end{cases}$$
2. *For $s = 1$, that is, under constant rate of return on the QoS level, $\beta_i \neq b_i v_i$, $i \in \mathcal{I}$, so that*
$$f_i^* = \begin{cases} 1 & \text{if } \beta_i < v_i b_i, \\ 0 & \text{if } \beta_i > v_i b_i. \end{cases}$$

Proof: For all $s > 0$, the function g_i is convex, this implies that the function U_i is concave. Hence, we need only show that the profit function of each provider i is jointly log-concave in (p_i, f_i) for $s = 1$. Recall that $\tilde{U}_i$ denotes $\log U_i$. Note that

$$\begin{aligned}
\frac{\partial \tilde{U}_i(p,\mathbf{f})}{\partial p_i} &= \frac{1}{(p_i - v_i f_i)} - \frac{b_i}{D_i(p^*,\mathbf{f})} \text{ and } \frac{\partial \tilde{U}_i(p,\mathbf{f})}{\partial f_i} = -\frac{v_i}{(p_i - v_i f_i)} + \frac{\beta_i}{D_i(p^*,\mathbf{f})} \\
\frac{\partial^2 \tilde{U}_i(p,\mathbf{f})}{\partial p_i^2} &= \frac{-1}{(p_i - v_i f_i)^2} - \frac{b_i^2}{D_i^2(p^*,\mathbf{f})} < 0, \; \frac{\partial^2 \tilde{U}_i(p,\mathbf{f})}{\partial p_i \partial f_i} = \frac{v_i}{(p_i - v_i f_i)^2} + \frac{b_i \beta_i}{D_i^2(p^*,\mathbf{f})} \\
\frac{\partial^2 \tilde{U}_i(p,\mathbf{f})}{\partial f_i^2} &= \frac{-v_i^2}{(p_i - v_i f_i)^2} - \frac{\beta_i^2}{D_i^2(p^*,\mathbf{f})} < 0.
\end{aligned}$$

Since $\frac{\partial^2 \tilde{U}_i}{\partial f_i^2} < 0$ and $\frac{\partial^2 \tilde{U}_i}{\partial p_i^2} < 0$, it suffices to show that the determinant of the Hessian matrix H is positive, where $H = \begin{pmatrix} \frac{\partial^2 \tilde{U}_i}{\partial p_i^2} & \frac{\partial^2 \tilde{U}_i}{\partial p_i \partial f_i} \\ \frac{\partial^2 \tilde{U}_i}{\partial f_i \partial p_i} & \frac{\partial^2 \tilde{U}_i}{\partial f_i^2} \end{pmatrix}$. We have

$$\begin{aligned}
\det H[\tilde{U}_i(\mathbf{p},\mathbf{f})] &= \frac{\partial^2 \tilde{U}_i}{\partial f_i^2}\frac{\partial^2 \tilde{U}_i}{\partial p_i^2} - \left(\frac{\partial^2 \tilde{U}_i}{\partial p_i \partial f_i}\right)^2 = \Big(D_i(p_i - v_i f_i)\Big)^{-4} \\
&\quad \Big([D_i^2 + b_i^2(p_i - v_i f_i)^2][v_i^2 D_i^2 + \beta_i^2(p_i - v_i f_i)^2] - [v_i D_i^2 + b_i \beta_i (p_i - v_i f_i)^2]^2\Big) \\
&= \frac{(\beta_i - b_i v_i)^2}{D_i^2 (p_i - v_i f_i)^2} > 0
\end{aligned}$$

The latter inequality follows from Assumption 6.

Next, we shall establish the uniqueness of the equilibrium. Note first that any solution to the following Kuhn-Tucker conditions is a Nash equilibrium: For every $i \in \mathcal{I}$, there exist non-negative Lagrange multipliers α^i and λ^i such that

$$0 = \frac{\partial \tilde{U}_i(p,\mathbf{f})}{\partial p_i} = \frac{1}{(p_i - v_i f_i^s)} - \frac{b_i}{D_i(p^*,\mathbf{f})} \tag{27}$$

$$\lambda^i - \alpha^i = \frac{\partial \tilde{U}_i(p,\mathbf{f})}{\partial f_i} = -\frac{s v_i f_i^{s-1}}{(p_i - v_i f_i^s)} + \frac{\beta_i}{D_i(p^*,\mathbf{f})} \tag{28}$$

$$\alpha^i f_i = 0, \;\; \lambda^i(1 - f_i) = 0, \tag{29}$$

where, as before, α^i and λ^i are the Lagrange multipliers corresponding to the min and max constraints on the values of f_i; note, also as before, that we do not need Lagrange multipliers for the constraints on p_i since p_i^* is obtained at a point where the partial derivative (21) is zero, possibly at the lower boundary, (9) and (12). By multiplying (27) by β_i and (28) by b_i, and summing the resulting equations, we obtain

$$b_i(\lambda^i - \alpha^i) = \frac{\beta_i - s v_i b_i f_i^{s-1}}{(p_i - v_i f_i^s)} \tag{30}$$

Now, we are going to detemine the QoS for $s > 1$. We have use the fact that the partial derivative $\frac{\partial \tilde{U}_i(p,\mathbf{f})}{\partial f_i}$ is positive. Hence we do not need the Langrage mulitiplier α^i for the constraint $f_i - f_i^{min} \geq 0$. According to this, (30) becomes $b_i \lambda^i = \frac{\beta_i - s v_i b_i f_i^{s-1}}{(p_i - v_i f_i^s)}$. We deduce that

$$f_i^* = \begin{cases} \left(\frac{\beta_i}{sv_ib_i}\right)^{\frac{1}{s-1}} & \text{if} \quad \frac{\beta_i}{sv_ib_i} \le 1, \\ 1 & \text{if} \quad \frac{\beta_i}{sv_ib_i} > 1. \end{cases}.$$

Letting now $s = 1$. It follows, then, from (30) with $s = 0$ that $f_i^* = \begin{cases} 0 & \text{if} \quad \beta_i < v_ib_i \\ 1 & \text{if} \quad \beta_i \ge v_ib_i \end{cases}$. Fixing $\mathbf{f} = \mathbf{f}^*$, we obtain that $\mathbf{p}^*$ is the *unique* solution of the linear system given by theorem 1. Thus $(\mathbf{p}^*, \mathbf{f}^*)$ is consequently is the unique Nash equilibrium. ∎

4.3. Session rejection probability: the Erlang loss formula

Consider now a model handling blocked calls in which there are K servers available, and each newly arriving customer is given his own server; if a customer arrives when all servers are occupied, that customer's request is lost. This system is of particular interest in telephone networks, and is known as Erlang's loss formula. In terms of our notation, we have:

$$P_{loss}^i = G_i(\rho_i) = \frac{(\rho_i)^{K_i}/K_i!}{\sum_{j=0}^{K_i}(\rho_i)^j/j!}$$

We take the first derivatives of $G_i(.)$ with respect to ρ. We obtain

$$G_i'(\rho_i) = \frac{\sum_{j=0}^{K_i-1}(K_i-j)\rho_i^{K_i+j-1}/K_i!j!}{[\sum_{j=0}^{K_i}(\rho_i)^j/j!]^2} > 0$$

Hence the function G_i^{-1} exists.

Let the concavity coefficient, $s = 1$, i.e., QoS measure f and loss probabilities are linearly related. Next we characterize the function g_i with respect to f_i.

Lemma 2 *The function $g_i(.) \in C^2$ is convex, increasing, and its derivative satisfies*

$$\lim_{f_i \to 1} g_i'(f_i) = +\infty, \text{ and } \lim_{f_i \to 0} g_i'(f_i) = \frac{1}{K_i}$$

Proof: See Appendix in the full version of the paper [5]. ∎

Combining the last lemma with Theorem 2 we obtain the following result.

Corollary 2 *The two-parameter price-QoS game (5) has a unique Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* the unique solution to the equations* $\begin{cases} g_i'(f_i^*) & = & \frac{\beta_i}{v_ib_i} \ \text{if} \ K_i \ge \frac{v_ib_i}{\beta_i} \\ f_i^* & = & 0 \ \text{if} \ K_i < \frac{v_ib_i}{\beta_i} \end{cases}$ *and $\mathbf{p}^*$ is obtained through Theorem 1.*

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